The Strong Brauer Group of a Cocommutative Coalgebra

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1 Introduction

The Brauer group of a cocommutative coalgebra C, denoted by $Br(C)$, was constructed in [15] by taking the Morita-Takeuchi equivalence relation on the set of Azumaya C-coalgebras. This theory presents some differences with respect to the Brauer group theory of commutative algebras, e.g. it is not a torsion group. There is not in general a good relation between $Br(C)$ and the Brauer group of the dual algebra $Br(C^*)$. This is due to the fact that the dual algebra of an Azumaya C-coalgebra is usually not an Azumaya algebra over C^*. However, it has been shown in [2] that when C is irreducible a complete duality does follow and $Br(C)$ is a subgroup of $Br(C^*)$.

The aim of this paper is to extend this result by finding a subgroup of $Br(C)$ which is a subgroup of $Br(C^*)$ for an arbitrary C. The key is to use strong equivalences, studied by Lin in [7], instead of Morita-Takeuchi equivalences. In this theory the finitely cogenerated comodules replace the quasi-finite ones. We define strong Azumaya C-coalgebras as those Azumaya C-coalgebras which are finitely cogenerated as C-comodules. By considering the strongly similar equivalence relation on the set of such coalgebras, we obtain a new group $Br^s(C)$, called the strong Brauer group of C, Theorem 4.3. It is proved that the dual of a strong Azumaya C-coalgebra is an Azumaya algebra over C^*, Theorem 4.6. This is done by showing that the dual of the Morita-Takeuchi context associated to a finitely cogenerated injective comodule P_C is exactly the derived Morita context of $P^*_{C^*}$, Proposition 3.6. Thus we have a group morphism $(-)^* : Br^s(C) \to Br(C^*), [D] \mapsto [D^*]$. Using the linear topology of all closed and cofinite left ideals and arguments from localization theory we may prove that $(-)^*$ is injective, Theorem 4.8. Hence $Br^s(C)$ is in particular a torsion group. This allows one to obtain several interesting generalizations of earlier results, Remark 4.13. Some cases where $(-)^* : Br(C) \to Br(C^*)$ is an isomorphism are studied (Theorems 4.12, 4.14) e.g. C being coreflexive.

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2 Notation and preliminaries

Throughout k is a fixed field. Unless otherwise stated, all vector spaces, algebras, coalgebras, unadorned $\otimes, Hom$, etc ... are over k.

Coalgebras and comodules (see [13], [1]): For a coalgebra C, we let Δ_C, ε_C denote the comultiplication and counit respectively, and C^* its dual algebra. The category of right comodules is denoted by $\mathcal{M}^C$. For $X, Y \in \mathcal{M}^C$, $Com_{-C}(X,Y)$ denote the space of all C-comodule maps from X to Y. By ρ_X we denote the C-comodule structure map of X. We use the usual sigma notation for coalgebras and comodules. We will also use the fact that right comodules are left rational C^*-modules. An $X \in \mathcal{M}^C$ is said to be *finitely cogenerated* if there is an injective C-comodule map $f : X \to W \otimes C$ for some finite dimensional space W. The C-comodule $W \otimes C$ is nothing but the direct sum $C^{(n)}$ where $n = dim(W)$.

Morita-Takeuchi theory (see [14]): $X \in \mathcal{M}^C$ is called *quasi-finite* if $Com_{-C}(Y,X)$ is finite dimensional for all finite dimensional $Y \in \mathcal{M}^C$. We recall from [14] the definition of the co-hom functor, co-endomorphism coalgebra and some of its properties.

Lemma 2.1 *Let ${}_DX_C$ be a bicomodule. X_C is quasi-finite if and only if the cotensor product functor $-\Box_D X : \mathcal{M}^D \to \mathcal{M}^C$ has a left adjoint functor, denoted by $h_{-C}(X,-)$. For $Y \in \mathcal{M}^C$,*

$$h_{-C}(X,Y) = \varinjlim_{\lambda\in\Lambda} Com_{-C}(Y_\lambda, X)^*,$$

where $\{Y_\lambda\}_{\lambda\in\Lambda}$ is a directed family of finite dimensional subcomodules of Y such that $Y = \varinjlim_{\lambda\in\Lambda} Y_\lambda$.

The functor $h_{-C}(X,-)$ is called the *co-hom functor*. Let $\theta_{X,Y} : Y \to h_{-C}(X,Y)\Box_D X$ denote the unit of the adjunction. Assuming that $D = k$ and X_C quasi-finite, $e_{-C}(X) = h_{-C}(X,X)$ is a coalgebra, called the *co-endomorphism coalgebra* of X, and X is a $(e_{-C}(X), C)$-bicomodule via $\theta_{X,X} : X \to e_{-C}(X) \otimes X$. From the adjoint situation we get an isomorphism

$$\zeta_X : e_{-C}(X)^* \cong Hom(h_{-C}(X,X),k) \cong Com_{-C}(X, k\otimes X) \cong Com_{-C}(X,X).$$

It is defined as $\zeta_X(u) = (u \otimes 1)\theta_{X,X}$ for all $u \in e_{-C}(X)^*$. Taking the opposite multiplication in $Com_{-C}(X,X)$, ζ_X becomes an algebra isomorphism, see [4, Lemma 1.11].

A *Morita-Takeuchi context* (C, D, P, Q, f, g) consists of coalgebras C, D, bicomodules ${}_CP_D$, ${}_DQ_C$, and bicolinear maps $f : C \to P\Box_D Q$, and $g : D \to Q\Box_C P$ such that

$$\sum_{(p)} p_{(0)} \otimes g(p_{(1)}) = \sum_{(p)} f(p_{(-1)}) \otimes p_{(0)}, \qquad \sum_{(q)} q_{(0)} \otimes f(q_{(1)}) = \sum_{(q)} g(q_{(-1)}) \otimes q_{(0)},$$

for all $p \in P, q \in Q$. The context is said to be *strict* if f and g are injective (equiv. isomorphisms). In this case, the functors $-\Box_C P$ and $-\Box_D Q$ establish an equivalence between $\mathcal{M}^C$ and $\mathcal{M}^D$. C and D are called *Morita-Takeuchi* equivalent coalgebras.

Hereditary pretorsion classes and equivalences (see [2]): The category of comodules $\mathcal{M}^C$ may be considered as the hereditary pretorsion class associated to the linear topology $\mathcal{F}_C$ of all closed cofinite left ideals of C^*, see [11]. This will be a key fact throughout this paper. For detail on torsion theory cf. [12]. Let R be an algebra, and ${}_R\mathcal{M}$ the category of left R-modules. A left linear topology $\mathcal{T}$ on R is said to be *symmetric* if for every $I \in \mathcal{T}$ there is a two-sided ideal J of R such that $J \subseteq I$ and $J \in \mathcal{T}$.

Suppose that R is a commutative algebra and A an R-algebra. If $\mathcal{T}$ is a linear topology on R, the family $\mathcal{T}A = \{J \leq {}_A A : IA \subseteq J \text{ for some } I \in \mathcal{T}\}$ is a symmetric linear topology on A. If $\mathcal{F}$ is a symmetric left linear topology on A, the family $\mathcal{F} \cap R = \{J \leq R : I \cap R \subseteq J$ for some two-sided ideal $I \in \mathcal{F}\}$, is a linear topology on R. When A is R-Azumaya, $\mathcal{T} = \mathcal{T}A \cap R$ and $\mathcal{F} = (\mathcal{F} \cap R)A$. This is due to the bijective correspondence between the lattice of ideals of R and the lattice of two-sided ideals of A, see [9, Corollary 2.11]. For the definition of Azumaya algebra, the Brauer group of a commutative ring and its more important properties we refer to [8], [9]. Finally we recall from [2, Theorem 3.3] the following result which will be very useful in the sequel.

Theorem 2.2 *Let R be a commutative algebra, $\mathcal{T}$ a linear topology on R and A, B two R-algebras. Let $\mathcal{C}_A$ and $\mathcal{C}_B$ be the hereditary pretorsion classes associated to the induced topologies $\mathcal{T}A, \mathcal{T}B$ on A and B respectively. If A and B are Morita equivalent over R, then the restriction is an equivalence between $\mathcal{C}_A$ and $\mathcal{C}_B$.*

3 Strong equivalences revisited

The strong equivalences, studied by Lin in [7], are a particular case of equivalences between categories of comodules. Given two coalgebras C and D, the categories of right C-comodules $\mathcal{M}^C$, $\mathcal{M}^D$ may be embedded, via rational modules, in ${}_{C^*}\mathcal{M}, {}_{D^*}\mathcal{M}$ respectively. A *strong equivalence* between $\mathcal{M}^C$ and $\mathcal{M}^D$ is an equivalence which is induced by an equivalence between ${}_{C^*}\mathcal{M}$ and ${}_{D^*}\mathcal{M}$. In this case, C and D are called *strongly equivalent*. These equivalences were characterized in [7] in terms of ingenerators. We recall that a right C-comodule P is an *ingenerator* if it is a finitely cogenerated injective cogenerator. An equivalence

$$\begin{tikzcd} \mathcal{M}^C \arrow[r, shift left, "F"] & \mathcal{M}^D \arrow[l, shift left, "G"] \end{tikzcd}$$

is strong if and only if $F(C), G(D)$ are ingenerators in $\mathcal{M}^C$ and $\mathcal{M}^D$ respectively. The main difference of this theory with respect to Takeuchi's theory is the use of finitely cogenerated comodules instead of quasi-finite ones.

The aim of this section is to study the relation between the Morita-Takeuchi context associated to an equivalence and the Morita context obtained by the dualization procedure. When the equivalence is strong, one is strict if and only if the other one is.

If M is a (D, C)-bicomodule, then the dual space M^* is a (D^*, C^*)-bimodule via the actions:

$$\langle d^* \cdot m^*, m \rangle = \sum_{(m)} \langle d^*, m_{(-1)} \rangle \langle m^*, m_{(0)} \rangle, \qquad \langle m^* \cdot c^*, m \rangle = \sum_{(m)} \langle m^*, m_{(0)} \rangle \langle c^*, m_{(1)} \rangle,$$

for all $c^* \in C^*, d^* \in D^*, m^* \in M^*$ and $m \in M$. We recall from [3, Lemma 4.3 i)] that $M \in \mathcal{M}^C$ is finitely cogenerated if and only if M^* is finitely generated as C^*-module.

Lemma 3.1 *Let M be a finitely cogenerated right C-comodule. M_C is injective if and only if $M^*_{C^*}$ is projective.*

Proof: Note that M is finitely cogenerated injective if and only if it is a direct summand of $C^{(n)}$ for some $n \in \mathbb{N}$. Hence M^* is a direct summand of $C^{*(n)}$, and thus it is projective.

Conversely, let $f : M \to C^{(n)}$ be the injective C-comodule map given by hypothesis. The dual map $f^* : M^* \to C^{*(n)}$ is a surjective C^*-module map. Since M^* is projective, there is a C^*-module map $g : C^{*(n)} \to M^*$ such that $f^*g = 1_{M^*}$. Then $C^{*(n)} = g(M^*) \oplus ker(f^*)$. From this, $C^{**(n)} = g(M^*)^{\perp} \oplus (ker(f^*))^{\perp}$, and hence $Rat(C^{**(n)}) = Rat(g(M^*)^{\perp}) \oplus Rat(ker(f^*)^{\perp})$. Denote by $\lambda_C : C \to C^{**}$ the canonical embedding. By [7, Lemma 1], $Rat(C^{**}) = \lambda_C(C)$. Then $Rat(C^{**(n)}) = \lambda_C^{(n)}(C^{(n)})$, where $\lambda_C^{(n)} : C^{(n)} \to C^{**(n)}$ is the canonical injection. On the other hand, $ker(f^*)^{\perp} = Im(f)^{\perp\perp} \subset C^{**(n)}$. We claim that $Rat(Im(f)^{\perp\perp}) = \lambda_C^{(n)}(Im(f))$. If $x \in Rat(Im(f)^{\perp\perp})$, then $x \in Rat(C^{**(n)})$. There is $d \in C^{(n)}$ such that $x = \lambda_C^{(n)}(d)$. For any $y \in Im(f)^{\perp}$, $\langle y, d\rangle = \langle \lambda_C^{(n)}(d), y\rangle = \langle x, y\rangle = 0$. Taking now $\perp$ in $C^{(n)}$, $d \in Im(f)^{\perp\perp} = Im(f) \subset C^{(n)}$. The other inclusion is clear. ∎

Lemma 3.2 *Let C, D be coalgebras and ${}_CP_D, {}_CQ_D$ bicomodules. If P_D (resp. ${}_CP$) is finitely cogenerated and injective and ${}_CQ$ (resp. Q_D) is finitely cogenerated, then ${}_Ch_{-D}(P,Q)$ (resp. $h_{C-}(P,Q)_D$) is finitely cogenerated.*

Proof: By the hypothesis, P is a direct summand of $D^{(n)}$. Let $i : P \to D^{(n)}, \pi : D^{(n)} \to P$ be respectively the inclusion and projection maps. These induce C-colinear maps $u : h_{-D}(P,Q) \to h_{-D}(D^{(n)},Q), v : h_{-D}(D^{(n)},Q) \to h_{-D}(P,Q)$ such that $vu = 1$. Now $h_{-D}(D^{(n)},Q) \cong Q^{(n)}$ as C-comodules. Since Q is a finitely cogenerated C-comodule, $h_{-D}(P,Q)$ is too. ∎

Lemma 3.3 *Let ${}_DM_C$ and ${}_CN_E$ be bicomodules. The map*

$$\eta_{M,N} : M^* \otimes_{C^*} N^* \to (M \square_C N)^*, \quad \langle \sum_i m_i^* \otimes n_i^*, \sum_j m_j \otimes n_j\rangle = \sum_{i,j} \langle m_i^*, m_j\rangle\langle n_i^*, n_j\rangle$$

is a (D^*, E^*)-bimodule map. If, in addition, $M_C, {}_CN$ are finitely cogenerated and injective, then it is an isomorphism.

Proof: An easy computation shows that $\eta_{M,N}$ is a bimodule map. We check the second claim. Let $n, m \in \mathbb{N}$, if $\gamma : C^{*(n)} \otimes_{C^*} C^{*(m)} \to C^{*(nm)}$ and $\xi : C^{(n)} \square_C C^{(m)} \to C^{(nm)}$ are the canonical isomorphisms, then it may be easily checked that $\xi^* \eta_{C^{(n)},C^{(m)}} = \gamma$. It is not hard to extend the result for direct summands of $C^{(n)}, C^{(m)}$. ∎

Definition 3.4 *A Morita-Takeuchi context (C, D, P, Q, f, g) is said to be strong if P_D, Q_C are finitely cogenerated and injective.*

Proposition 3.5 *Let (C, D, P, Q, f, g) be a Morita-Takeuchi context. It is strong and strict if and only if $(C^*, D^*, P^*, Q^*, f^*\eta_{P,Q}, g^*\eta_{Q,P})$ is a strict Morita context. As a consequence, if C and D are strongly equivalent, then C^* and D^* are Morita equivalent.*

Proof: $\Rightarrow$) Given $p \in P$, we may write $f(p_{(-1)}) = \sum_i m_i \otimes n_i \in P\Box_D Q$ and $g(p_{(1)}) = \sum_j m'_j \otimes n'_j \in Q\Box_C P$. The condition of being a Morita-Takeuchi context transforms to:

$$\sum_{(p)} \sum_i m_i \otimes n_i \otimes p_{(0)} = \sum_{(p)} \sum_j p_{(0)} \otimes m'_j \otimes n'_j.$$

Write $\phi^*_C : C^* \otimes_{C^*} P^* \to P^*$ and $\phi^*_D : P^* \otimes_{D^*} D^* \to P^*$ for the canonical isomorphisms. Let $p^*, r^* \in P^*$ and $q^* \in Q^*$,

$$\begin{aligned}\langle[\phi_{C^*}((f^*\eta_{P,Q}) \otimes 1)](p^* \otimes q^* \otimes r^*), p\rangle &= \textstyle\sum_{(p)} \langle \eta_{P,Q}(p^* \otimes q^*), f(p_{(-1)})\rangle\langle r^*, p_{(0)}\rangle \\ &= \textstyle\sum_{(p)} \sum_i \langle p^*, m_i\rangle\langle q^*, n_i\rangle\langle r^*, p_{(0)}\rangle \\ &= \textstyle\sum_{(p)} \sum_j \langle p^*, p_{(0)}\rangle\langle q^*, m'_j\rangle\langle r^*, n'_j\rangle \\ &= \textstyle\sum_{(p)} \langle p^*, p_{(0)}\rangle\langle \eta_{Q,P}(q^* \otimes r^*), g(p_{(1)})\rangle \\ &= \langle[\phi_{D^*}(1 \otimes (g^*\eta_{Q,P}))](p^* \otimes q^* \otimes r^*), p\rangle.\end{aligned}$$

Hence $\phi_{C^*}((f^*\eta_{P,Q}) \otimes 1) = \phi_{D^*}(1 \otimes (g^*\eta_{Q,P}))$. Similarly $\phi_{D^*}((g^*\eta_{Q,P}) \otimes 1) = \phi_{C^*}(1 \otimes (f^*\eta_{Q,P}))$ where now ϕ_{C^*} and ϕ_{D^*} denote the canonical isomorphisms for Q^*. Note that we does not need for this the context to be strict.

As the context is strict, [14, Theorem 2.5] implies that P, Q are left and right injective, $P \cong h_{-D}(Q, D)$ and $Q \cong h_{C-}(P, C)$. Lemma 3.2 yields that ${}_CP, {}_DQ$ are finitely cogenerated, and by Lemma 3.3, $\eta_{P,Q}, \eta_{Q,P}$ are isomorphisms. From this, it follows that $f^*\eta_{P,Q}, g^*\eta_{Q,P}$ are isomorphisms.

$\Leftarrow$) The Morita theorem implies that $P^*_{D^*}, Q^*_{C^*}$ are finitely generated. Then P_D, Q_C are finitely cogenerated, see [3, Lemma 4.3]. Since the maps $f^*\eta_{P,Q}, g^*\eta_{Q,P}$ are isomorphisms, f, g are injective.

Assume that C and D are strongly equivalent. By [14, Proposition 2.1, Theorem 3.5], there is a Morita-Takeuchi context (C, D, P, Q, f, g) where P_D, Q_C are finitely cogenerated. Now it suffices to apply the foregoing result and the classical Morita theorem. ∎

The Morita context $(C^*, D^*, P^*, Q^*, f^*\eta_{P,Q}, g^*\eta_{Q,P})$ will be called the *dual context* of (C, D, P, Q, f, g). According to [14, Theorem 3.5], strong equivalences are given by an strong and strict Morita-Takeuchi context. The preceding proposition provides a different proof of [7, Theorem 5] from Takeuchi's results.

Let P be a quasi-finite right C-comodule and $D = e_{-C}(P)$. Consider the Morita-Takeuchi context associated to it ([14, page 639]), $D = e_{-C}(P), Q = h_{-C}(P, C)$ and the bicolinear maps $\theta_{P,C} : C \to h_{-C}(P, C)\Box_D P$ and $\delta : D \to P\Box_C Q$. Recall that δ is the unique bicolinear map verifying $(1\Box\theta_{P,C})\rho_P = (\delta\Box 1)\theta_{P,P}$.

Proposition 3.6 *If P_C is finitely cogenerated and injective, then the dual Morita context of $(C, D, P, Q, \theta_{P,C}, \delta)$ may be identified with the Morita context associated to $P^*_{C^*}$.*

Proof: We recall that the Morita context associated to P^* is given by the following data: $R = End_{-C^*}(P^*), C^*, P^*, \bar{Q} = Hom_{-C^*}(P^*, C^*), g : \bar{Q} \otimes_R P^* \to C^*, \bar{q} \otimes p^* \mapsto \bar{q}(p^*)$ and $f : P^* \otimes_{C^*} \bar{Q} \to R$ defined as $f(p^* \otimes \bar{q})(m^*) = m^*\bar{q}(p^*)$ for all $p^*, m^* \in P^*$ and $\bar{q} \in \bar{Q}$.

We prove that, under the suitable identifications, this Morita context is the dual of the Morita-Takeuchi context associated to P. We first establish these identifications. Recall from [7, Lemma 4] that there is a natural transformation

$$ad : Hom_{C^*-}(-, P) \to Hom_{-C^*}(P^*, (-)^*).$$

Given $Q \in \mathcal{M}^C$, it is defined as $\langle ad(\varphi)(p^*), q\rangle = \langle p^*, \varphi(q)\rangle$ for any $q \in Q, \varphi \in Com_{-C}(Q, P)$. In case P, Q are bicomodules, ad is a bimodule map. For $Q = P$, ad is an algebra isomorphism when taking the opposite multiplication in $Com_{-C}(P, P)$.

Let $\zeta_C : h_{-C}(P, C)^* \to Com_{-C}(C, P)$ be the isomorphism from the adjoint situation in Lemma 2.1. Recall that it is defined as $\zeta(u) = (u \otimes 1)\theta_{P,C}$ for all $u \in h_{-C}(P, C)^*$. Let $\Phi : h_{-C}(P, C)^* \to Hom_{-C^*}(P^*, C^*)$ be the composition $ad\zeta_C$. Explicitly, $\langle \Phi(u)(p^*), c\rangle = \langle p^*, (u\otimes 1)\theta_{P,C}(c)\rangle$. Replacing C by P, we get an algebra isomorphism $\Psi : D^* \to End_{-C^*}(P^*)$ given by $\langle \Psi(d^*)(p^*), p\rangle = \langle d^* \cdot p^*, p\rangle$ for all $d^* \in D^*, p^* \in P^*$ and $p \in P$.

We check that $\theta^*_{P,C}\eta_{Q,P}(\Phi^{-1} \otimes 1) = g$ and $\Psi\delta^*\eta_{P,Q}(1 \otimes \Phi^{-1}) = f$. Let $p^*, p_1^*, p_2^* \in P^*, p \in P$ and $\varphi \in Hom_{-C^*}(P^*, C^*)$. Assume that $\Phi(q^*) = \varphi$ for some $q^* \in h_{-C}(P, C)^*$.

$$\begin{aligned}
\langle[\theta^*_{P,C}\eta_{Q,P}(\Phi^{-1} \otimes 1)](\varphi \otimes p^*), c\rangle &= \langle \eta_{Q,P}(q^* \otimes p^*), \theta_{P,C}(c)\rangle \\
&= \langle p^*, (q^* \otimes 1)\theta_{P,C}(c)\rangle \\
&= \langle \Phi(q^*)(p^*), c\rangle \\
&= \langle \varphi(p^*), c\rangle \\
&= \langle g(\varphi \otimes p^*), c\rangle. \\
\langle[\Psi\delta^*\eta_{P,Q}(1 \otimes \Phi^{-1})](p_1^* \otimes \varphi)(p_2^*), p\rangle &= \langle(((\delta^*\eta_{P,Q})(p_1^* \otimes q^*)) \cdot p_2^*, p\rangle \\
&= \textstyle\sum_{(p)}\langle \eta_{P,Q}(p_1^* \otimes q^*), \delta(p_{(-1)})\rangle\langle p_2^*, p_{(0)}\rangle \\
&= \textstyle\sum_{(p)}\langle p_1^*, p_{(0)}\rangle\langle \eta_{Q,P}(q^* \otimes p_2^*), \theta_{P,C}(p_{(1)})\rangle \\
&= \textstyle\sum_{(p)}\langle p_1^*, p_{(0)}\rangle\langle p_2^*, (q^* \otimes 1)\theta_{P,C}(p_{(1)})\rangle \\
&= \textstyle\sum_{(p)}\langle p_1^*, p_{(0)}\rangle\langle \Phi(q^*)(p_2^*), p_{(1)}\rangle \\
&= \langle p_1^* \cdot [\varphi(p_2^*)], p\rangle \\
&= \langle f(p_1^* \otimes \varphi)(p_2^*), p\rangle. \quad \blacksquare
\end{aligned}$$

Corollary 3.7 *If P_C is an ingenerator, then C and $e_{-C}(P)$ are strongly equivalent.*

Proof: In view of [14, Theorem 3.5] the problem is reduced to proving that $Q = h_{D-}(P, D)$ is finitely cogenerated as a right C-comodule. As P_C is an ingenerator, $P^*_{C^*}$ is a progenerator, [7, page 319]. Thus P^* is finitely generated as left $End_{-C^*}(P^*)$-module. Identifying D^* with $End_{-C^*}(P^*)$ via Ψ, one sees that the $End_{-C^*}(P^*)$-module structure of P^* is induced by the D-comodule structure of P. Hence P is finitely cogenerated as D-comodule. Lemma 3.2 gives that Q_C is finitely cogenerated. $\blacksquare$

In the rest of the paper C will be a cocommutative coalgebra. We recall that a coalgebra D is said to be a *coalgebra over C or a C-coalgebra* if there is a coalgebra map $\epsilon : D \to C$

(called the *C-counit*) such that

$$\sum_{(d)} \epsilon(d_{(1)}) \otimes d_{(2)} = \sum_{(d)} \epsilon(d_{(2)}) \otimes d_{(1)} \qquad \forall d \in D.$$

The coalgebra D becomes a C-comodule via ϵ and the dual algebra D^* is an algebra over C^* via $\epsilon^* : C^* \to D^*$. Let D, E be two C-coalgebras with C-counits ϵ_D and ϵ_E respectively. A (D, E)-bicomodule M is called a *bicomodule over* C if the following diagram is commutative,

$$\begin{tikzcd} M \arrow[r, "\rho_E"] \arrow[d, "\rho_D"'] & M \otimes E \arrow[r, "1 \otimes \epsilon_E"] & M \otimes C \arrow[d, equal] \\ D \otimes M \arrow[r, "\epsilon_D \otimes 1"] & C \otimes M \arrow[r, "\tau"] & M \otimes C \end{tikzcd}$$

where τ is the twist map. We know from [14, Proposition 2.1] that any (resp. strong) equivalence $F : \mathcal{M}^D \to \mathcal{M}^E$ is of the form $-\square_D P$ for a suitable (D, E)-bicomodule P. We will say that F is an *equivalence over* C if P is a bicomodule over C. In this case D, E will be called *Morita-Takeuchi (or strongly) equivalent over* C.

Proposition 3.8 *Let D, E be C-coalgebras. Suppose that*

$$\begin{tikzcd} {}_{D^*}\mathcal{M} \arrow[r, shift left, "F"] & {}_{E^*}\mathcal{M} \arrow[l, shift left, "G"] \end{tikzcd}$$

is an equivalence over C^* verifying that $F(\mathcal{M}^D) \subseteq \mathcal{M}^E$ and $G(\mathcal{M}^E) \subseteq \mathcal{M}^D$. Then D and E are strongly equivalent over C.

Proof: By the Morita theorem, $F(-) \cong - \otimes_{D^*} P$ where P is a (D^*, E^*)-bimodule centralized by C^*, that is, $\epsilon_D^*(c^*)p = p\epsilon_E^*(c^*)$ for all $p \in P, c^* \in C^*$. The restriction of F, G to $\mathcal{M}^D, \mathcal{M}^E$, denoted by $\bar{F}, \bar{G}$ respectively, establishes an equivalence between $\mathcal{M}^C$ and $\mathcal{M}^D$. In view of [14, Proposition 2.1], $\bar{F} \cong -\square_D M, \bar{G} \cong -\square_E N$ where $M = \bar{F}(D), N = \bar{G}(E)$ are (D, E) and (E, D)-bicomodules respectively. From [7, Proposition 2], it follows that M_E and N_D are finitely cogenerated, and thus C and D are strongly equivalent. We have to check that the equivalence is over C, that is, M is a bicomodule over C.

By definition, $M = D \otimes_{D^*} P$ and its D-comodule structure map ρ_D is $F(\Delta_D)$, see [14, Proposition 2.1]. For $m = \sum_i d_i \otimes p_i \in M$, $\rho_D(m) = \sum_i d_{i(1)} \otimes d_{i(2)} \otimes p_i$. Let

$$a = \tau(\epsilon_D \otimes 1)\rho_D(m) = \textstyle\sum_{(d)} d_{i(2)} \otimes p_i \otimes \epsilon_D(d_{i(1)}),$$
$$b = (1 \otimes \epsilon_E)\rho_E(m) = \textstyle\sum_{(m)} m_{(0)} \otimes \epsilon_E(m_{(1)}).$$

Taking $c^* \in C^*$ arbitrary, we have:

$$
\begin{aligned}
(1\otimes c^*)(b) &= \textstyle\sum_{(m)}\langle c^*, \epsilon_E(m_{(1)})\rangle m_{(0)} \\
&= (\textstyle\sum_i d_i\otimes p_i)\epsilon_E^*(c^*) \\
&= \textstyle\sum_i d_i\otimes (p_i\epsilon_E^*(c^*)) \\
&= \textstyle\sum_i d_i\otimes (\epsilon_D^*(c^*)p_i) \\
&= \textstyle\sum_i (d_i\epsilon_D^*(c^*))\otimes p_i \\
&= \textstyle\sum_i\sum_{(d_i)}\langle c^*, \epsilon_D(d_{i(1)})\rangle d_{i(2)}\otimes p_i \\
&= (1\otimes c^*)(a).
\end{aligned}
$$

Consider $\lambda_C : C \to C^{*0}$ the canonical injection defined by $\langle \lambda_C(c), c^*\rangle = \langle c^*, c\rangle$ for all $c \in C, c^* \in C^*$. Then the map $1\otimes\lambda_C : M\otimes C \to M\otimes C^{*0}$ is injective. With this notation, the foregoing equality yields that $(1\otimes\lambda_C)(a) = (1\otimes\lambda_C)(b)$. Therefore $a = b$ and thus M is a bicomodule over C. ∎

4 The strong Brauer group

The Brauer group of a cocommutative coalgebra C, denoted by $Br(C)$, was introduced in [15] by considering the Morita-Takeuchi equivalence relation on the set of Azumaya C-coalgebras (see loc. cit. for further details). If we deal with strong equivalences instead of Morita-Takeuchi equivalences, a new subgroup of $Br(C)$ appears, the strong Brauer group. In this section, we introduce this subgroup and study some of its properties.

Definition 4.1 *A coalgebra D is said to be a strong Azumaya C-coalgebra if D is an Azumaya C-coalgebra and D is finitely cogenerated as C-comodule.*

Lemma 4.2 *Let $B^s(C)$ denote the set of isomorphism classes of strong Azumaya C-coalgebras.*

i) If $P \in \mathcal{M}^C$ is an ingenerator, then $e_{-C}(P) \in B^s(C)$.

ii) If $D, E \in B^s(C)$, then $D^{cop}, D\square_C E \in B^s(C)$.

iii) If C' is a cocommutative coalgebra and $f : C' \to C$ a coalgebra map, then $D\square_C C' \in B^s(C')$.

Proof: From [15, Example 2.8, Corollary 3.1], it follows that $e_{-C}(P), D^{cop}, D\square_C E, D\square_C C'$ are Azumaya coalgebras. We have only to prove that they are strong.

i) $e_{-C}(P)$ is a C-coalgebra via the map $\epsilon : e_{-C}(P) \to C$ defined as the unique coalgebra map ϵ making the following diagram commutative:

$$
\begin{tikzcd}
P \arrow[r, "\rho"] \arrow[dr, "\theta_{P,P}"'] & P\otimes C \arrow[r, "\tau"] & C\otimes P \\
& e_{-C}(P)\otimes P \arrow[ur, "\epsilon\otimes 1"'] &
\end{tikzcd}
$$

The C-comodule structure of $e_{-C}(P)$ via ϵ coincides with the C-comodule structure induced by P. The claim now follows from Lemma 3.2.

ii), iii) D^{cop} is finitely cogenerated since $D = D^{cop}$ as a C-comodule. Assume that D, E embed in $C^{(n)}, C^{(m)}$ respectively. The left exactness of the cotensor product implies that $D\Box_C E, D\Box_C C'$ embed in $C^{(nm)}, C'^{(n)}$ respectively. ∎

We say that $D, E \in B^s(C)$ are *strongly similar*, denoted by $\sim^s$, if there are ingenerators $P, Q \in \mathcal{M}^C$ such that $D\Box_C e_{-C}(P) \cong E\Box_C e_{-C}(Q)$ as C-coalgebras. It is not hard to check that $\sim^s$ is an equivalence relation.

Theorem 4.3 *The quotient set $Br^s(C) = B^s(C)/\sim^s$ is a subgroup of $Br(C)$. Moreover, a map of cocommutative coalgebras $f : C \to C'$ induces an homomorphism $f_* : Br^s(C') \to Br^s(C), [D] \mapsto [D\Box_C C']$.*

Proof: Follows from Lemma 4.2. ∎

Proposition 4.4 *$D, E \in B^s(C)$ are strongly similar if and only if D and E are strongly equivalent coalgebras over C. $[D] = [C] \in Br^s(C)$ if and only if there is a ingenerator $P \in \mathcal{M}^C$ such that $D \cong e_{-C}(P)$.*

Proof: Analogous to [15, Proposition 4.4, Corollary 4.5] taking into account that we are dealing with strong equivalences. ∎

The group $Br^s(C)$ is called the *strong Brauer group of* C. The quotient group $Br(C)/Br^s(C)$ represents the influence of the difference between strong equivalences and the usual ones.

Proposition 4.5 *If C has finite dimensional coradical, then $Br^s(C) = Br(C)$.*

Proof: In this case every quasi-finite comodule is finitely cogenerated. Hence every equivalence is an strong equivalence. See [7, page 322] ∎

A coalgebra D may be viewed as a right D^e-comodule where D^e is the enveloping C-coalgebra $D^e = D\Box_C D^{cop}$. The co-endomorphism coalgebra $C = e_{-D^e}(D)$ is the cocenter of D, see [15, Theorem 3.14]. Consider the Morita-Takeuchi context

$$(C, D^e, D, h_{-D^e}(D, D^e), f, g), \qquad (2)$$

associated to D_{D^e}.

Theorem 4.6 *Let D be a coalgebra. The following assertions are equivalent:*

i) D is a strong Azumaya coalgebra.

ii) The Morita-Takeuchi context (2) *is strong and strict.*

iii) ${}_C D$ is a ingenerator and $e_{C-}(D) \cong D^e$.

iv) D^* is an Azumaya algebra over C^*.

Proof: *i) $\Rightarrow$ ii)* (2) is strict by [15, Theorem 3.14]. Since D^e is a C-coalgebra and D is finitely cogenerated as a C-comodule, D is finitely cogenerated as a D^e-comodule. Lemma 3.2 now applies.

ii) $\Rightarrow$ iii) Follows from [14, Theorem 2.5].

iii) $\Rightarrow$ i) This is [15, Theorem 3.14] combined with the fact that D is finitely cogenerated as a C-comodule.

In order to prove *ii) $\Leftrightarrow$ iv)*, we first recall from [3, Corollary 2.4] that C^* is canonically isomorphic to the center of D^*. On the other hand, it is well-known that D^* is an Azumaya algebra over C^* if and only if the associated context

$$(End_{D^{*e}}(D^*), D^{*e}, D^*, Hom_{D^{*e}}(D^*, D^{*e}), \bar{f}, \bar{g})$$

is strict. Here D^{*e} denotes the C^*-enveloping algebra of D^*, $D^* \otimes_{C^*} D^{*op}$.

ii) $\Rightarrow$ iv) By Proposition 3.5, the dual context of (2) is strict. But, from Proposition 3.6, it is the Morita context associated to D^*. Hence D^* is an Azumaya algebra over C^*.

iv) $\Rightarrow$ ii) The hypothesis entails that D^* is finitely cogenerated and projective as a C^*-module. Combining [3, Lemma 4.3 i)] and Lemma 3.1, D is finitely cogenerated and injective as a C-comodule. Thus (2) is strong. By Proposition 3.6, the dual context of (2) is identified with the Morita context associated to D^*. From Proposition 3.5, (2) is strict. ∎

Our next goal is to prove that the strong Brauer group embeds in the Brauer group of the dual algebra. The method used in [2] may be adapted for our purpose.

For a coalgebra D, $\mathcal{F}_D$ denotes the symmetric linear topology consisting of all left ideals in D^* which are closed and cofinite, see [2]. The hereditary pretorsion class associated to it is the category of right comodules over D. If D is a C-coalgebra with C-counit $\epsilon_D : D \to C$, then D^* is an algebra over C^* via $\epsilon^* : C^* \to D^*$. We may consider the linear topologies $\mathcal{F}_C D^*$ and $\mathcal{F}_D \cap C^*$.

Lemma 4.7 *Let D be a strong Azumaya C-coalgebra. Then $\mathcal{F}_C = \mathcal{F}_D \cap C^*$ and $\mathcal{F}_C D^* = \mathcal{F}_D$.*

Proof: The proof follows the lines of [2, Lemma 3.5]. We include it here for completeness and to emphasize the importance of D to be finitely cogenerated. Since the inclusion $\mathcal{F}_C \supseteq \mathcal{F}_D \cap C^*$ always holds, we have to prove that $\mathcal{F}_C \subseteq \mathcal{F}_D \cap C^*$. There is an injective C-comodule map $h : D \to W \otimes C$ for some finite dimensional space W. Let $J \in \mathcal{F}_C$, and V a finite dimensional subcoalgebra of C with $J = V^{\perp(C^*)}$. For $d \in D$ we may set $h(d) = \sum_{j=1}^r w_j \otimes c_j$ for some $w_j \in W, c_j \in C$. Any $d^* \in h^*(W^* \otimes V^{\perp(C^*)})$ may be expressed as $d^* = h^*(\sum_{i=1}^n w_i^* \otimes c_i^*)$ for $w_i^* \in W^*$ and $c_i^* \in V^{\perp(C^*)}$. Let $d_i^* = h^*(w_i^* \otimes \varepsilon_C)$. Then,

$$\begin{aligned}
\langle \textstyle\sum_{i=1}^n d_i^* \epsilon^*(c_i^*), d\rangle &= \textstyle\sum_{i=1}^n \sum_{(d)} \langle h^*(w_i^* \otimes \varepsilon_C), d_{(1)}\rangle \langle \epsilon^*(c_i^*), d_{(2)}\rangle \\
&= \textstyle\sum_{i=1}^n \sum_{(d)} \langle w_i^* \otimes \varepsilon_C, h(d_{(1)})\rangle \langle c_i^*, \epsilon(d_{(2)})\rangle \\
&= \textstyle\sum_{i=1}^n \sum_{j=1}^r \sum_{(c_j)} \langle w_i^* \otimes \varepsilon_C, w_j \otimes c_{j(1)}\rangle \langle c_i^*, c_{j(2)})\rangle \\
&= \textstyle\sum_{i=1}^n \sum_{j=1}^r \sum_{(c_j)} \langle w_i^*, w_j\rangle \langle \varepsilon_C, c_{j(1)}\rangle \langle c_i^*, c_{j(2)})\rangle \\
&= \textstyle\sum_{i=1}^n \sum_{j=1}^r \langle w_i^*, w_j\rangle \langle c_i^*, c_j\rangle \\
&= \langle d^*, d\rangle,
\end{aligned}$$

where in the third equality we have used the C-colinearity of h. We have that $d^* = \sum_{i=1}^{n} d_i^* \epsilon^*(c_i^*) \in D^*\epsilon^*(V^{\perp(C^*)})$. Hence $h^*(W^* \otimes V^{\perp(C^*)}) \subseteq D^*\epsilon^*(V^{\perp})$. But $h^*(W^* \otimes V^{\perp(C^*)}) = h^*((W \otimes V)^{\perp}) = h^{-1}(W \otimes V)^{\perp(D^*)}$ and $h^{-1}(W \otimes V)$ is a finite dimensional right coideal in D. This yields that $D^*\epsilon^*(J)$ is a closed cofinite two-sided ideal in D^*. Since D^* is an Azumaya C^*-algebra, $J = (\epsilon^*(J)D^*) \cap C^*$, and consequently $J \in \mathcal{F}_D \cap C^*$. This proves the first part.

As $\mathcal{F}_D$ is a symmetric linear topology on D^* and D^* is an Azumaya algebra over C^*, there is a linear topology $\mathcal{T}$ on C^* such that $\mathcal{T}D^* = \mathcal{F}_D$. Now $\mathcal{T} = (\mathcal{T}D^*) \cap C^* = \mathcal{F}_D \cap C^* = \mathcal{F}_C$. ∎

The following theorem generalizes [2, Corollary 4.1, 4.2] where the coalgebra C was assumed to be irreducible. Under this hypothesis, $Br^s(C) = Br(C)$.

Theorem 4.8 *The map $(-)^* : Br^s(C) \to Br(C^*), [D] \mapsto [D^*]$ is a group monomorphism. Hence $Br^s(C)$ is a torsion group.*

Proof: We know from Theorem 4.6 that if D is an Azumaya C-coalgebra, then D^* is an Azumaya algebra over C^*. Let $D, E \in B^s(C)$ with $[D] = [E]$ in $Br^s(C)$. By Proposition 4.4, D and E are strong equivalent over C. Then D^* and E^* are Morita equivalent over C^*. Thus $[D^*] = [E^*]$ in $Br(C^*)$ and so the map $(-)^* : Br^s(C) \to Br(C^*)$ is well-defined. One may check that the isomorphism $\eta_{D,E} : (D \Box_C E)^* \to D^* \otimes_{C^*} E^*$ is a C^*-algebra map. Hence, $(-)^*$ is a group homomorphism.

From Lemma 4.7, $\mathcal{F}_C D^* = \mathcal{F}_D$ and $\mathcal{F}_C E^* = \mathcal{F}_E$. Suppose now that $[D^*] = [E^*]$ in $Br(C^*)$. Then D^* and E^* are Morita equivalent over C^*. If

$$\begin{tikzcd} {}_{D^*}\mathcal{M} \arrow[r, shift left, "F"] & {}_{E^*}\mathcal{M} \arrow[l, shift left, "G"] \end{tikzcd}$$

are the inverse equivalences, then Theorem 2.2 establishes that $F(\mathcal{M}^D) \subseteq \mathcal{M}^E$ and $G(\mathcal{M}^E) \subseteq \mathcal{M}^D$. In view of Proposition 3.8, D and E are strongly equivalent over C. Hence $[D] = [E]$ in $Br^s(C)$.

Since the Brauer group of any commutative ring is a torsion group, $Br^s(C)$ is a torsion group. ∎

Example 4.9 Let C be the group-like coalgebra indexed by the natural numbers over the rational number field. It was proved in [15, page 564] that $Br(C)$ is not a torsion group. By the above theorem $Br^s(C)$ is a torsion group. Hence $Br^s(C) \neq Br(C)$.

The sequel of the paper is devoted to studying some conditions under which the map $(-)^* : Br^s(C) \to Br(C^*)$ is surjective. We first need some results about completions with respect the cofinite topology.

For an algebra A, the cofinite topology $\mathcal{T}_A$ is a directed system. Since it is symmetric, we may take a basis $\mathcal{B}$ of two-sided ideals. Given $I, J \in \mathcal{B}$ with $I \subseteq J$, there is a surjective algebra map $f_{I,J} : A/I \to A/J$ such that $f_{I,J} p_I(a) = p_J(a) \quad \forall a \in A$, where p_I, p_J denote the canonical projections. It makes sense to consider the completion $\widehat{A}$ of A with respect to $\mathcal{T}_A$,

$$\widehat{A} = \varprojlim_{I\in\mathcal{T}_A} A/I = \{(a_I + I) \in \prod_{I\in\mathcal{T}_A} A/I : f_{I,J}p_I(a_I) = p_J(a_J)\}.$$

We say that A is *complete with respect to* $\mathcal{T}_A$ if the natural map $\nu_A : A \to \widehat{A}, a \mapsto (a+I)_{I\in\mathcal{T}_A}$ is an isomorphism.

Lemma 4.10 *Let R be a commutative algebra, and A an Azumaya R-algebra. If R is complete with respect to $\mathcal{T}_R$, then A is complete with respect to $\mathcal{T}_A$.*

Proof: The basis of $\mathcal{T}_A$ is given by $\mathcal{B} = \{IA\}_{I\in\mathcal{T}_R}$. Since A is Azumaya, we may find an ideal J of R such that $\cap_{I\in\mathcal{T}_R} IA = JA$ and $J \subseteq \cap_{I\in\mathcal{T}_R} I$. The completeness of R implies that $J = \{0\}$, and thus $\cap_{I\in\mathcal{T}_R} IA = \{0\}$. This proves the injectivity of ν_A.

In order to prove the surjectivity, take $(a_{IA}+IA)_{I\in\mathcal{T}_R} \in \widehat{A}$. Because A is finitely generated as an R-module put $A = Ra_1+...+Ra_n$ for some $a_l \in A$. For any $I \in \mathcal{T}_R$, $IA = Ia_1+...+Ia_n$. Then $a_{IA} = \sum_{l=1}^n r_{I,l}a_l$, where $r_{I,l} \in I$. Fixed l, the family $(r_{I,l} + I)_{I\in\mathcal{T}_R} \in \widehat{R}$. As R is complete, there is $r_l \in R$ such that $r_l + I = r_{I,l} + I$ for all $I \in \mathcal{T}_R$. Let $a = \sum_{l=1}^n r_l a_l$. Then $a - a_{IA} = \sum_{l=1}^n (r_l - r_{I,l})a_l \in IA$ for all $I \in \mathcal{T}_R$. ∎

Lemma 4.11 *Let C be a coalgebra and A an algebra.*

i) A^{0*} is the completion of A with respect to $\mathcal{T}_A$.

ii) If C is coreflexive, then C^* is complete with respect to $\mathcal{T}_{C^*}$.

Proof: *i)* The finite dual of A,

$$A^0 = \varinjlim_{I\in\mathcal{T}_A} (A/I)^*. \text{ Now, } A^{0*} = Hom(\varinjlim_{I\in\mathcal{T}_A} (A/I)^*, k) \cong \varprojlim_{I\in\mathcal{T}_A} (A/I)^{**} \cong \varprojlim_{I\in\mathcal{T}_A} A/I = \widehat{A}.$$

ii) If C is coreflexive, then the canonical embedding $\lambda_C : C \to C^{*0}$ is an isomorphism. Hence $C^* \cong C^{*0*} \cong \widehat{C^*}$. ∎

Theorem 4.12 *Let C be a cocommutative coreflexive coalgebra. The duality map $(-)^*$: $Br^s(C) \to Br(C^*)$ is an isomorphism.*

Proof: In view of Theorem 4.8, it suffices to prove the surjectivity. Let A be an Azumaya algebra over C^*. By Lemma 4.10, A is complete with respect to the cofinite topology. From Lemma 4.11, it follows that $A^{0*} \cong A$. The coalgebra A^0 is a C-coalgebra, and it is a strong Azumaya because of Theorem 4.6. ∎

Remark 4.13 Theorem 4.12 may be viewed as a generalization of the finite dimensional case, [15, Proposition 4.6]. It is also a generalization of [16, Theorem 3.10] where the coalgebras were assumed irreducible. In that case, a dual version of the crossed product theorem was needed to prove the surjectivity of $(-)^*$. The general proof presented here is more straightforward.

Theorem 4.14 *Let C be a cocommutative coalgebra with separable and coreflexive coradical. If $J = Rad(C^*)$ is a nil-ideal, in particular nilpotent, then the duality map $(-)^* : Br^s(C) \to Br(C^*)$ is an isomorphism.*

Proof: Let $i : C_0 \to C$ be the inclusion map and $i_* : Br^s(C) \to Br^s(C_0)$ the induced homomorphism. By [2, Theorem 4.5], i_* is injective. On the other hand, since C_0 is separable, the Malcev-Wedderburn decomposition ([1, Theorem 2.3.11]) gives the existence of a coalgebra map $\pi : C \to C_0$ such that $\pi i = 1_{C_0}$. The functorial behaviour of $Br^s(-)$ establishes that i_* is surjective.

The dual map $p = i^* : C^* \to C_0^* \cong C^*/J$ turns out to be exactly the canonical projection. Let $p_* : Br(C^*) \to Br(C^*/J)$ be the induced homomorphism. We have a commutative diagram

$$\begin{array}{ccc} Br^s(C) & \xrightarrow{i_*} & Br(C_0) \\ \downarrow{\scriptstyle (-)^*_C} & & \downarrow{\scriptstyle (-)^*_{C_0}} \\ Br(C^*) & \xrightarrow{p_*} & Br(C^*/J) \end{array}$$

where i_* and $(-)^*_{C_0}$ are isomorphisms. Since J is nilpotent, [5, Corollary 3] yields that p_* is an isomorphism. We conclude that $(-)^*_C$ is an isomorphism. ∎

Remark 4.15 1.- Some conditions for a coalgebra to be coreflexive are studied in [6], [10]. Assuming that the ground field is perfect, the separability condition of the coradical always holds.

2.- $Rad(C^*)$ is nilpotent if and only if the coradical filtration of C is finite, [2, Lemma 4.12]. If the ground field is of characteristic zero, $Rad(C^*)$ is nilpotent if and only if it is a nil-ideal, [11, Proposition 3.4].

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