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Some comments on Hurst exponent and the long memory processes on capital markets

M.A. Sánchez Granero [a,1], J.E. Trinidad Segovia [b,*,2], J. García Pérez [c,3]

[a] Department of Geometry and Topology, University of Almeria[4], Spain
[b] Department of Accounting and Finance, University of Almeria, Spain
[c] Department of Applied Economic, University of Almeria, Spain

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ABSTRACT

The analysis of long memory processes in capital markets has been one of the topics in finance, since the existence of the market memory could implicate the rejection of an efficient market hypothesis. The study of these processes in finance is realized through Hurst exponent and the most classical method applied is R/S analysis. In this paper we will discuss the efficiency of this methodology as well as some of its more important modifications to detect the long memory. We also propose the application of a classical geometrical method with short modifications and we compare both approaches.

1. Introduction

It is well known and accepted that some of the human and natural phenomena show long memory, and there are a wide amount of papers about this topic in natural sciences. In economy, as it happened with other processes observed in physics and in general in natural sciences, the study of long memory caught the interest of researchers during the seventies [20,18,19]. Ref. [21] contains many of the early papers that Mandelbrot wrote on the application of the Hurst exponent in financial time series.

Since those days, the application of the long memory processes in economy has been extended from macroeconomics to finance. Examples are Refs. [8,4,12,13,26,3,25,6] and recently Ref. [7] to quote some of them.

In finance, the discussion as to whether or not the stock market prices display long memory properties still continues since this fact has important consequences on the capital market theories. So, if stock prices show long memory this means that predictability is not a dream but a possibility. The main implication of this circumstance is that an efficient market hypothesis is clearly rejected because stock market prices do not follow a random walk.

The study of the long memory processes is normally realized through the Hurst exponent that can be estimated using three methods[5]:

* Corresponding address: Department of Accounting and Finance, University of Almeria, Carretera de Sacramento, S/N La Canada de San Urbano, 04120 Almeria, Spain. Tel.: +34 950015817; fax: +34 950633179.

E-mail addresses: misanche@ual.es (M.A. Sánchez Granero), jetrini@ual.es (J.E. Trinidad Segovia), jgarcia@ual.es (J. García Pérez).

1 Tel.: +34 950015311; fax: +34 950015481.

2 Part of this paper has been developed during the stay of Juan E. Trinidad and Miguel A. Sanchez in the Department of Finance of the University of Rotterdam, The Netherlands.

3 Tel.: +34 950015173; fax: +34 950251795.

4 Carretera de Sacramento, S/N La Cañada de San Urbano 04120 Almería.

5 Ref. [29] contains an interesting description of these methods.

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1. R/S Analysis.
2. Detrended Fluctuation Analysis.
3. Periodogram Regression.

Our main objective in this paper is to prove that the most popular method used to estimate the Hurst exponent, the R/S analysis, exhibits serious problems and as a result it is possible to obtain evidence of long memory in random series. To avoid these problems, two geometrical methods are introduced to be applied in capital markets. One of them (GM1) is based on a classical formulae of Hurst exponent while the other one (GM2) is based on a new formulae which is introduced in Section 4 and developed further in the Appendix. We will show how this methodology will allow us to study the presence of the long memory processes in the capital markets from a stricter point of view. Regarding the R/S analysis, a new approach to the application of the Anis and Lloyd's formula in [1] is introduced to solve the problems presented when subintervals of small lengths are used.

The organization of the paper is as follows: In Section 2 we introduce the Hurst exponent basis as well as the R/S analysis to calculate it. In Section 3 we study R/S analysis behavior in the particular case of random series. Section 4 presents a different approach to obtain the Hurst exponent and it is compared with R/S and the modified R/S analysis. The paper continues with an empirical application using the different approaches and finally Section 6 presents the main conclusions.

2. The Hurst exponent and long memory processes. Classical estimation via R/S analysis

The Hurst exponent is the classical test to detect long memory in time series. This analysis was introduced by English hydrologist H.E. Hurst in 1951, based on Einstein's contributions regarding Brownian motion of physical particles, to deal with the problem of reservoir control near Nile River Dam. R/S analysis in economy was introduced by Mandelbrot [18,19,21], who argued that this methodology was superior to the autocorrelation, the variance analysis and to the spectral analysis.

As we pointed out there are a wide range of papers about this topic in economy in general and in finance in particular. It is curious that most of them concluded finding evidence of memory in financial data. Examples are Ref. [10] in New York Stock Exchange, Ref. [24] in several stocks and Indexes such us S&P 500 Index, Ref. [22] in Athens Stock Exchange, Ref. [11] in S&P 500 return series before the market crashed in 1987 and 1990 or Ref. [15] in London Stock Exchange. Only Refs. [16,17] reported no evidence of long range dependence in stock returns.

The eldest and best-known method to estimate the Hurst exponent is R/S analysis. It was proposed by Mandelbrot and Wallis [20], based on the previous work of Hurst [14].

The procedure is as follows. The time series (of returns) of length L has to be divided into d sub series ($Z_{i,m}$) of length n, and for each sub series $m = 1, \ldots, d$. Then,

1. It is necessary to find the mean (E_m) and the standard deviation (S_m) of the sub series ($Z_{i,m}$).
2. The data of the sub series ($Z_{i,m}$) has to be normalized by subtracting the sample mean $X_{i,m} = Z_{i,m} - E_m$ for $i = 1, \ldots, n$.
3. Create the cumulative time series $Y_{i,m} = \sum_{j=1}^{i} X_{j,m}$ for $i = 1, \ldots, n$.
4. Find the range $R_m = \max\left\{Y_{1,m}, \ldots, Y_{n,m}\right\} - \min\left\{Y_{1,m}, \ldots, Y_{n,m}\right\}$.
5. Rescale the range (R_m/S_m).
6. Calculate the mean value $(R/S)_n$ of the rescaled range for all sub series of length n.

Considering that the R/S statistic asymptotically follows the relation $(R/S)_n \approx cn^H$, the value of H can be obtained by running a simple linear regression over a sample increasing time horizons.

$$\log\,(R/S)_n = \log c + H \log n. \tag{1}$$

When the process is a Brownian motion, H has to be 0.5, when it is persistent H will be greater than 0.5, and finally when it is anti-persistent H will be less than 0.5. For a white noise, $H = 0$, while for a simple linear trend, $H = 1$. Note that H must lie between 0 and 1.

3. Testing R/S analysis

To test the R/S analysis we have applied the Monte Carlo method making 10,000 random walk series. Table 1 contains the mean and standard deviation of the Hurst exponent considering different n values (n is the minimum length of the sub series used in R/S analysis). Results (for $n = 2$) shows that the Hurst exponent average value is 0.68 with a standard deviation of 0.02. It is clear that this alteration in average is a consequence of n value, because when n is large enough the mean value is nearer to 0.5.

As it can be observed, the length of the series influences the standard deviation obtained, but also the mean. To obtain values near to the real 0.5 it is needed to choose a large n, which is impossible for short series.

Table 2 compares the influence of the series length on the calculation of the Hurst exponent for 1000 random walks. For the R/S analysis, n were chosen so that the estimation of the mean were the most accurate (but then the standard deviation was greater).

Table 1
Mean and standard deviation of Hurst exponent, using R/S analysis, for different n values (where n is the length of the shortest subinterval used in the calculation of the Hurst exponent)

n	Mean	Standard deviation	Series length
2	0.68	0.02	1,000
4	0.63	0.02	1,000
8	0.6	0.03	1,000
16	0.58	0.04	1,000
16	0.53	0.01	100,000[a]
32	0.56	0.06	1,000
32	0.54	0.03	5,000
64	0.55	0.1	1,000
128	0.53	0.1	2,000
256	0.52	0.12	3,000
512	0.51	0.13	5,000

[a] Only 100 random walks were simulated.

Table 2
Influence of the series length in Hurst exponent using R/S analysis (n is the length of the shortest subinterval used in the calculation of the Hurst exponent)

Length	R/S analysis		
	Mean	Std	n
100	0.67	0.12	8
200	0.61	0.12	16
500	0.57	0.11	32
1,000	0.55	0.10	64
5,000	0.51	0.08	256
10,000	0.50	0.13	1,024

It seems that R/S analysis shows important problems to study long range dependence when series is not large enough. In this line Lo [16] indicated some inconvenience of the methodology and he proposed a new statistic based on R/S analysis. However, Teverovsky, Taqqu and Willinger [27] proved that Lo's modification of R/S statistic is too strict.

The loss of accuracy in some specific cases has also been remarked by other authors. In this line Anis and Lloyd [1] and afterwards Peters [23] introduced a new formulation to improve the performance for small n. Weron [29] indicates that the procedure consists of obtaining $\mathrm{E(R/S)}_n$ as is shown in the next formula:

$$E(R/S)_n = \begin{cases} \dfrac{n-\frac{1}{2}}{n} \dfrac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)} \displaystyle\sum_{i=1}^{n-1} \sqrt{\frac{n-i}{i}} & \text{for } n \leq 340 \\ \dfrac{n-\frac{1}{2}}{n} \dfrac{1}{\sqrt{n\frac{\pi}{2}}} \displaystyle\sum_{i=1}^{n-1} \sqrt{\frac{n-i}{i}} & \text{for } n \geq 340. \end{cases} \tag{2}$$

Following Weron, once (2) is calculated, the Hurst exponent H will be 0.5 plus the slope of $(R/S)_n - E(R/S)_n$. However, if we calculate this modified R/S analysis in this way, results show a Hurst exponent, for some random series, with values higher than 1, which makes no sense. For this reason, we have followed a different procedure than in Ref. [29]. This procedure[6] lies in adding a final step to the classical R/S analysis which consist in calculating

$$\log H_n = \log\,(R/S)_n - \log E\,(R/S)_n + \log(n)/2$$

where $E\,(R/S)_n$ is given by (2).

Then find H by linear regression on

$$\log H_n = \log c + H \log n. \tag{3}$$

The distribution for the Hurst exponent calculated as stated previously (which we will note by R/S-AL), resembles in this case a normal[7] one with a mean of 0.49 and a standard deviation of 0.04 (with $n = 16$). Note that the distribution of the Hurst exponent calculated using standard R/S analysis cannot be approximated by a normal distribution.

We would like to remark that since formula (2) was derived for the series with underlying normal distribution, modified R/S analysis should be studied deeply to check its correctness for other series (for example, the series with Hurst exponent different from 0.5).

To conclude this section, Tables 3 and 4 show the sensibility to the length of the series and to n of classical and modified R/S analysis.

[6] As far as we know this methodology has not been applied before. Only in Ref. [29] we have found how to apply [1] formulation of R/S analysis.

[7] It is even interesting that the data obtained passed the Kolmogorov–Smirnov good-of-fitness test.

Table 3
Hurst exponent values using R/S and modified R/S analysis (R/S-AL) (n is the length of the shortest subinterval used in the calculation of the Hurst exponent)

n	R/S classical		R/S-AL		Series length
	Mean	Std	Mean	Std	
2	0.68	0.02	–	–	1,000
4	0.63	0.02	0.46	0.02	1,000
8	0.6	0.03	0.48	0.03	1,000
16	0.58	0.04	0.49	0.04	1,000
16	0.53	0.01	0.50	0.01	100,000[a]
32	0.56	0.06	0.49	0.06	1,000
32	0.54	0.03	0.50	0.03	5,000
64	0.55	0.1	0.49	0.11	1,000
128	0.53	0.1	0.50	0.10	2,000
256	0.52	0.12	0.50	0.12	3,000
512	0.51	0.13	0.49	0.14	5,000

[a] Only 100 random walks were simulated.

Table 4
Influence of series length in Hurst exponent values using R/S and modified R/S analysis (n is the length of the shortest subinterval used in the calculation of the Hurst exponent)

Length	Classical R/S			R/S-AL		
	n	Mean	Std	n	Mean	Std
100	8	0.67	0.12	8	0.46	0.12
200	16	0.61	0.12	16	0.49	0.12
500	32	0.57	0.11	16	0.49	0.11
1,000	64	0.55	0.10	16	0.49	0.10
5,000	256	0.51	0.08	16	0.50	0.08
10,000	1,024	0.50	0.13	16	0.50	0.13

4. Hurst exponent estimation via geometrical interpretation

In geometry there is a well-known method used to estimate the Hurst exponent which is based in the following formulae[8]

$$\overline{\Delta B}\ \alpha\ T_S^H \tag{4}$$

where: $\overline{\Delta B} = \overline{B(t+T_S) - B(t)}$ and it represents the mean of the variation of B on intervals of length T_S.

B: the price series in log.
T_S: the length of the time intervals.
H: Hurst exponent.
α: "is proportional to".

The procedure to estimate H is as follows. The time series (X_i) (of log-prices) of length L has to be divided into d sub series of length n, and for each sub series $m = 1, \ldots, d$:

(1) $D_m = X_{mn} - X_{(m-1)n+1}$
(2) Calculate the mean $H_n = \text{mean}\{D_m : m = 1, \ldots, d\}$.

Now the Hurst exponent H can be calculated by running a simple linear regression

$$\log H_n = \log c + H \log n. \tag{5}$$

Note that while R/S analysis uses the (log) return of the series, the previous formula uses the log price values.
We will use the notation GM1 to refer to this method of calculating the Hurst exponent hereafter.
If we have more information (maximum and minimum price for each period), then we propose to apply the following modification of GM1.

$$\overline{\text{range}(B)}\quad \alpha \quad T_S^H \tag{6}$$

where:

$$\overline{\text{range}(B)} = \overline{\max\{B(s) : t \le s \le t + T_s\} - \min\{B(s) : t \le s \le t + T_s\}}$$

and it represents the mean of the range B in each interval.

[8] See for example Ref. [2].

Table 5
Influence of series length in Hurst exponent values using R/S analysis and geometrical methods (n is the length of the shortest subinterval used in the calculation of the Hurst exponent)

Length	GM1		GM2		R/S analysis		
	Mean	Std	Mean	Std	Mean	Std	n
100	0.48	0.12	0.52	0.04	0.67	0.12	8
200	0.48	0.09	0.52	0.03	0.61	0.12	16
500	0.49	0.06	0.51	0.02	0.57	0.11	32
1,000	0.49	0.05	0.51	0.02	0.55	0.10	64
5,000	0.49	0.04	0.51	0.02	0.51	0.08	256
10,000	0.49	0.04	0.51	0.02	0.50	0.13	1,024

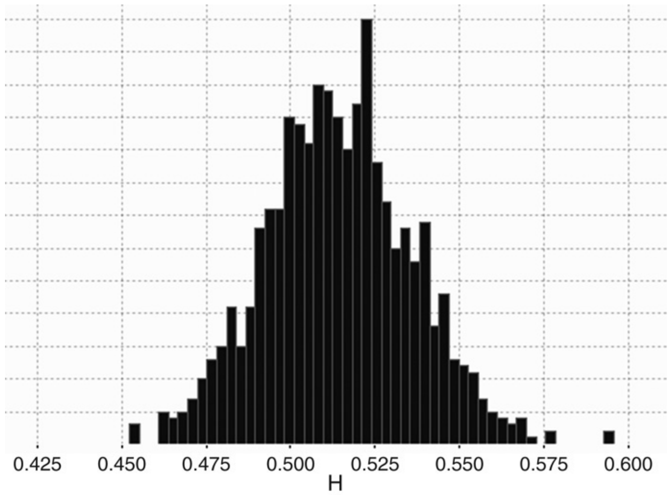

Fig. 1. Hurst exponent distribution using GM2.

The detailed procedure is as in GM1, except that D_m is defined as the maximum of the log-price in the period $X\,[(m-1)\,n, m \cdot n]$ minus the minimum of the log-price in the same period. Note that this is similar to the usual R/S analysis, but the maximum and minimum of each period is used, instead of just the closed prices. See the Appendix for a mathematical justification of the method which will be referred to as GM2.

Before applying this formulation in capital markets we will test its accuracy via the Monte Carlo Method as follows. In the case of GM1, we have generated 10,000 random walks with a length of 1000 and have calculated the Hurst exponent. The mean of the Hurst exponent was 0.49 and the standard deviation 0.05.

The same study was carried out for GM2, generating 2000 random walks with lengths of 1000, considering that prices change 100 times per day. Fig. 1 shows that the distribution of Hurst exponent resembles in this case a normal[9] one with a mean of 0.51 and a standard deviation of 0.02. This is obviously interesting since it will allow us to build confidence intervals.

This means, first of all, that both, original approach GM1 as well as its modification GM2, work appropriately because the value of the Hurst exponent is around 0.5 in the case of random samples. On the other hand, in spite that GM2 needs more data (maximum and minimum of each period), it has a better accuracy than GM1, since the standard deviation is lower.

Table 5 compares the influence of the length of the series using GM1, GM2 and R/S analysis (see also Table 4 for a comparison of the modified R/S analysis).

Another point to remark is that the Hurst exponent can be greater than 0.5 even with a "random walk", but with an average distinct from 0 (that is, for a process of the form $x_{t+1} = x_t + \varepsilon$, where ε follows a normal distribution with a non-null mean). Table 6 and Fig. 2 show the effect of the mean (in fact, the variation coefficient, that is, the standard deviation over the mean) on the Hurst exponent. Results were calculated using GM1 for 1000 random walks with a length of 1000.

Note that when there exists an average distinct from 0, the series does not verify (4), so it makes no sense of the Hurst exponent. Indeed, the previous table was intended only to prove that to calculate the Hurst exponent in a series with a mean distinct from 0, yields values distinct from 0.5. In this case, we should eliminate the mean from the series and perform the calculation of the Hurst exponent to the resulting data. Note that R/S and R/S-AL analysis eliminate the mean from the series in their own formula.

[9] It is even interesting that the data obtained passed the Kolmogorov–Smirnov good-of-fitness test.

Table 6
Effect of the mean in Hurst exponent values

Variation coefficient	H
0.1	1
1	0.98
5	0.7
10	0.58
20	0.52
50	0.5
100	0.49

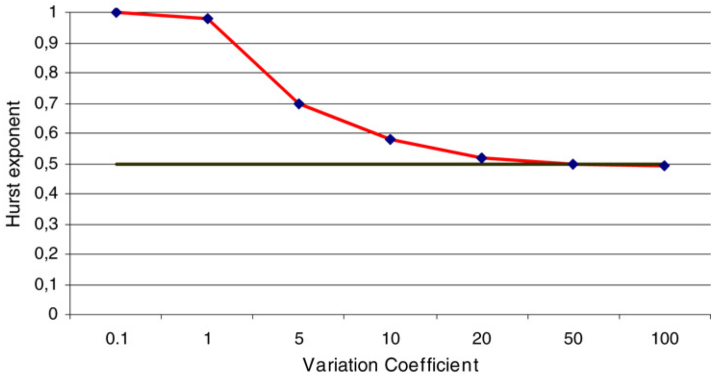

Fig. 2. Effect of the mean on the Hurst exponent.

Table 7
Hurst exponent calculation for different indexes

	Groups				
	All data	I group	II group	III group	IV group
Cac 40	0.54	0.58	0.53	0.56	0.53
FTSE 100	0.52	0.6	0.52	0.55	0.49
Nikkei	0.53	0.56	0.53	0.58	–
Nasdaq	0.54	0.55	–	–	–
Ibex 35	0.57	0.64	0.62	0.51	–
S&P 500	0.55	0.55	0.47	0.57	0.52

Table 8
Hurst Exponent calculation using different approaches

	GM1	GM2	R/S analysis	R/S-AL
Cac 40	0.51	0.54	0.57	0.50
FTSE 100	0.5	0.52	0.47	0.46
Nikkei	0.5	0.53	0.55	0.48
Nasdaq	0.52	0.54	0.55	0.49
Ibex 35	0.55	0.57	0.55	0.53
S&P 500	0.55	0.55	0.45	0.45

5. An empirical application

In this section, we calculate the Hurst exponent for several international indexes: Cac40 (From 1989), FTSE (From 1987), Nikkei (From 1992), Nasdaq (From 1998), Ibex 35 (From 1991) and S&P500 (From 1987).

Table 7 shows the results of the analysis (using GM2). It can be observed that the analysis is presented for all data and different groups. Each of these groups includes 1000 observations. The realization of groups will allow us to identify the memory in different trends.

If the indexes are random variables, the Hurst exponent value must be, with a confidence level of 99%, in the interval (0.464; 0.556).

If results obtained in this first approach are analysed we can observe that in 70% of cases the indexes behaviour is random. Only IBEX35 shows some evidence of long memory when all the data is considered, but the value is quite close to the interval limit.

Table 8 shows the Hurst exponent comparison using GM1, GM2, R/S analysis (with n greater enough, depending on the series length) and the modified R/S analysis (with $n = 16$).

Table 9
Mean influence in Hurst exponent results

	Mean	Std	Variation coefficient
Cac 40	0.00025	0.0137	55
FTSE 100	0.00024	0.0115	48
Nikkei	−0.0002	0.0145	−73
Nasdaq	0.00023	0.0246	107
Ibex 35	0.00041	0.0132	32
S&P 500	0.00033	0.0126	38

Table 10
Refined Hurst exponent values

	H (GM1)
Cac 40	0.48
FTSE 100	0.45
Nikkei	0.48
Nasdaq	0.51
Ibex 35	0.53
S&P 500	0.5

Finally, to analyze the average influence in the Hurst exponent results, average values have been calculated. Table 9 shows the results.

It can be observed that average influences are low but it does exist. Table 10 shows the new Hurst exponent for GM1, once we have eliminated the mean in the series.

Then, once we eliminated the mean in the series, we cannot conclude that the Hurst exponent is distinct from 0.5 in any case.

6. Conclusions

To conclude, we will like to point out that this paper presents several theoretical and empirical results about the calculation of the Hurst exponent.

Two methods, GM1 and GM2, are proposed. We prove that in both cases they are correct in an asymptotic way (Appendix contains the proof for GM2) and show empirical evidence via Monte Carlo simulations that they are also accurate for short series. Any other result provided, for example the distribution of the Hurst exponent calculated by GM2 can be approximated by a normal one; that GM1 and specially GM2 are more accurate than R/S analysis, or the sensibility of H to a non-zero mean, is also empirical.

A theoretical proof of these results should be very interesting as a base for further research. Since the calculation of H is used in other fields, the two methods proposed can be useful, especially if the series contains few data.

From our point of view, it is clear that using R/S analysis to determine the Hurst exponent can lead to incorrect results if it is not carried out carefully, and evidence of long memory can be obtained in random series.

Results contained in this paper shows that the influence of series length is very important in the Hurst exponent calculation when R/S analysis is used. Table 1 shows that Hurst exponent is nearer 0.5 when more than 5000 data are used, which, in financial series, means data from about 20 years. However, even in this case, standard deviation is 0.13 which means that a Hurst exponent of 0.64 does not necessarily involve the presence of memory in the series. It is clear that exhibited deviations by R/S analysis are more significant if we bear in mind that several articles such as Refs. [15,22,24] or Ref. [14], obtain the Hurst exponent values close to 0.7. In the case of the Spanish market, for example, Blasco and Santamaria [5] reported evidence of long range dependence in Madrid Stock Exchange with the Hurst exponent values around 0.60 in the different sectorial indexes. The study represents a sample of 14 years, which means around 3500 data. If we observe Table 3 in the case of 3000 data the Hurst exponent calculated using R/S analysis has an average of 0.52 with a standard deviation of 0.12.

With regard to our formulation, by the opposite, gives satisfactory results when it is applied over random series, so it seems to be suitable.

On the other hand, we proved that if GM2 is used, the distribution of the Hurst exponent of a random series can be approximated by a normal one, which allows determining confidence intervals for H. Previous papers in the line of studying the underlying distribution of the Hurst exponent of random walks are Refs. [6,28].

It seems clear that H, calculated using GM1 and GM2 is sensible to the series average value (in fact, to the variation coefficient), so this fact has to be considered in robust analysis, because this means that a random series could apparently show memory, when what really happens is that the mean is not zero. In financial series, it means that in large horizons there will be a positive (or negative if the mean was negative) gain.

Regarding the methodology proposed to apply modified R/S analysis introduced by Anis and Lloyd [1], the obtained results show better accuracy than classic R/S analysis, but, as it can be observed in Tables 4 and 5, GM2 gives better results, especially with short series.

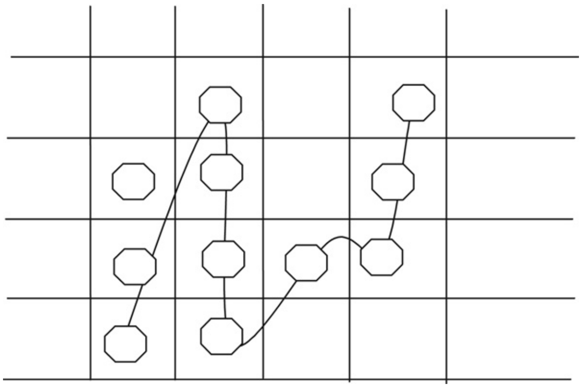

Fig. 3. Computation of H by GM2.

The empirical application shows that, once the influence of the average is deleted, there is no evidence of long memory in the analysed financial data, so we cannot conclude that the series are not random. If these results are accepted as valid, Indexes follows a random walk in which the mean of the returns can be different from 0.

Finally, we would like to remark that the main goal of this paper is just to indicate that results obtained for H using R/S analysis can lead to erroneous conclusions, as well as to provide simple tools for more consistent and accurate results in financial data.

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Appendix. Justification of GM2 method

Let $f : [0, 1] \to R$ be a Brownian sample function with Hurst exponent equal to H.

We will use Theorem 16.7 of [9], which can be stated as follows:

Theorem. *With probability* 1 *a Brownian sample function* $f : [0, 1] \to R$ *with Hurst exponent* H *has graph with Hausdorff and box dimension* $2 - H$.

Then $H = 2 - d$, where d is the fractal (box-counting or Hausdorff) dimension of the graph of the function.

On the other hand, by Section 3.1 of [9], d can be computed by

$$d = \lim_{h\to 0} \frac{\log N_h}{-\log h}$$

where h is the time increment, and N_h is the number of h-mesh cubes that intersect the graph of the function.

Let us take h of the form $\frac{1}{2^n}$. Let R_h be the mean of the range of the function in all intervals $[0, h], \dots, [kh, (k + 1)h], \dots, [1 - h, 1]$.

The Hurst exponent given by GM2 is

$$H' = \lim_{h\to 0} \frac{\log R_h}{\log h}.$$

Assuming that the function f is continuous, in each subinterval $[kh, (k + 1)h]$ there will be a mean of approximately $\frac{R_h}{h}$ cubes. Since there are $\frac{1}{h}$ subintervals, that makes a total of approximately $\frac{R_h}{h^2}$ cubes, and hence it follows that $R_h = N_h h^2$.

Then

$$H' = \lim_{h\to 0} \frac{\log R_h}{\log h} = \lim_{h\to 0} \frac{\log N_h + \log h^2}{\log h} = \lim_{h\to 0} \frac{\log N_h}{\log h} + 2 = 2 - d = H$$

and therefore GM2 computes the right Hurst exponent.

Let us see it with an example:

For the function in Fig. 3, $h = 0.25$, $N_h = 12$. In each interval there are a mean of 3 cubes (marked by a star), and there are $4 = 1/h$ subintervals. Note that R_h is approximately $3/4$, so it approximately follows that $R_h = N_h h^2$.

Fig. 3 shows an illustration of the example.

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