The Brauer Group of the Dihedral Group

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Abstract

Let p^m be a power of a prime number p, $\mathbb{D}_{p^m}$ be the dihedral group of order $2p^m$ and k be a field where p is invertible and containing a primitive $2p^m$-th root of unity. The aim of this paper is computing the Brauer group $BM(k, \mathbb{D}_{p^m}, R_z)$ of the group Hopf algebra of $\mathbb{D}_{p^m}$ with respect to the quasi-triangular structure R_z arising from the group Hopf algebra of the cyclic group $\mathbb{Z}_{p^m}$ of order p^m, for z coprime with p. The main result states that $BM(k, \mathbb{D}_{p^m}, R_z) \cong \mathbb{Z}_2 \times k^{\cdot}/k^{\cdot 2} \times Br(k)$ when p is odd and when $p = 2$, $BM(k, \mathbb{D}_{2^m}, R_z) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \times Br(k)$.

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Introduction

Let k be a ring with unity and H be a Hopf algebra over k with bijective antipode. In [2] S. Caenepeel, F. Van Oystaeyen, and Y.H. Zhang defined the Brauer group of the Hopf algebra H, denoted by $BQ(k,H)$, consisting of Brauer equivalence classes of H-Azumaya algebras. The Brauer group $BQ(k,H)$ generalizes to arbitrary Hopf algebras the Brauer-Long group of a commutative and cocommutative Hopf algebra introduced in [10]. Thus the class of Hopf algebras with a Brauer group theory is enlarged. In particular, it makes sense to think about the Brauer group of the group Hopf algebra of a non abelian group. For G a finite abelian group the Brauer-Long group of the Hopf algebra kG, denoted by $BD(k,G)$ and studied in [9], was proposed as a generalization of previous existing Brauer groups of graded algebras like the Brauer-Wall group [20] or the Brauer group $B_\phi(k,G)$ of G-graded algebras with respect to a pairing $\phi : G \times G \to k$, see [6], [5], [7]. The Brauer group $BD(k,G)$ contains these other Brauer groups as subgroups.

In the generalization proposed in [2], the Brauer group $B_\phi(k,G)$ may be recognized as the Brauer group of a coquasi-triangular Hopf algebra, see [3, Lemma 1.2]. For a coquasi-triangular Hopf algebra (H,r) the Brauer group $BQ(k,H)$ contains a subgroup $BC(k,H,r)$ consisting of classes of H^{op}-comodule algebras with H-action stemming from the coquasi-triangular structure r. Dually, if (H,R) is a quasi-triangular Hopf algebra, $BQ(k,H)$ contains a subgroup $BM(k,H,R)$ consisting of classes of H-module algebras with H-coaction arising from the quasi-triangular structure R.

Let n be a nonnegative integer, let k be a field containing a primitive n-th root of unity ω and such that n is invertible in k. In this paper we study the Brauer group $BM(k,\mathbb{D}_n,R_z)$ of the group Hopf algebra of the dihedral group $\mathbb{D}_n = \langle g,h : g^n = h^2 = 1, gh = hg^{n-1}\rangle$ with respect to the quasi-triangular structures

$$R_z = \frac{1}{n}\Big(\sum_{0\le l,m<n} \omega^{-lm} g^l \otimes g^{zm}\Big), \quad (0 \le z \le n-1)$$

for z coprime with n. These quasi-triangular structures arise from the quasi-triangular structure on the group Hopf algebra $k\mathbb{Z}_n$. For $n = p^m$ a power of a prime number p a concrete description of $BM(k, \mathbb{D}_n, R_z)$ is given. It is proved in Theorem 3.5 that $BM(k, \mathbb{D}_n, R_z) \cong \mathbb{Z}_2 \times k^{\cdot}/k^{\cdot 2} \times Br(k)$ if p is odd and $BM(k, \mathbb{D}_n, R_z) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \times Br(k)$ if $p = 2$. Here $Br(k)$ denotes the usual Brauer group of k and $k^{\cdot}/k^{\cdot 2}$ is the multiplicative group of k modulo squares. For the case $p = 2$ the assumption that $\omega = \theta^2$ for a primitive $2n$-root of unity θ is needed.

The underlying idea in our study of $BM(k, \mathbb{D}_n, R_z)$ is to relate it to the Brauer groups $BM(k, \mathbb{Z}_2, R_0)$ and $BM(k, \mathbb{Z}_n, R_z)$ which belong to the theory of the Brauer-Long group and describe $BM(k, \mathbb{D}_n, R_z)$ from the knowledge of them. The cases n odd and n even are different and need to be treated separately. The inclusion map $i : \mathbb{Z}_n \to \mathbb{D}_n$ induces a group homomorphism $i^* : BM(k, \mathbb{D}_n, R_z) \to BM(k, \mathbb{Z}_n, R_z)$. It is shown in Theorem 2.10 that $Ker(i^*) \cong k^{\cdot}/k^{\cdot 2}$ when n is odd and $Ker(i^*) \cong k^{\cdot}/k^{\cdot 2} \times \mathbb{Z}_2$ when n is even. Any $[\beta] \in k^{\cdot}/k^{\cdot 2}$ and $\bar{a} \in \mathbb{Z}_2$ is represented in $Ker(i^*)$ by the algebra $A(\beta, \omega^a)$. As an algebra $A(\beta, \omega^a)$ is the 2×2 matrix algebra $M_2(k)$ and the $\mathbb{D}_n$-action is defined by letting g and h act by conjugation by the elements

$$u = \begin{pmatrix} \omega^a & 0 \\ 0 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix},$$

respectively. The algebra $C(1) = k\langle \delta : \delta^n = 1 \rangle$ with g-action given by $g \cdot \delta = \omega^{z^{-1}} \delta$ is $\mathbb{Z}_n$-Azumaya. The class of $C(1)$ in $BM(k, \mathbb{Z}_n, R_z)$ lies in the image of i^* since the g-action may be extended to a $\mathbb{D}_n$-action by setting $h \cdot \delta = \omega^r \delta^{n-1}$ for $0 \leq r \leq n-1$. With this $\mathbb{D}_n$-action $C(1)$ is $\mathbb{D}_n$-Azumaya. When n is odd the isomorphism class of this $\mathbb{D}_n$-module algebra is independent of r while when n is even there are exactly two inequivalent $\mathbb{D}_n$-Azumaya algebra structures on $C(1)$ depending on the parity of r (Proposition 2.12). If k is algebraically closed and n is a power of a prime p not dividing z it is known that $BM(k, \mathbb{Z}_n, R_z) \cong \mathbb{Z}_2$ and it is generated by $[C(1)]$. From these facts it is derived that $BM(k, \mathbb{D}_n, R_z) \cong BM(k, \mathbb{Z}_n, R_z) \cong \mathbb{Z}_2$ if p is odd (Corollary 2.13), and $BM(k, \mathbb{D}_n, R_z) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ if $p = 2$ (Corollary 2.16).

This result is used to determine $BM(k, \mathbb{D}_n, R_z)$ for k arbitrary by going to its algebraic closure $\overline{k}$. The inclusion map $\iota : k \to \overline{k}$ induces a group homomorphism $\iota_* : BM(k, \mathbb{D}_n, R_z) \to BM(\overline{k}, \mathbb{D}_n, R_z)$. When n is odd the kernel of ι_* is the subgroup $BAz(k, \mathbb{D}_n, R_z)$ consisting of classes of $BM(k, \mathbb{D}_n, R_z)$ containing a representative element which is classically Azumaya. It is shown in Proposition 3.2 that $Ker(\iota_*) \cong k^{\cdot}/k^{\cdot 2} \times Br(k)$. The group $k^{\cdot}/k^{\cdot 2}$ is represented by the algebras $A(\beta, 1)$ for $[\beta] \in k^{\cdot}/k^{\cdot 2}$. When $n = 2q$ with q even, $Ker(\iota_*) \cong k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \times Br(k)$. For q odd, $Ker(\iota_*)$ is isomorphic to the direct product of $k^{\cdot}/k^{\cdot 2}$ and the group extension $k^{\cdot}/k^{\cdot 2} \times_{\{-,-\}} Br(k)$ where $\{-,-\} : k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \to Br(k)$ is the 2-cocycle mapping $([a],[b])$ to $[\{a,b\}]$. Here $\{a,b\}$ denotes the quaternion algebra generated by x, y subject to the relations $x^2 = a, y^2 = b$ and $xy = -yx$. In both cases the first copy of $k^{\cdot}/k^{\cdot 2}$ is represented by the algebras $A(\beta, 1)$ and the second copy is represented by the algebra $A(t)$ defined as follows: for $[t] \in k^{\cdot}/k^{\cdot 2}$, $A(t) = M_2(k)$ as an algebra and the $\mathbb{D}_n$-action is given by h acting trivially and g acting by conjugation by

$$u = \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}.$$

When n is a power of a prime number the map ι_* is surjective and split and its image commutes with $Ker(\iota_*)$ (Theorem 3.5).

1 Preliminaries

Throughout k will be a field and H a finite dimensional Hopf algebra over k. For general facts on Hopf algebras and related notions we refer the reader to [17], [14], and [8]. In this section we recall the construction of the Brauer group $BM(k, H, R)$ of a finite dimensional quasi-triangular Hopf algebra (H, R) over a field k, see [2], [3].

Let (H, R) be a quasi-triangular Hopf algebra with quasi-triangular structure $R = \sum R^{(1)} \otimes R^{(2)} \in H \otimes H$. Any left H-module algebra A is naturally endowed with a standard right H-comodule algebra structure

$$\rho : A \to A \otimes H^{op},\ a \mapsto \sum (R^{(2)} \cdot a) \otimes R^{(1)}. \tag{1}$$

The *braided product* $A \# B$ of two left H-module algebras A, B is again a left H-module algebra and it is defined as follows: as a vector space $A \# B = A \otimes B$, with multiplication and H-action defined by

$$(a \# b)(a' \# b') = \sum a(R^{(2)} \cdot a') \#(R^{(1)} \cdot b)b'$$

$$h \cdot (a \otimes b) = \sum (h_{(1)} \cdot a) \otimes (h_{(2)} \cdot b)$$

for all $a, a' \in A, b, b' \in B, h \in H$. The H*-opposite* algebra $\overline{A}$ of a left H-module algebra A is equal to A as a left H-module but with multiplication given by

$$\overline{a} * \overline{b} = \sum \overline{(R^{(2)} \cdot b)(R^{(1)} \cdot a)}$$

for all $\overline{a}, \overline{b} \in \overline{A}$. For a finite dimensional left H-module M, the endomorphism algebra $End_k(M)$ becomes a left H-module algebra with H-action

$$(h \cdot f)(m) = \sum h_{(1)} \cdot f(S(h_{(2)}) \cdot m),$$

for all $h \in H, f \in End_k(M)$, and $m \in M$, where S denotes the antipode of H. Similarly, the usual opposite algebra $End_k(M)^{op}$ becomes a left H-module algebra with H-action

$$(h \cdot f)(m) = \sum h_{(2)} \cdot f(S^{-1}(h_{(1)}) \cdot m),$$

for all $h \in H, f \in End_k(M)^{op}$, and $m \in M$.

A finite dimensional left H-module algebra A is called H*-Azumaya* if the following two left H-module algebra maps are isomorphisms:

$$F : A\#\overline{A} \to End_k(A), \; F(a\#\overline{b})(c) = \sum a(R^{(2)} \cdot c)(R^{(1)} \cdot b),$$
$$G : \overline{A}\#A \to End_k(A)^{op}, \; G(\overline{a}\#b)(c) = \sum (R^{(2)} \cdot a)(R^{(1)} \cdot c)b,$$

for all $a, b, c \in A$ and $\overline{a}, \overline{b} \in \overline{A}$. Let $Az(H, R)$ denote the set of isomorphism classes of left H-Azumaya algebras. The following equivalence relation in $Az(H, R)$ is introduced: two H-Azumaya module algebras A, B are called *Brauer equivalent*, denoted by $A \sim B$, if there are two finite dimensional left H-modules M, N such that $A\#End(M) \cong B\#End(N)$ as left H-module algebras. The quotient set $BM(k, H, R) = Az(H, R)/\sim$ turns out to be a group with product induced by the braided product, that is, for $[A], [B] \in BM(k, H, R)$, $[A][B] = [A\#B]$. The inverse of $[A]$ is $[\overline{A}]$ and the identity element is $[k]$. Note that for a finite dimensional left H-module M, $End(M)$ is a representative element of $[k]$. The group $BM(k, H, R)$ is called the *Brauer group of H with respect to the quasi-triangular structure R.*

The Brauer group $BM(k, H, R)$ has a functorial behaviour at the field level and at the Hopf algebra level. Any field homomorphism $f : k \to k'$ induces a group homomorphism $f_* : BM(k, H, R) \to BM(k, H \otimes_k k', R_{k'})$

by mapping the class $[A]$ into the class $[A \otimes_k k']$. Any quasi-triangular map $\chi : (H, R) \to (H', R')$ induces a group homomorphism $\chi^* : BM(k, H', R') \to BM(k, H, R)$, $[A] \mapsto [A]$ by pulling back the action of H' on A along the map χ.

For a coquasi-triangular Hopf algebra (H, r) a dual construction of the Brauer group holds; one considers right H^{op}-comodule algebras and use the coquasi-triangular structure in order to define a braiding, braided product, H-opposite algebras, and H-Azumaya algebras. The group obtained in this way is denoted by $BC(k, H, r)$ and it is called the *Brauer group of H with respect to the coquasi-triangular structure r*. For a quasi-triangular Hopf algebra (H, R), H^* is a coquasi-triangular Hopf algebra with coquasi-triangular structure r induced on H^* by R. Then $BM(k, H, R) \cong BC(k, H^*, r)$. If H is commutative and cocommutative, then r induces a pairing ϕ on H and the Brauer group $BC(k, H, r)$ is isomorphic to the Brauer group $B_\phi(k, H)$ of ϕ-Azumaya algebras, see [3, Lemma 1.1], [4, page 329], [7], [15] for more details.

Let $(D(H), \mathcal{R})$ the Drinfel'd double of H equipped with its canonical quasi-triangular structure $\mathcal{R}$. The Brauer group $BQ(k, D(H), \mathcal{R})$ is usually denoted by $BQ(k, H)$ and it is called the *Brauer group of H*. If H admits a quasi-triangular structure R, then $BM(k, H, R)$ is a subgroup of $BQ(k, H)$. Similarly, if (H, r) is a coquasi-triangular structure, then $BC(k, H, r)$ is a subgroup of $BQ(k, H)$. All these Brauer groups are particular cases of Brauer groups of a braided monoidal category, see [19].

When H is the group algebra $H = kG$ of some group G we will denote $BM(k, H, R)$ by $BM(k, G, R)$.

2 The Brauer group $BM(k, \mathbb{D}_n, R)$

From now on k is a field containing a primitive $2n$-th root of unity θ and n is invertible in k. Let $k^{\cdot}$ denote the multiplicative group of k. Consider the dihedral group $\mathbb{D}_n = \langle g, h : g^n = h^2 = 1, gh = hg^{n-1}\rangle$. We identify $\mathbb{Z}_n$ with $\langle g\rangle$. The quasi-triangular structures on $k\mathbb{D}_n$ were studied in [21]. It is proved in [21, Proposition 3.2] that *for $n \neq 4$, $(k\mathbb{D}_n, R)$ is a quasi-triangular Hopf algebra if and only if $(k\mathbb{Z}_n, R)$ is quasi-triangular.* For $n = 4$ there are more quasi-triangular structures arising from the subgroups $\langle h, g^2\rangle, \langle hg, g^2\rangle$

which are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The quasi-triangular structures on $k\mathbb{Z}_n$ are computed in [16, page 219], and these are of the form,

$$R_z = \frac{1}{n}\Big(\sum_{0\le l,m<n} \omega^{-lm} g^l \otimes g^{zm}\Big),$$

for $0 \le z \le n-1$, where ω is a primitive n-th root of unity. Let $i : k\mathbb{Z}_n \to k\mathbb{D}_n$ be the inclusion map and $p : k\mathbb{D}_n \to k\mathbb{Z}_2, g \mapsto \bar{0}, h \mapsto \bar{1}$ be the canonical projection map. We have quasi-triangular maps,

$$(k\mathbb{Z}_n, R_z) \xrightarrow{\;i\;} (k\mathbb{D}_n, R_z) \xrightarrow{\;p\;} (k\mathbb{Z}_2, R_0),$$

where $R_0 = 1 \otimes 1$ is the trivial quasi-triangular structure on $k\mathbb{Z}_2$. The functorial behaviour of the Brauer group $BM(k, -)$ yields a sequence

$$BM(k, \mathbb{Z}_2, R_0) \xrightarrow{\;p^*\;} BM(k, \mathbb{D}_n, R_z) \xrightarrow{\;i^*\;} BM(k, \mathbb{Z}_n, R_z).$$

We describe explicitly these homomorphisms. Any $\mathbb{D}_n$-Azumaya algebra is a $\mathbb{Z}_n$-Azumaya algebra by forgetting the action of h. Indeed, *a $\mathbb{D}_n$-module algebra is $\mathbb{D}_n$-Azumaya if and only if it is $\mathbb{Z}_n$-Azumaya.* This is due to the fact that the quasi-triangular structures on $k\mathbb{Z}_n$ and $k\mathbb{D}_n$ are the same. Thus we get a map $i^* : BM(k, \mathbb{D}_n, R_z) \to BM(k, \mathbb{Z}_n, R_z), [A] \mapsto [A]$ but with the latter A considered as a $\mathbb{Z}_n$-module algebra. Similarly, *any $\mathbb{Z}_2$-Azumaya module algebra is a $\mathbb{D}_n$-Azumaya module algebra* via p, and we have a map $p^* : BM(k, \mathbb{Z}_2, R_0) \to BM(k, \mathbb{D}_n, R_z), [A] \mapsto [A]$.

The rest of this section is devoted to study the above sequence. Let us first note that for the case $z = 0$, i.e., $R_0 = 1 \otimes 1$, the Brauer group $BM(k, \mathbb{D}_n, R_0)$ is already known. It consists of classes of $\mathbb{D}_n$-module algebras which are classically Azumaya. By [10, Theorem 1.12], $BM(k, \mathbb{D}_n, R_0) \cong Br(k) \times H^2(\mathbb{D}_n, k)$ where $H^2(\mathbb{D}_n, k)$ is the second cohomology group of $\mathbb{D}_n$ with values in k. We will concentrate on the case $z \neq 0$ and we will describe $BM(k, \mathbb{D}_n, R_z)$ in terms of $BM(k, \mathbb{Z}_n, R_z)$ and $BM(k, \mathbb{Z}_2, R_0)$. These two groups belong to the classical theory of the Brauer group of an abelian group, see [4], [9], [10]. The Brauer group $BM(k, \mathbb{Z}_2, R_0) \cong k^{\cdot}/k^{\cdot 2} \times Br(k)$, see [10, Theorem 1.12]. The Brauer group $BM(k, \mathbb{Z}_n, R_z)$ is just the group $B_{\phi_z}(k, \mathbb{Z}_n)$ of ϕ_z-Azumaya algebras with $\phi_z : \mathbb{Z}_n \times \mathbb{Z}_n \to k$ being the pairing induced by R_z, see [3, Lemma 1.2], [4, pages 329, 341, 434]. For this description we have identified $k\mathbb{Z}_n$ and

$(k\mathbb{Z}_n)^*$ as Hopf algebras. The Brauer group $B_{\phi_z}(k, \mathbb{Z}_n)$ was first defined by Child, Garfinkel and Orzech in [5] and it can be described by an exact sequence due to Childs, see [6] .

Recall that the action of a Hopf algebra H on an algebra A is called *inner* if there is a convolution invertible linear map $\pi : H \to A$ such that

$$h \cdot a = \sum \pi(h_{(1)}) a \pi^{-1}(h_{(2)})$$

for all $h \in H, a \in A$. The action is called *strongly inner* if π may be chosen as an algebra map. The Skolem-Noether Theorem for Hopf algebras claims that the action of any Hopf algebra on a classically Azumaya algebra is inner, see [12]. The following lemma will be very useful in the sequel.

Lemma 2.1 *Let (H, R) be a quasi-triangular Hopf algebra and A be a matrix algebra which is an H-Azumaya module algebra. Then $[A]$ is trivial in $BM(k, H, R)$ if and only if the action of H on A is strongly inner.*

Proof: This is proved in [18, Lemma 2] for the Drinfel'd double of a Hopf algebra with its canonical quasi-triangular structure. The same proof works for any quasi-triangular Hopf algebra. ∎

Proposition 2.2 *Let A be a $\mathbb{D}_n$-module algebra which is classically Azumaya. The following statements hold:*

i) A contains a subalgebra generated by u, v subject to the relations $u^n = \alpha,\ v^2 = \beta,\ uv = \gamma v u^{n-1}$, with $\alpha, \beta, \gamma \in k^{\cdot}$ satisfying $\gamma^n \alpha^{n-2} = 1$.

ii) The action of $\mathbb{D}_n$ on A is strongly inner if and only if there are $s, t \in k^{\cdot}$ such that $\alpha = t^n, \beta = s^2$ and $\gamma = (t^{-1})^{n-2}$.

iii) If $n = 2q$ is even, then the action of $\mathbb{D}_n$ on A is strongly inner if and only if there are $s, t \in k^{\cdot}$ such that $\alpha = t^n, \beta = s^2$ and $\gamma^q = \alpha^{1-q}$.

Proof: *i)* Since A is classically Azumaya, the Skolem-Noether Theorem yields that the action of $\mathbb{D}_n$ on A is inner. Let $\pi \in Hom_k(k\mathbb{D}_n, A)$ be a convolution invertible map such that $\sigma \cdot a = \pi(\sigma) a \pi^{-1}(\sigma)$ for all $\sigma \in \mathbb{D}_n$. As σ is a group-like element, $\pi^{-1}(\sigma) = \pi(\sigma)^{-1}$.

Let $u = \pi(g)$ and $v = \pi(h)$. Then, $a = 1 \cdot a = g^n \cdot a = u^n a (u^{-1})^n$ for all $a \in A$. Since A is central, there is $\alpha \in k^{\cdot}$ such that $u^n = \alpha$. Similarly, $v^2 = \beta$ for some $\beta \in k^{\cdot}$. From the equalities,

$$\begin{aligned}(gh) \cdot a &= g \cdot (h \cdot a) = uvav^{-1}u^{-1}, \\ (gh) \cdot a &= (hg^{n-1}) \cdot a = h \cdot (g^{n-1} \cdot a) = vu^{n-1}a(u^{-1})^{n-1}v^{-1},\end{aligned}$$

we deduce that there exists $\gamma \in k^{\cdot}$ satisfying $uv = \gamma v u^{n-1}$. Multiplying this latter equality on the left by u^{n-1} we get $\alpha v = \gamma^n v u^{n(n-1)} = \gamma^n \alpha^{n-1} v$. Hence $\gamma^n \alpha^{n-2} = 1$.

ii) Assume that the action of $\mathbb{D}_n$ on A is strongly inner, and let $\zeta : k\mathbb{D}_n \to A$ be a convolution invertible algebra map such that $\sigma \cdot a = \zeta(\sigma) a \zeta(\sigma)^{-1}$ for all $\sigma \in G, a \in A$. The elements $\bar{u} = \zeta(g)$ and $\bar{v} = \zeta(h)$ satisfy:

$$\bar{u}^n = 1, \qquad \bar{v}^2 = 1, \qquad \bar{u}\bar{v} = \bar{v}\bar{u}^{n-1}.$$

Since $g \cdot a = uau^{-1} = \bar{u}a\bar{u}^{-1}$ for all $a \in A$, there is an element $t \in k^{\cdot}$ such that $u = t\bar{u}$. Then, $\alpha = u^n = t^n \bar{u}^n = t^n$. Similarly, there is $s \in k^{\cdot}$ such that $v = s\bar{v}$, and $\beta = s^2$. Now, $\gamma s t^{n-1} \bar{v} \bar{u}^{n-1} = \gamma v u^{n-1} = uv = ts\bar{u}\bar{v} = ts\bar{v}\bar{u}^{n-1}$. Therefore, $\gamma = (t^{-1})^{n-2}$.

Conversely, suppose that $\alpha = t^n, \beta = s^2$, and $\gamma = (t^{-1})^{n-2}$ for some $s, t \in k^{\cdot}$. Define

$$\zeta(g) = \frac{1}{t} u, \qquad \zeta(h) = \frac{1}{s} v,$$

and extend it to an algebra map from $\mathbb{D}_n$ into A. This map is well-defined and gives the same action as π.

iii) If the action of $\mathbb{D}_n$ is strongly inner, then from *ii)* we obtain

$$\alpha^{1-q} = (t^{2q})^{1-q} = (t^{-1})^{2q(q-1)} = \gamma^q.$$

Conversely, if $\alpha = t^n$, $\beta = s^2$ and $\gamma^q = \alpha^{1-q}$ then

$$\alpha = (\alpha\gamma)^q, \qquad \gamma = ((\alpha\gamma)^{-1})^{q-1}.$$

By part *ii)* it is enough to show that $\alpha\gamma$ is a square in k. Since $\alpha = t^{2q} = (\alpha\gamma)^q$ there exists a q-th root of unity $\xi = \theta^{4r}$ for some r such that $\alpha\gamma = \xi t^2 = (\theta^{2r} t)^2$, hence the statement. ∎

Remark 2.3 The elements u, v of Proposition 2.2 i) are unique up to scalar multiples. The subalgebra generated by them is completely determined by the $\mathbb{D}_n$-action and we will call it the *induced subalgebra on A by the* $\mathbb{D}_n$*-action.* If we take different generators u' and v', then $u' = tu$ and $v' = sv$ for some nonzero scalars t and s and the corresponding constants will be $\alpha' = t^n\alpha$, $\beta' = s^2\beta$ and $\gamma' = (t^{-1})^{n-2}\gamma$.

The set $G = \{(\alpha, \gamma) \in k^{\cdot} \times k^{\cdot} : \gamma^n\alpha^{n-2} = 1\}$ is a group with the multiplication induced from $k^{\cdot} \times k^{\cdot}$. We introduce the following equivalence relation on G. Two elements $(\alpha, \gamma), (\alpha', \gamma') \in G$ are equivalent, denoted by $(\alpha, \gamma) \sim (\alpha', \gamma')$, if there is $t \in k^{\cdot}$ such that $\alpha' = t^n\alpha$ and $\gamma' = (t^{-1})^{n-2}\gamma$. The quotient set $\mathcal{G} = G/\sim$ is a group. Any $\mathbb{D}_n$-module algebra which is classically Azumaya has associated a unique invariant $([\beta], [(\alpha, \gamma)]) \in k^{\cdot}/k^{\cdot 2} \times \mathcal{G}$.

Remark 2.4 Note from the proof of Proposition 2.2 that the action of g is strongly inner if and only if α is a n-th power in k and that in this case one can always choose u and v such that $u^n = 1$ and $uv = \gamma vu^{-1}$ with $\gamma^n = 1$.

Lemma 2.5 *i) If n is odd, then $\mathcal{G}$ is trivial.*

ii) If n is even, then $\mathcal{G} \cong k^{\cdot}/k^{\cdot 2} \times \mathbb{Z}_2$.

Proof: *i)* We only need to show that if n is odd we can always find $t \in k^{\cdot}$ such that $\alpha = t^n$ and $\gamma = t^{2-n}$. Since $\gamma^n\alpha^n = \alpha^2$ this is equivalent to $\alpha = t^n$ and $\alpha\gamma = t^2$. As $(2, n) = 1$, there exist integers a and b for which $1 = 2a + nb$. Then $\alpha = \alpha^{2a}\alpha^{nb} = (\alpha\gamma)^{an}\alpha^{bn}$ and $\alpha\gamma = (\alpha\gamma)^{2a}(\alpha\gamma)^{nb} = (\alpha\gamma)^{2a}\alpha^{2b}$ so we may take $t = \alpha^{a+b}\gamma^a$.

ii) Suppose that $n = 2q$ and let $[(\alpha, \gamma)] \in \mathcal{G}$. From $\gamma^n\alpha^{n-2} = 1$, it follows that $\gamma^q\alpha^{q-1} = \pm 1$. It may be checked that the map

$$\Phi : \mathcal{G} \to k^{\cdot}/k^{\cdot 2} \times \mathbb{Z}_2, \ [(\alpha, \gamma)] \mapsto ([\gamma\alpha], \gamma^q\alpha^{q-1})$$

is an isomorphism. ∎

Corollary 2.6 *With notation as in Proposition 2.2 i), for n odd we can always choose u such that $u^n = 1$ and $uv = vu^{n-1}$.*

Any $\mathbb{D}_n$-module algebra A becomes a $\mathbb{Z}_n$-comodule algebra with comodule structure as in (1) for the quasi-triangular structure R_z. Hence A is a $\mathbb{Z}_n$-graded algebra. An element $a \in A$ has degree r, denoted by $deg(a) = r$, if $\rho(a) = a \otimes g^r$. Equivalently, $g^z \cdot a = \omega^r a$. If A, B are $\mathbb{D}_n$-module algebras, then the multiplication in the braided product $A\#B$ is

$$(a\#b)(a'\#b') = aa'\#(g^{deg(a')} \cdot b)b' \tag{2}$$

for homogeneous $a, a' \in A$ and $b, b' \in B$.

Lemma 2.7 *Let A, B be $\mathbb{D}_n$-module algebras and let B be a classically Azumaya algebra such that g acts strongly innerly on it. Then, $A\#B \cong A \otimes B$ as $\mathbb{D}_n$-module algebras. In particular, if A and B are both classically Azumaya with a strongly inner g-action, $A\#B$ is again so.*

Proof: The proof is inspired by [9, Lemma 2.2]. Since the action of g is strongly inner on the Azumaya algebra B there exists $u_B \in B$ with $g \cdot b = u_B b u_B^{-1}$ for every $b \in B$ and $u_B^n = 1$. Similarly, there exists $v_B \in B$ such that $h \cdot b = v_B b v_B^{-1}$ for every $b \in B$ with $u_B v_B = \gamma v_B u_B^{-1}$ and $\gamma^n = 1$. Let $\zeta = \theta^r \in k^\cdot$ be a $2n$-th root of unity for which $\zeta^2 = \gamma$. We check that the map

$$\Phi\colon A\#B \to A \otimes B,\ a\#b \mapsto a \otimes \zeta^{\deg(a)} u_B^{-\deg(a)} b,$$

for $a \in A$ homogeneous, is a $\mathbb{D}_n$-module algebra isomorphism. For a, $a' \in A$ homogeneous, and b, $b' \in B$,

$$\begin{aligned}
\Phi((a\#b)(a'\#b')) &= \Phi(aa'\#(g^{\deg(a')} \cdot b)b') \\
&= aa' \otimes \zeta^{\deg(a)+\deg(a')} u_B^{-\deg(a)} u_B^{-\deg(a')} (u_B^{\deg(a')} b u_B^{-\deg(a')}) b' \\
&= (a \otimes \zeta^{\deg(a)} u_B^{-\deg(a)} b)(a' \otimes \zeta^{\deg(a')} u_B^{-\deg(a')} b') \\
&= \Phi(a\#b)\Phi(a'\#b').
\end{aligned}$$

So the map Φ is an algebra homomorphism and it is clearly bijective because u_B is invertible. The inverse $\Phi^{-1} : A \otimes B \to A\#B$ is defined as $\Phi^{-1}(a \otimes b) = a\#\zeta^{-\deg(a)} u_B^{\deg(a)} b$ for $a \in A$ homogeneous and $b \in B$. We next show that Φ is a $\mathbb{D}_n$-module isomorphism. Notice that the action of g does not change the degree of an element in A and the action of h maps elements of a given degree into elements of opposite degree. Then,

$$\begin{aligned} g\cdot\Phi(a\#b) &= g\cdot(a\otimes\zeta^{\deg(a)}u_B^{-\deg(a)}b) \\ &= (g\cdot a\otimes\zeta^{\deg(a)}u_Bu_B^{-\deg(a)}bu_B^{-1}) \\ &= (g\cdot a\otimes\zeta^{\deg(g\cdot a)}u_B^{-\deg(g\cdot a)}g\cdot b) \\ &= \Phi(g\cdot(a\#b)). \end{aligned}$$

$$\begin{aligned} h\cdot\Phi(a\#b) &= (h\cdot a)\otimes\zeta^{\deg(a)}v_Bu_B^{-\deg(a)}bv_B^{-1} \\ &= (h\cdot a)\otimes\zeta^{\deg(a)}\gamma^{-\deg(a)}u_B^{\deg(a)}(h\cdot b) \\ &= (h\cdot a)\otimes\zeta^{-\deg(a)}u_B^{\deg(a)}(h\cdot b) \\ &= (h\cdot a)\otimes\zeta^{\deg(h\cdot a)}u_B^{-\deg(h\cdot a)}(h\cdot b) \\ &= \Phi(h\cdot(a\#b)). \end{aligned}$$

To prove the last statement of the lemma, assume that A is also a classically Azumaya algebra with a strongly inner g-action, and let u_A, v_A be generators of the induced subalgebra such that $u_A^n = 1$. Then $A\#B \cong A\otimes B$ is again classically Azumaya and $u := \Phi^{-1}(u_A\otimes u_B)$ satisfies $g\cdot(a\#b) = u(a\#b)u^{-1}$ for every $a\in A$ and $b\in B$ and $u^n = 1\#1$. ∎

Corollary 2.8 *The subset $BAz^g(k,\mathbb{D}_n,R_z)$ of classes in $BM(k,\mathbb{D}_n,R_z)$ which can be represented by an Azumaya algebra with strongly inner g-action is an abelian subgroup of $BM(k,\mathbb{D}_n,R_z)$. If n is odd, $BAz^g(k,\mathbb{D}_n,R_z)$ coincides with $BAz(k,\mathbb{D}_n,R_z)$, the subgroup of $BM(k,\mathbb{D}_n,R_z)$ of elements which can be represented by an Azumaya algebra.*

Proof: The last statement follows by Corollary 2.6. ∎

Lemma 2.9 *If $[A]$ in $BM(k,\mathbb{D}_n,R_z)$ may be represented by a classically Azumaya algebra A, then all other representatives will be also classically Azumaya. Moreover, with notation as in Remark 2.3, we may associate to $[A]$ the invariant $([\beta_A],[(\alpha_A,\gamma_A)])\in k^\cdot/k^{\cdot 2}\times\mathcal{G}$ and this assignment does not depend of the representative of $[A]$.*

Proof: If B is any other representative of the class $[A]$ then there are $\mathbb{D}_n$-modules P and Q such that $A\#End(P)\cong B\#End(Q)$. Using Lemma 2.7,

$$A\otimes End(P)\cong A\#End(P)\cong B\#End(Q)\cong B\otimes End(Q).$$

Therefore $B \otimes End(Q)$ is classically Azumaya. Then the algebra B is also Azumaya because it is the centralizer of $End(Q)$ in a classically Azumaya algebra. This gives the first statement. We prove the second one. By Lemma 2.7, $u_{A\#End(P)} = \Phi^{-1}(u_A \otimes u_{End(P)})$ and $v_{A\#End(P)} = \Phi^{-1}(v_A \otimes v_{End(P)})$ are generators for the induced subalgebra of $A\#End(P)$. Similarly for $B\#End(Q)$. Since the $\mathbb{D}_n$-action on $End(P)$ and $End(Q)$ is strongly inner, then

$$\begin{array}{ll} \alpha_{A\#End(P)} = \alpha_A \alpha_{End(P)} = \alpha_A t^n, & \alpha_{B\#End(Q)} = \alpha_B \alpha_{End(Q)} = \alpha_B t'^n \\ \beta_{A\#End(P)} = \beta_A \beta_{End(P)} = \beta_A s^2, & \beta_{B\#End(Q)} = \beta_B \beta_{End(Q)} = \beta_B s'^2 \\ \gamma_{A\#End(P)} = \gamma_A \gamma_{End(P)} = \gamma_A t^{2-n} & \gamma_{B\#End(Q)} = \gamma_B \gamma_{End(Q)} = \gamma_B t'^{2-n} \end{array}$$

for some $t, t', s, s' \in k^{\cdot}$. By Remark 2.3, there are $\tilde{s}, \tilde{t} \in k^{\cdot}$ such that $\alpha_A t^n = \tilde{t}^n \alpha_B t'^n$, $\beta_A s^2 = \tilde{s}^2 \beta_B s'^2$ and $\gamma_A t^{2-n} = \tilde{t}^{2-n} \gamma_B t'^{2-n}$, hence the statement. ∎

Theorem 2.10 *There are two exact sequences of groups,*

$$1 \longrightarrow k^{\cdot}/k^{\cdot 2} \longrightarrow BM(k, \mathbb{D}_n, R_z) \xrightarrow{i^*} BM(k, \mathbb{Z}_n, R_z), \tag{3}$$

for n odd and

$$1 \longrightarrow k^{\cdot}/k^{\cdot 2} \times \mathbb{Z}_2 \longrightarrow BM(k, \mathbb{D}_n, R_z) \xrightarrow{i^*} BM(k, \mathbb{Z}_n, R_z), \tag{4}$$

for n even.

Proof: The kernel of i^* is given by elements which can be represented by a matrix algebra with a strongly inner g-action, therefore it is a subgroup of the abelian group $BAz^g(k, \mathbb{D}_n, R_z)$. Let A be a representative of an element in $Ker(i^*)$. Its induced subalgebra is generated by u_A, v_A such that $u_A^n = 1, v_A^2 = \beta_A$ and $u_A v_A = \gamma_A v_A u_A^{-1}$, for $\beta_A \in k^{\cdot}$ and $\gamma_A \in k^{\cdot}$ an n-th root of unity. For n odd we can always make sure that $\gamma_A = 1$ by Corollary 2.6. For $n = 2q$ even, $\gamma_A^q = \pm 1$. In light of Lemma 2.9, the maps $Inv_o\colon Ker(i^*) \to k^{\cdot}/k^{\cdot 2}$, $[A] \mapsto [\beta_A]$ for n odd, and $Inv_e\colon Ker(i^*) \to k^{\cdot}/k^{\cdot 2} \times \mathbb{Z}_2$, $[A] \mapsto ([\beta_A], \gamma_A^q)$ for $n = 2q$ even are well defined. We check that they are are group homomorphisms. If A, B are in $Ker(i^*)$ and have induced subalgebras generated by u_A, v_A and u_B, v_B respectively, then by Lemma 2.7, the induced subalgebra of $A\#B$ is generated by $u = \Phi^{-1}(u_A \otimes u_B)$ and $v = \Phi^{-1}(v_A \otimes v_B)$. Hence $v^2 = \beta_A\beta_B$ and $uv = \gamma_A\gamma_B vu^{-1}$. The injectivity follows by Lemma 2.1, Remark 2.4 and Proposition 2.2 ii), iii).

Finally we prove the surjectivity of these two maps. Let γ be an n-th root of unity. Consider the matrix algebra $A(\beta,\gamma) = M_2(k)$. Let

$$u = \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}, \qquad v = \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}.$$

It is easy to verify that $u^n = 1, v^2 = \beta$ and $uv = \gamma vu^{-1}$. Thus the conjugation by u and v provide A of a $\mathbb{D}_n$-module algebra structure. Consider the $\mathbb{Z}_n$-action induced by restriction. Since $A(\beta,\gamma)$ is classically Azumaya and it has a $\mathbb{Z}_n$-trivial graded center, it is $\mathbb{Z}_n$-Azumaya. Hence $A(\beta,\gamma)$ is $\mathbb{D}_n$-Azumaya. Clearly, if n is odd, $Inv_o(A(\beta,\gamma)) = [\beta]$ and if $n = 2q$ is even $Inv_e(A(\beta,\gamma)) = ([\beta],\gamma^q)$. Hence both maps are surjective. ∎

Remark 2.11 The Brauer group $BM(k,\mathbb{Z}_n,R_z)$ may be identified with the Brauer group $B_{\phi_z}(k,\mathbb{Z}_n)$ where $\phi_z : \mathbb{Z}_n \times \mathbb{Z}_n \longrightarrow k, (g^i, g^j) \mapsto \omega^{zij}$ is the pairing induced by the quasi-triangular structure R_z, [3, Lemma 1.2]. When $n = p^m$ is a power of a prime number p with p invertible in k, k containing a primitive $2n$-th root of unity and ϕ_z is non-degenerated (equivalently, z is coprime with n), the multiplication rules of $B_{\phi_z}(k,\mathbb{Z}_n)$ are known, see [4, Corollary 13.12.36]. As a set $B_{\theta_z}(k,\mathbb{Z}_n) = \mathbb{Z}_2 \times k^{\cdot}/k^{\cdot n} \times Br(k)$. The product is given by

$$\begin{aligned} (\pm, S, A)(+, S', A') &= (\pm, SS', AA'|S'\#S|) \\ (\pm, S, A)(-, S', A') &= (\mp, S^{-1}S', AA'|S'\#S^{-1}|). \end{aligned}$$

We identify these rules in $B_{\phi_z}(k,\mathbb{Z}_n)$, see [1, page 235]. For $\alpha \in k^{\cdot}$, the algebra $C(\alpha) = k\langle \delta : \delta^n = \alpha\rangle$ with $\mathbb{Z}_n$-action given by $g{\cdot}\delta = \omega^{z^{-1}}\delta$ is $\mathbb{Z}_n$-Azumaya. The symbol $-$ is represented by $[C(1)]$. Each $[\alpha] \in k^{\cdot}/k^{\cdot n}$ is viewed in $B_{\phi_z}(k,\mathbb{Z}_n)$ as $[C(\alpha)\#(k\mathbb{Z}_n)^*]$. For $[\alpha],[\beta] \in k^{\cdot}/k^{\cdot n}$, the braided product $C(\alpha)\#C(\beta)$ is an Azumaya algebra, see [11, Proposition 2.1], [4, page 359]. By $|C(\alpha)\#C(\beta)|$ we denote the underlying algebra. It is generated by two elements x, y subject to the relations $x^n = \alpha, y^n = \beta, yx = \omega^{z^{-1}}xy$. The Brauer group $Br(k)$ is embedded as usual as the subgroup of ordinary Azumaya algebras with trivial $\mathbb{Z}_n$-action. In particular, if k is algebraically closed, $BM(k,\mathbb{Z}_n,R_z) \cong \mathbb{Z}_2$ and it is generated by $[C(1)]$.

By the Remark above, if k is algebraically closed and n is a power of a prime p not dividing z, then the exact sequences (3), (4) in Theorem 2.10 become

$$1 \longrightarrow BM(k, \mathbb{D}_n, R_z) \xrightarrow{i^*} \mathbb{Z}_2 \tag{5}$$

for n odd and

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow BM(k, \mathbb{D}_n, R_z) \xrightarrow{i^*} \mathbb{Z}_2 \tag{6}$$

for n even. In this setting $BM(k, \mathbb{D}_n, R_z)$ is thus always an abelian group. In particular, for n odd, we can prove that $BM(k, \mathbb{D}_n, R_z) \cong \mathbb{Z}_2$ by showing that it is nontrivial. The even case is slightly more complicated. We will prove that $BM(k, \mathbb{D}_{2^m}, R_z) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ by showing that i^* is surjective and split. For this purpose, we study all possible lifts of the $\mathbb{Z}_n$-action on $C(\alpha)$ to a $\mathbb{D}_n$-action.

In the sequel we will assume that z is coprime with n and we will denote by s the inverse of z modulo n.

Proposition 2.12 *Consider the algebra $C(\alpha) = k\langle \delta : \delta^n = \alpha \rangle$ with $\mathbb{Z}_n$-action given by $g \cdot \delta = \omega^s \delta$. Then, $C(\alpha)$ is $\mathbb{D}_n$-Azumaya if and only if there is $\lambda \in k^{\cdot}$ such that $\lambda^n \alpha^{n-2} = 1$. In this case, $h \cdot \delta = \lambda \delta^{n-1}$. Furthermore:*

i) If n is odd all possible lifts of the $\mathbb{Z}_n$-action give isomorphic $\mathbb{D}_n$-module algebras.

ii) If $n = 2q$, there are either 0 or 2 possible isomorphism classes of lifts of the $\mathbb{Z}_n$-action according on the existence of a λ as above. Two lifts corresponding to λ and λ' are isomorphic if and only if $\lambda^q = (\lambda')^q$.

Proof: From [11, page 442], $C(\alpha)$ is $\mathbb{Z}_n$-Azumaya. Recall that an algebra is $\mathbb{D}_n$-Azumaya if and only if it is $\mathbb{Z}_n$-Azumaya. So it is enough to check whether we can provide $C(\alpha)$ of a $\mathbb{D}_n$-module algebra structure. It is easy to see that for $\lambda, \alpha \in k^{\cdot}$ satisfying $\lambda^n \alpha^{n-2} = 1$, the action given by $g \cdot \delta = \omega^s \delta$, $h \cdot \delta = \lambda \delta^{n-1}$ makes $C(\alpha)$ into a $\mathbb{D}_n$-module algebra.

Conversely, the h-action on $C(\alpha)$ maps eigenvectors of the g-action of eigenvalue ω^t into eigenvectors of eigenvalue ω^{-t}. As s is coprime with n, the eigenspaces for the g-action are 1-dimensional. Thus, necessarily $h \cdot \delta = \lambda \delta^{n-1}$.

From the equality

$$\begin{aligned}\delta &= h^2\cdot\delta = h\cdot(h\cdot\delta) = h\cdot(\lambda\delta^{n-1}) = \lambda(h\cdot\delta)^{n-1} = \lambda(\lambda\delta^{n-1})^{n-1} = \lambda^n\delta^{(n-1)^2}\\ &= \lambda^n\alpha^{n-2}\delta,\end{aligned}$$

it follows that $\lambda^n\alpha^{n-2} = 1$.

For $\lambda \in k^{\cdot}$ such that $\lambda^n\alpha^{n-2} = 1$ let $C_\lambda(\alpha)$ denote the lift of $C(\alpha)$ with $h\cdot\delta = \lambda\delta^{n-1}$. Consider two lifts $C_\lambda(\alpha)$ and $C_{\lambda'}(\alpha)$. Then $(\lambda')^n = \lambda^n$. So that $\lambda' = \zeta\lambda$ for an n-th root of unity $\zeta = \omega^r$ for some integer r. It is easy to check that if $r = 2t$ is even, then the map $\Psi\colon C_\lambda(\alpha) \to C_{\lambda'}(\alpha),\ \delta \mapsto \omega^t\delta$ is a $\mathbb{D}_n$-module algebra isomorphism.

i) For n odd, we can always make sure that r is even.

ii) For $n = 2q$ even, r is even if and only if $\lambda^q = (\lambda')^q$. Hence if $\lambda^q = (\lambda')^q$, then $C_\lambda(\alpha)$ and $C_{\lambda'}(\alpha)$ are isomorphic as $\mathbb{D}_n$-module algebras. Conversely, suppose now that $\Psi : C_\lambda(\alpha) \to C_{\lambda'}(\alpha)$ is an isomorphism of $\mathbb{D}_n$-module algebras. Then $\Psi(\delta) = \omega^r\delta$ for some r because $(s,n) = 1$ and $\delta^n = \alpha$. Since the elements $\Psi(h\cdot\delta) = \lambda'\omega^{-t}\delta^{n-1}$ and $h\cdot\Psi(\delta) = \omega^t\lambda\delta^{n-1}$ coincide, it follows that $\lambda' = \omega^{2t}\lambda$. Therefore $\lambda^q = \lambda'^q$. ∎

For n a power of an odd prime number and k algebraically closed the computation of $BM(k, \mathbb{D}_n, R_z)$ derives from the sequence (5) and Proposition 2.12 i).

Corollary 2.13 *Let $n = p^m$ for an odd prime p and let k be algebraically closed. Then, for every z not divisible by p, $BM(k, \mathbb{D}_n, R_z) \cong \mathbb{Z}_2$. The non trivial element is $[C_1(1)]$.*

For n a power of 2 and k algebraically closed more work is needed to compute $BM(k, \mathbb{D}_n, R_z)$.

Proposition 2.14 *Let $n = 2q$ and let $C_\lambda(\alpha), C_{\lambda'}(\alpha)$ as above. Then, $[C_{\lambda'}(\alpha)] = [C_\lambda(\alpha)]$ in $BM(k, \mathbb{D}_n, R_z)$ if and only if $\lambda^q = \lambda'^q$.*

Proof: If $\lambda^q = \lambda'^q$, we know from Proposition 2.12 ii) that $C_\lambda(\alpha)$ and $C_{\lambda'}(\alpha)$ are indeed isomorphic. Conversely, suppose that $C_\lambda(\alpha)$ and $C_{\lambda'}(\alpha)$

represent the same element in $BM(k,\mathbb{D}_n,R_z)$, and let P,Q be two $\mathbb{D}_n$-modules such that $C_\lambda(\alpha)\#End(P)\cong C_{\lambda'}(\alpha)\#End(Q)$ as $\mathbb{D}_n$-module algebras. It follows from Lemma 2.7 that $C_\lambda(\alpha)\otimes End(P)\cong C_{\lambda'}(\alpha)\otimes End(Q)$ as $\mathbb{D}_n$-module algebras. Then the centres $C_\lambda(\alpha)\otimes k$ and $C_{\lambda'}(\alpha)\otimes k$ of these two algebras are isomorphic as $\mathbb{D}_n$-module algebras. By Proposition 2.12 ii), $\lambda^q=\lambda'^q$. ∎

From now on the algebra $C_1(1)$ will be denoted by $C_{\bar{0}}(1)$ both for n even or odd. For n even, $C_{\bar{1}}(1)$ will denote $C_{\omega^s}(1)$.

Lemma 2.15 *With notation as above, the classes* $[C_{\bar{0}}(1)]$ *(n even or odd),* $[C_{\bar{1}}(1)]$ *and* $[C_{\bar{0}}(1)\#C_{\bar{1}}(1)]$ *have all order 2 in the corresponding* $BM(k,\mathbb{D}_n,R_z)$. *Moreover,* $[C_{\bar{0}}(1)]$ *commutes with* $[C_{\bar{1}}(1)]$.

Proof: As the braided product of $\mathbb{D}_n$-module algebras coincides with the braided product of $\mathbb{Z}_n$-module algebras, the algebra $C_{\bar{a}}(1)\#C_{\bar{b}}(1)$ is a matrix algebra ([11, Proposition 2.4]) with strongly inner g-action. We prove that the $\mathbb{D}_n$-action on $C_{\bar{a}}(1)\#C_{\bar{b}}(1)$ for $a,b=0,1$ is strongly inner if and only if $a=b$. Let δ,η denote generators of $C(1)$. Let $u=\zeta(\delta^{n-1}\#\eta)$ with ζ an n-th (respectively $2n$-th) root of unity for n odd (respectively even) for which $\zeta^2=\omega^s$. By induction, $u^r=\zeta^{2r-r^2}\delta^{n-r}\#\eta^r$, so that $u^n=1$. It may be checked that the g-action on $C_{\bar{a}}(1)\#C_{\bar{b}}(1)$ is given by conjugation by u. The h-action on $C_{\bar{a}}(1)$ and $C_{\bar{b}}(1)$ is defined by

$$h\cdot\delta^j=\omega^{saj}\delta^{-j},\qquad h\cdot\eta^j=\omega^{sbj}\eta^{-j}.$$

Let

$$v=\begin{cases}\frac{1}{n}\sum_{i,j=0}^{n-1}\zeta^{ij}\delta^i\#\eta^j & \text{if } n \text{ is odd,}\\ \frac{1}{q}\sum_{i,j=0}^{q-1}\omega^{-sai-sbj+2sij}\delta^{2i}\#\eta^{2j} & \text{if } n=2q.\end{cases}$$

We claim that the element v satisfies $v^2=1$ and $h\cdot(\delta^i\#\eta^j)=v(\delta^i\#\eta^j)v^{-1}$. We prove it for $n=2q$, the odd case is proved similarly.

$$\begin{aligned}v^2 &= \tfrac{1}{q^2}\textstyle\sum_{i,j=0}^{q-1}\sum_{l,m=0}^{q-1}\omega^{-sa(i+l)-sb(j+m)+2sij+2slm+4sjl}\delta^{2(i+l)}\#\eta^{2(j+m)}\\ &= \tfrac{1}{q^2}\textstyle\sum_{r,t=0}^{q-1}\sum_{l,m=0}^{q-1}\omega^{-sar-sbt+2sr(t-m)+2slt}\delta^{2r}\#\eta^{2t}\\ &= \tfrac{1}{q^2}\textstyle\sum_{r,t=0}^{q-1}\omega^{-sar-sbt+2str}(\sum_{l,m=0}^{q-1}\omega^{-2srm+2stl})\delta^{2r}\#\eta^{2t}\\ &= 1\#1.\end{aligned}$$

In order to prove that the h-action is conjugation by v we show that $v(\delta^i\#\eta^j) = \omega^{sai+sbj}(\delta^{-i}\#\eta^{-j})v$. We do so for the even case, the odd case is done similarly.

$$\begin{aligned} v(\delta^i\#\eta^j) &= \tfrac{1}{q}\textstyle\sum_{l,m=0}^{q-1}\omega^{-sal-sbm+2slm+2sim}\delta^{2l+i}\#\eta^{2m+j} \\ &= \tfrac{1}{q}\textstyle\sum_{l'=i}^{q-1+i}\sum_{m'=j}^{q-1+j}\omega^{-sal'-sbm'+sai+sbj+2sl'm'-2sl'j}\delta^{2l'-i}\#\eta^{2m'-j} \\ &= \omega^{sai+sbj}(\delta^{n-i}\#\eta^{n-j})(\tfrac{1}{q}\textstyle\sum_{l'=0}^{q-1}\sum_{m'=0}^{q-1}\omega^{-sal'-sbm'+2sl'm'}\delta^{2l'}\#\eta^{2m'}) \\ &= \omega^{sai+sbj}(\delta^{n-i}\#\eta^{n-j})v, \end{aligned}$$

where in the second equality the limits of the sums are reduced modulo q if necessary. Hence, for n odd, $[C_{\bar{0}}(1)]^2 = 1$ because $v^2 = 1\#1$ is a square in k. For $n = 2q$ we still have to compute γ^q where γ is defined as usual. Using the commutation rules for v and $\delta^i\#\eta^j$ and the expression of powers of u we find:

$$vu^{n-1} = \zeta^{-3}v(\delta\#\eta^{n-1}) = \zeta^{-3}\omega^{s(a-b)}(\delta^{n-1}\#\eta)v = \omega^{-2s}\omega^{s(a-b)}uv.$$

Thus $\gamma = \omega^{2s}\omega^{s(b-a)}$. Hence $\gamma^q = 1$ if and only if $a = b$. It follows that $[C_{\bar{0}}(1)]^2 = [C_{\bar{1}}(1)]^2 = 1$ while $[C_{\bar{0}}(1)\#C_{\bar{1}}(1)] = [C_{\bar{0}}(1)][C_{\bar{1}}(1)] \neq 1$. The algebra $C_{\bar{0}}(1)\#C_{\bar{1}}(1)$ is a matrix algebra with strongly inner g-action. So $[C_{\bar{0}}(1)\#C_{\bar{1}}(1)]$ is in the kernel of i^*. Its image through the map Inv_e of Theorem 2.10 is $([1], -1)$. A similar argument applies to $[C_{\bar{1}}(1)\#C_{\bar{0}}(1)] = [C_{\bar{1}}(1)][C_{\bar{0}}(1)]$. Since Inv_e is injective, both classes coincide. ∎

Corollary 2.16 *Let k be algebraically closed, $n = 2^m$ and let z be odd. Then $BM(k, \mathbb{D}_n, R_z) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. It is generated by $[C_{\bar{0}}(1)]$ and $[C_{\bar{1}}(1)]$.*

Proof: By Lemma 2.15 the map i^* in sequence (4) is surjective and split. Hence $BM(k, \mathbb{D}_n, R_z) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with generators $[C_{\bar{0}}(1)]$ and $[C_{\bar{1}}(1)]$. ∎

3 The map ι_*

In this section we study the Brauer group $BM(k, \mathbb{D}_n, R_z)$ when the field k is not necessarily algebraically closed. Let $\overline{k}$ denote the algebraic closure of k. The inclusion map $\iota : k \to \overline{k}$ induces a group homomorphism $\iota_* : BM(k, \mathbb{D}_n, R_z) \to BM(\overline{k}, \mathbb{D}_n, R_z)$, $[A] \mapsto [A \otimes_k \overline{k}]$. We describe the kernel of ι_*.

Lemma 3.1 *If n is odd there is an exact sequence*

$$1 \longrightarrow BAz(k, \mathbb{D}_n, R_z) \longrightarrow BM(k, \mathbb{D}_n, R_z) \xrightarrow{\iota_*} BM(\overline{k}, \mathbb{D}_n, R_z),$$

where $BAz(k, \mathbb{D}_n, R_z) = BAz^g(k, \mathbb{D}_n, R_z)$ is the set consisting of classes of $BM(k, \mathbb{D}_n, R_z)$ represented by classically Azumaya algebras.

If $n = 2q$ is even, then $Ker(\iota_*)$ consists of classically Azumaya algebras with α, γ in the induced subalgebra satisfying $\gamma^q\alpha^{q-1} = 1$.

Proof: The kernel of ι_* consists of classes of $\mathbb{D}_n$-Azumaya algebras $[A]$ such that $[A \otimes_k \overline{k}]$ becomes Brauer-trivial in $BM(\overline{k}, \mathbb{D}_n, R_z)$. Hence $A \otimes_k \overline{k}$ is a matrix algebra over $\overline{k}$ with strongly inner $\mathbb{D}_n$-action, and consequently, an Azumaya algebra over $\overline{k}$. But it is well-known that A is Azumaya over k if and only if $A_{\bar{k}} = A \otimes_k \overline{k}$ is Azumaya over $\overline{k}$.

If n is odd, then $[A] \in BAz(k, \mathbb{D}_n, R_z)$. Conversely, for n odd and A a $\mathbb{D}_n$-Azumaya module algebra which is classically Azumaya, $A \otimes_k \overline{k}$ is Azumaya over $\overline{k}$. But the only Azumaya algebras over an algebraically closed field are matrix algebras. Moreover, from Proposition 2.2, the $\mathbb{D}_n$-action on $A \otimes_k \overline{k}$ is strongly inner since $\bar{k}$ is algebraically closed. Then $A \otimes_k \overline{k}$ is Brauer-trivial in $BM(\overline{k}, D_n, R_z)$ by Lemma 2.1.

If $n = 2q$ and $[A] \in Ker(\iota_*)$, then $A_{\bar{k}} = A \otimes_k \bar{k}$ is a matrix algebra over $\overline{k}$. So A is Azumaya over k. The induced subalgebra B on $A_{\bar{k}}$ is generated by u and v such that $u^n = \alpha$ and $uv = \gamma v u^{n-1}$ with $\alpha, \gamma \in \bar{k}$ satisfying $\gamma^q\alpha^{q-1} = 1$ by Proposition 2.2. On the other hand, $B = B' \otimes_k \overline{k}$ where B' is the induced subalgebra on A. Let u', v' be the generators of B'. The elements u, v in B must be scalar multiples of u', v'. If $u = tu'$ and $v = sv'$ for some s, $t \in \bar{k}$, then $\alpha' = t^n\alpha$ and $\gamma' = (t^{-1})^{n-2}\gamma$ so

$$\gamma'^q\alpha'^{q-1} = (t^{2-n})^q\gamma^q(t^n)^{q-1}\alpha^{q-1} = (t^{q-1})^{2-n}t^{2-n}(t^{q-1})^{n-2}(t^{q-1})^2\gamma^q\alpha^{q-1} = \gamma^q\alpha^{q-1}.$$

Conversely, if A is a $\mathbb{D}_n$-Azumaya module algebra which is classically Azumaya and satisfying $\gamma^q\alpha^{q-1} = 1$, then $A \otimes_k \bar{k}$ is Brauer trivial in $BM(\bar{k}, \mathbb{D}_n, R_z)$ because $\bar{k}$ is algebraically closed. ∎

Proposition 3.2 *i) For n odd, $BAz^g(k, \mathbb{D}_n, R_z) \cong k^{\cdot}/k^{\cdot 2} \times Br(k)$.*

ii) For n even, $BAz^g(k, \mathbb{D}_n, R_z) \cong \mathbb{Z}_2 \times k^{\cdot}/k^{\cdot 2} \times Br(k)$.

Proof: We know from Corollary 2.8 that $BAz^g(k, \mathbb{D}_n, R_z)$ is abelian. The assignment $\tau\colon BAz^g(k, \mathbb{D}_n, R) \to Br(k)$ which maps $[A]$ into $[A]$ by forgetting the $\mathbb{D}_n$-action is a group homomorphism by Lemma 2.7. Moreover, any k-Azumaya algebra may be endowed with the trivial $\mathbb{D}_n$-action becoming clearly $\mathbb{D}_n$-Azumaya. Thus the map so defined splits τ. Hence $BAz^g(k, \mathbb{D}_n, R_z) \cong Br(k) \times Ker(\tau)$. As in the proof of Theorem 2.10 we can show that $Ker(\tau) \cong k^{\cdot}/k^{\cdot 2}$ for n odd, and $Ker(\tau) \cong k^{\cdot}/k^{\cdot 2} \times \mathbb{Z}_2$ for n even. In both cases $Ker(\tau)$ is represented by the classes of the algebras $A(\beta, \gamma)$ for $\beta \in k^{\cdot}$ and γ an n-th root of unity. ∎

For $a, b \in k^{\cdot}$ let $\{a, b\}$ denote the quaternion algebra generated by x, y such that $x^2 = a, y^2 = b$ and $xy = -yx$. Since this algebra is also generated by x and $\theta^q bxy^{-1}$, we have that $\{a, b\} = \{a, ab\}$. When $b = 1$, $\{a, 1\}$ is a matrix algebra. For more details on these algebras see [11], [13, Section 15].

For any $t \in k^{\cdot}$ let $A(t)$ denote the $\mathbb{D}_n$-module algebra constructed in the following way: as an algebra $A(t) = M_2(k)$, and the $\mathbb{D}_n$-action is given by h acting trivially and g acting by conjugation by

$$u = \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}.$$

Lemma 3.3 *With $A(t)$ as above and $n = 2q$ even, the following assertions hold:*

i) $A(t)$ is a $\mathbb{D}_n$-Azumaya module algebra.

ii) $A(t) \cong A(tr^2)$ as $\mathbb{D}_n$-module algebras for any $r \in k^{\cdot}$.

iii) If q is even, then $A(t)\#A(r) \cong M_2(k) \otimes A(tr)$ as $\mathbb{D}_n$-module algebras where $M_2(k)$ has trivial $\mathbb{D}_n$-action. If q is odd, then $A(t)\#A(r) \cong \{t, r\} \otimes A(tr)$ as $\mathbb{D}_n$-module algebras where $\{t, r\}$ has trivial $\mathbb{D}_n$-action.

iv) $[A(t)]$ belongs to $Ker(\iota_*)$ and it has order two.

Proof: i) We show that $A(t)$ is a $\mathbb{Z}_n$-Azumaya algebra, hence a $\mathbb{D}_n$-Azumaya algebra. We observe that since $u^2 = t$ and since z is odd in this case, the action of g^z is again conjugation by u. Therefore

$$g^z \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = g \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & tc \\ t^{-1}b & a \end{pmatrix}.$$

There are only elements of degree 0 and degree q in $A(t)$, so $A(t)$ is in fact $\mathbb{Z}_2$-graded. The elements of degree 0 (even elements) and the elements of degree q (odd elements) are given by matrices of the form

$$\begin{pmatrix} a & tc \\ c & a \end{pmatrix}, \qquad \begin{pmatrix} a & -tc \\ c & -a \end{pmatrix},$$

respectively. It is easy to check that the graded center of $A(t)$ is k, and consequently, $A(t)$ is $\mathbb{Z}_n$-Azumaya.

ii) The elements

$$x = \theta^q \begin{pmatrix} 0 & -t \\ 1 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7}$$

are generators for $A(t)$. These satisfy $x^2 = t, y^2 = 1, xy = -yx$ and $g \cdot x = -x, g \cdot y = -y$. For $r \in k^{\cdot}$, the isomorphism of $\mathbb{D}_n$-module algebras from $A(t)$ to $A(tr^2)$ is given by mapping x into rx and y into y.

iii) Let $M, M' \in A(t)$ and $N, N' \in A(r)$ be homogeneous. From (2),

$$(M\#N)(M'\#N') = MM'\#(g^{deg(M')} \cdot N)N'.$$

As we saw in i), $deg(M')$ is equal to 0 or q. If q is even, then the action by g^q is trivial. Thus $A(t)\#A(s) = A(t) \otimes A(s)$. Let x, y be generators for $A(t)$ and x', y' generators for $A(r)$ as in (7). Let

$$X = x\#y',\ Y = y\#y',\ Z = 1\#y',\ W = \theta^q(xy\#x').$$

A computation shows that these elements satisfy the following relations:

$$\begin{array}{ll} X^2 = t,\ Y^2 = 1,\ XY = -YX, & Z^2 = 1, W^2 = tr, ZW = -WZ, \\ XZ = ZX,\ XW = WX, & YZ = ZY,\ YW = WY, \\ g \cdot X = X,\ g \cdot Y = Y, & g \cdot Z = -Z,\ g \cdot W = -W \end{array}$$

This yields that $A(t) \otimes A(r) \cong \{t, 1\} \otimes A(tr)$ as $\mathbb{D}_n$-module algebras with $\{t, 1\}$ having trivial g-action. Since $\{t, 1\} \cong M_2(k)$ as algebras, the statement follows.

Assume now that q is odd. Then the action by g^q is the same as the action by g. Thus $g^q \cdot N = (-1)^{deg(N)}N$. The product takes the form

$$(M\#N)(M'\#N') = MM'\#(-1)^{deg(M')deg(N)}NN'. \tag{8}$$

Let $X = \theta^q(xy\#1),\ Y = \theta^q(x\#x'),\ Z = 1\#y'$ and $W = \theta^q(xy\#x')$. Using the formula (8), it may be checked that

$$\begin{array}{ll} X^2 = t,\ Y^2 = tr,\ XY = -YX, & Z^2 = 1, W^2 = tr, ZW = -WZ, \\ XZ = ZX,\ XW = WX, & YZ = ZY,\ YW = WY, \\ g\cdot X = X,\ g\cdot Y = Y, & g\cdot Z = -Z,\ g\cdot W = -W. \end{array}$$

From these relations, $A(t)\#A(r) \cong \{t, tr\} \otimes A(tr)$ as $\mathbb{D}_n$-module algebras. Notice now that $\{t, tr\} \cong \{t, r\}$ as algebras.

iv) The elements $\alpha_{A(t)}, \beta_{A(t)}$, and $\gamma_{A(t)}$ of the induced subalgebra on $A(t)$ are $\alpha_{A(t)} = t^q, \beta_{A(t)} = 1$ and $\gamma_{A(t)} = t^{1-q}$. As $\gamma^q_{A(t)}\alpha^{q-1}_{A(t)} = 1$ and $A(t)$ is a matrix algebra, $[A(t)]$ belongs to $Ker(\iota_*)$.

The algebra $A(t)\#A(t)$ is classically Azumaya since it belongs to $Ker(\iota_*)$. Moreover, it has strongly inner $\mathbb{D}_n$-action. Note that $u_{A(t)\#A(t)} = u_{A(t)}\#u_{A(t)}$ and $v_{A(t)\#A(t)} = 1$ because $u_{A(t)}$ has degree 0 and the h-action is trivial on $A(t)$. From this, $\alpha_{A(t)\#A(t)} = t^n, \beta_{A(t)\#A(t)} = 1$ and $\gamma_{A(t)\#A(t)} = t^{2-n}$. If q is even, then $A(t)\#A(t) \cong M_2(k) \otimes A(t^2)$, and so $A(t)\#A(t)$ is a matrix algebra. If q is odd, then $A(t)\#A(t) \cong \{t, t^2\} \otimes A(t^2)$. But $\{t, t^2\} \cong \{t, 1\}$ and $\{t, 1\}$ is a matrix algebra. Hence, in this case also $A(t)\#A(t)$ is a matrix algebra. Finally, Lemma 2.1 implies that $[A(t)\#A(t)]$ is trivial. ∎

The map $\{-,-\} : k^\cdot/k^{\cdot 2} \times k^\cdot/k^{\cdot 2} \to Br(k),\ ([a],[b]) \mapsto [\{a,b\}]$ is a 2-cocycle, see [13, page 146]. Let $k^\cdot/k^{\cdot 2} \times_{\{-,-\}} Br(k)$ denote the extension of $k^\cdot/k^{\cdot 2}$ and $Br(k)$ by this cocycle.

Theorem 3.4 *With notation as above*

$$Ker(\iota_*) \cong \begin{cases} k^\cdot/k^{\cdot 2} \times Br(k) & \textit{for } n \textit{ odd} \\ k^\cdot/k^{\cdot 2} \times k^\cdot/k^{\cdot 2} \times Br(k) & \textit{for } n = 2q,\ q \textit{ even} \\ k^\cdot/k^{\cdot 2} \times (k^\cdot/k^{\cdot 2} \times_{\{-,-\}} Br(k)) & \textit{for } n = 2q,\ q \textit{ odd} \end{cases}$$

Proof: For n odd, Lemma 3.1 and Corollary 2.8 establish that $Ker(\iota_*) = BAz^g(k, \mathbb{D}_n, R_z)$. Now Proposition 3.2 i) applies. The case n even is more complicated and requires a different argument. The elements of $Ker(\iota_*)$ may all be represented by classically Azumaya algebras with α, γ in the induced subalgebra satisfying $\gamma^q\alpha^{q-1} = 1$ by Lemma 3.1.

Suppose that $n = 2q$ is even. Let $[A] \in Ker(\iota_*)$, then A is classically Azumaya and the elements α_A, β_A, and γ_A in the induced subalgebra satisfy $\gamma_A^q\alpha_A^{q-1} = 1$. For $t_A = (\alpha_A\gamma_A)^{-1}$, the algebra $A\#A(t_A)$ represents an element of $Ker(\iota_*)$ because A and $A(t_A)$ do. Hence it is classically Azumaya. Moreover, it has strongly inner g-action because $u_{A\#A(t_A)} = u_A\#u_{A(t_A)}$ and $u^n_{A\#A(t_A)} = u^n_A\#u^n_{A(t_A)} = \alpha_A(\alpha_A\gamma_A)^{-q} = 1$ (the degree of u in the induced subalgebra is always zero). So $[A\#A(t_A)] \in BAz^g(k, \mathbb{D}_n, R_z)$. By Proposition 3.2,

$$[A\#A(t_A)] = [A(\beta, \gamma)][|A\#A(t_A)|] \in Ker(\iota_*)$$

where $[\beta] \in k^{\cdot}/k^{\cdot 2}$, γ is an n-th root of unity, and $|A\#A(t_A)|$ denotes the underlying algebra of $A\#A(t_A)$ with trivial action. By Lemma 2.9 we obtain $[\gamma] = [\gamma_{A\#A(t_A)}]$ and $\gamma^q = 1$. By the proof of Theorem 2.10, $[A(\beta, \gamma)] = [A(\beta, 1)]$ so we may assume that the g-action on the right hand side is trivial and that the braided product of the representative of elements of the right hand side with $A(t_A)$ is trivial. Hence

$$[A] = [A(\beta, 1) \otimes |A\#A(t_A)| \otimes A(t_A)]$$

were both representatives are classically Azumaya. By Lemma 2.9, $[\beta] = [\beta_A] \in k^{\cdot}/k^{\cdot 2}$. Thus the three classes $[A(\beta_A, 1)], [A(t_A)]$ and $|A\#A(t_A)|$ are uniquely determined by $[A]$.

Assume that q is even. We prove that the map

$$\begin{aligned} \Psi : Ker(\iota_*) &\longrightarrow k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \times Br(k) \\ [A] &\mapsto ([\beta_A], [(\alpha_A\gamma_A)^{-1}], [|A\#A((\alpha_A\gamma_A)^{-1})|]) \end{aligned}$$

is an isomorphism. We first check that it is well-defined. Assume that $[A] = [B]$ in $Ker(\iota_*)$. Let $t_A = (\alpha_A\gamma_A)^{-1}$ and $t_B = (\alpha_B\gamma_B)^{-1}$. By Lemma 2.9 and Lemma 2.5, $[\beta_A] = [\beta_B]$ and $[t_A] = [t_B]$ in $k^{\cdot}/k^{\cdot 2}$. By Lemma 3.3 ii), $A(t_A) \cong A(t_B)$. Then $[A\#A(t_A)] = [B\#A(t_B)]$ in $BM(k, \mathbb{D}_n, R_z)$. There are finite dimensional $\mathbb{D}_n$-modules P, Q such that

$$(A\#A(t_A))\#End(P) \cong (B\#A(t_B))\#End(Q)$$

as $\mathbb{D}_n$-module algebras. Since $End(P), End(Q)$ are classically Azumaya with strongly inner g-action, from Lemma 2.7 it follows that

$$(A\#A(t_A)) \otimes End(P) \cong (B\#A(t_B)) \otimes End(Q)$$

as algebras. Hence $[|A\#A(t_A)|] = [|B\#A(t_B)|]$ in $Br(k)$. This proves that Ψ is well-defined. Secondly, we show that Ψ is a group homomorphism. Let $[A], [B] \in Ker(\iota_*)$ and assume that

$$[A] = [A(\beta_A, 1)][|A\#A(t_A)|][A(t_A)], \quad [B] = [A(\beta_B, 1)][|B\#A(t_B)|][A(t_B)],$$
$$[A\#B] = [A(\beta_{A\#B}, 1)][|(A\#B)\#A(t_{A\#B})|][A(t_{A\#B})].$$

Observe that when q is even $[A(t_A)]$ commutes with $[A(t_B)]$ in light of Lemma 3.3, $[A(t_A)]$ commutes with the elements $[A(\beta, 1)]$ and with the elements of $Br(k)$ since these have trivial g-action. This implies that $[B][A(t_A)] = [A(t_A)][B]$. Then,

$$\begin{aligned}[A\#B][A(t_{A\#B})] &= [A][B][A(t_A)][A(t_B)] \\ &= [A][A(t_A)][B][A(t_B)] \\ &= [(A\#A(t_A))\#(B\#A(t_B))] \\ &= [(A\#A(t_A)) \otimes (B\#A(t_B))]\end{aligned}$$

where in the last equality we have used Lemma 2.7 since the g-action on $B\#A(t_B)$ is strongly inner. Hence

$$[|(A\#B)\#A(t_{A\#B})|] = [|(A\#A(t_A))| \otimes |(B\#A(t_B))|]$$

in $Br(k)$. Using all the preceding facts, we have,

$$\begin{aligned}[A\#B] &= [A][B] \\ &= [A(\beta_A, 1)][|A\#A(t_A)|][A(t_A)][A(\beta_B, 1)][|B\#A(t_B)|][A(t_B)] \\ &= [A(\beta_A, 1)][A(\beta_B, 1)][|A\#A(t_A)|][|B\#A(t_B)|][A(t_A)][A(t_B)] \\ &= [A(\beta_A, 1)\#A(\beta_B, 1)][|A\#A(t_A)| \otimes |B\#A(t_B)|][A(t_A)\#A(t_B)] \\ &= [A(\beta_A\beta_B, 1)][|(A\#A(t_A))|][|(B\#A(t_B))|][A(t_At_B)]\end{aligned} \tag{9}$$

where in the last equality we have used Lemma 3.3 iii) and Theorem 2.10. Finally we show that Ψ is bijective. It is clearly surjective since to any $([\beta], [\lambda], [D]) \in k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \times Br(k)$ we can associate $[A(\lambda^{-1}) \otimes A(\beta, 1) \otimes |D|] \in Ker(\iota_*)$. To prove the injectivity, let $[A] \in Ker(\Psi)$. Then β_A, t_A are squares and $|A\#A(t_A)|$ is a matrix algebra. Thus $[A] = [A(1,1)][|M_m(k)|][A(s^2)]$ for some $m \in \mathbb{N}$ and $s \in k^{\cdot}$ such that $t_A = s^2$. Then $[A]$ is represented by a matrix algebra with strongly inner $\mathbb{D}_n$-action. Lemma 2.1 implies that $[A]$ is trivial.

For q odd, the same proof works but we have to modify the multiplication on $k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \times Br(k)$. With notation as in (9), for q odd, $A(t_A)\#A(t_B) \cong \{t_A, t_B\} \otimes A(t_A t_B)$ by Lemma 3.3. Then,

$$[|(A\#B)\#A(t_{A\#B})|] = [|(A\#A(t_A))| \otimes |(B\#A(t_B))| \otimes \{t_A, t_B\}].$$

Notice that $[B][A(t_A)] = [A(t_A)][B]$ is true in this case because $\{t_A, t_B\} \cong \{t_B, t_A\}$. ∎

Theorem 3.5 *Let p be a prime number not dividing z, $m \in \mathbb{N}$, and $n = p^m$. Let k be a field containing a primitive $2n$-th root of unity and let n be invertible in k. Then*

$$BM(k, \mathbb{D}_n, R_z) \cong \begin{cases} k^{\cdot}/k^{\cdot 2} \times Br(k) \times \mathbb{Z}_2 & \text{if } p \text{ is odd,} \\ k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \times Br(k) \times \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } p = 2. \end{cases}$$

Proof: By Corollary 2.13, Corollary 2.16, Lemma 3.1 and Theorem 3.4 we have two exact sequences

$$1 \longrightarrow k^{\cdot}/k^{\cdot 2} \times Br(k) \longrightarrow BM(k, \mathbb{D}_n, R_z) \xrightarrow{\iota_*} \mathbb{Z}_2$$

for p odd and

$$1 \longrightarrow k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \times Br(k) \longrightarrow BM(k, \mathbb{D}_n, R_z) \xrightarrow{\iota_*} \mathbb{Z}_2 \times \mathbb{Z}_2$$

for $p = 2$.

Let $C_{\bar{a}}(1)_{\overline{k}} = C_{\bar{a}}(1) \otimes_k \overline{k}$ for $a = 0, 1$. The nontrivial element of the latter term in the first exact sequence is represented by $C_{\bar{0}}(1)_{\overline{k}}$. The latter term in the second exact sequence is given by the group generated by $[C_{\bar{a}}(1)_{\overline{k}}]$ with $a = 0, 1$. Hence ι_* is surjective in both cases. Mapping $[C_{\bar{a}}(1)_{\overline{k}}]$ to $[C_{\bar{a}}(1)]$ we obtain a group homomorphism in light of Lemma 2.15, which splits ι_*. Then $BM(k, \mathbb{D}_n, R_z)$ is a semidirect product of $k^{\cdot}/k^{\cdot 2} \times Br(k)$ and $\mathbb{Z}_2$ for n odd and a semidirect product of $k^{\cdot}/k^{\cdot 2} \times k^{\cdot}/k^{\cdot 2} \times Br(k)$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ for n even. If n is odd, since the elements representing $BAz(k, \mathbb{D}_n, R_z)$ have trivial g-action, the braided product of such an element and $C_{\bar{0}}(1)$ is just the usual tensor product. Thus the elements of $BAz(k, \mathbb{D}_n, R_z)$ commute with $[C(1)_{\bar{0}}]$ and we have the direct product decomposition for $BM(k, \mathbb{D}_n, R_z)$. If n is even the elements representing the first copy of $k^{\cdot}/k^{\cdot 2}$ and those representing $Br(k)$ have trivial g-action hence they commute with the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. The second copy of $k^{\cdot}/k^{\cdot 2}$ is represented by the algebras $A(t)$ defined in the proof of Theorem 3.4, with $\mathbb{Z}_n$-grading inducing a $\mathbb{Z}_2$-grading, which we will denote by $\deg'$. Let δ be the generator of $C(1)$ and let M, $N \in A(t)$ with M homogeneous. By formula (2),

$$(\delta^i \# M)(\delta^j \# N) = \delta^{i+j} \# (g^{j \bmod 2} \cdot M)N = (-1)^{(j \bmod 2)\deg'(M)} \delta^{i+j} \# MN.$$

Thus $C_{\bar{a}}(1) \# A(t) \cong C_{\bar{a}}(1) \otimes_2 A(t)$. Here $\otimes_2$ denotes the $\mathbb{Z}_2$-graded tensor product. Similarly,

$$\begin{aligned}(M \# \delta^i)(N \# \delta^j) &= MN \# (g^{\deg(N)} \cdot \delta^i)\delta^j \\ &= \omega^{siq \deg'(N)} MN \# \delta^{i+j} \\ &= (-1)^{(i \bmod 2)\deg'(N)} MN \# \delta^{i+j}.\end{aligned}$$

Since $A(t) \otimes_2 C_{\bar{a}}(1) \cong C_{\bar{a}}(1) \otimes_2 A(t)$ as $\mathbb{D}_n$-module algebras, $[A(t)]$ commutes with $[C_{\bar{a}}(1)]$ for $a = 0, 1$. Therefore the kernel of ι_* commutes with $\mathbb{Z}_2 \times \mathbb{Z}_2$ and we are done. ∎

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