Graded Almost Noetherian Rings and Applications to Coalgebras

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Introduction

It is well-known that the dual algebra of a coalgebra C is a topological algebra with the weak-$*$ topology. In this paper we study some finiteness conditions relative to the topological structure of C^* in terms of the category $Rat({}_{C^*}\mathcal{M})$ of rational left C^*-modules. In particular, we focus on the problem whether $Rat({}_{C^*}\mathcal{M})$ is closed under extensions. In torsion theoretic terms this raises the question of deciding when $Rat({}_{C^*}\mathcal{M})$ is a torsion theory or a localizing subcategory in ${}_{C^*}\mathcal{M}$, the category of all left C^*-modules (the notion of localizing subcategory used here is as in [5], [19]). This problem has been previously treated in [9], [11], and [18], where a coalgebra satisfying this property is said to be a coalgebra having a torsion rat functor.

It was proved in [9, Lemma 2.3] (see also [12, Theorem 3.3]) that if C is right semiperfect, then C has a torsion rat functor. Determining when an arbitrary coalgebra C has a torsion rat functor seems to be a difficult problem. However, there are some classes of coalgebras where a characterization is possible. For C being almost connected, i.e., the coradical C_0 is finite dimensional (equivalently, $Rat({}_{C^*}\mathcal{M})$ has a finite number of types of simple objects), we prove that C has a torsion rat functor if and only if C^* is left almost noetherian, that is, every cofinite left ideal is finitely generated, see Theorem 2.8. This result may be deduced by combining Lemma 2.10 and [8, Theorem 2.2.6]. We offer here a different approach by linking the notion of almost noetherian to the concept of gr-almost noetherian for a graded algebra. Let $gr(C^*)$ be the graded algebra associated to C^* by considering the filtration on C^* induced by the coradical filtration on C. The idea is to lift the property of C^* being left almost noetherian from $gr(C^*)$ being left gr-almost noetherian. This is one of the motivations to consider gr-almost noetherian algebras. This kind of rings are studied in Section 1 in a general framework of graded rings but with an eye to the applications for coalgebras. The basic properties of graded almost noetherian rings are developed, observing that for suitable filtrations the property of being almost noetherian lifts (Proposition 1.6) from the associated graded ring to the completion of the filtered ring.

A sufficient condition for C to have a torsion rat functor is that every closed and cofinite left ideal of C^* is finitely generated, see [16, 2.5]. Such coalgebras are called $\mathcal{F}$-noetherian. In Section 2 we establish some results about $\mathcal{F}$-noetherian coalgebras. When C is a direct sum of almost connected coalgebras, C has a torsion rat functor if and only if C is $\mathcal{F}$-noetherian, cf. Theorem 2.8. Another class of $\mathcal{F}$-noetherian coalgebras consists of right semiperfect coalgebras, see Theorem 2.12. A necessary condition to have a torsion rat functor is given. We prove that if C has a torsion rat functor, then C is a locally finite coalgebra in the sense of [8]. The converse is not true in general, although it holds when C is a direct sum of almost connected coalgebras (in particular cocommutative), cf. Theorem 2.11. In this case being locally finite, being $\mathcal{F}$-noetherian, and having a torsion rat functor are equivalent notions for C.

In Section 3 we characterize when C^* is left noetherian. This is equivalent to C being almost connected and $Rat({}_{C^*}\mathcal{M})$ being a stable torsion theory in ${}_{C^*}\mathcal{M}$, see Theorem 3.2. This characterization is a consequence of the Artin-Rees property for the Jacobson radical of C^*. The latter is perhaps an esthetically satisfying characterization, blending the algebraic and topological ingredients of the theory in categorical (torsion theoretic) phrasing. In this case we show that the homological dimension of C may be computed from the homological dimension of C^*, Proposition 3.4.

1 Graded almost noetherian k-algebras

In the sequel k will denote a field and all vector spaces, algebras, coalgebras, etc, unless otherwise stated, will be over k. This section is devoted to the study of graded almost noetherian k-algebras. Although many of the results of this section are meant to be applied in the coalgebra setting they may have independent interest.

Let R be a k-algebra. A left R-module M is called *almost noetherian* if every cofinite submodule of M is finitely generated. Submodules, quotients, and extensions of almost noetherian modules are almost noetherian. The algebra R is *left almost* *noetherian* if its almost noetherian as a left module. We call R *almost noetherian* if it is left and right almost noetherian. See [8, Section 1.1] for further details on almost noetherian modules and algebras.

Let $R = \oplus_{n\geq 0} R_n$ be a positively graded k-algebra. A left graded R-module $M = \oplus_{n\geq 0} M_n$ is said to be *gr-almost noetherian* if every cofinite graded submodule of M is finitely generated. If ${}_RR$ is gr-almost noetherian, then R is called *left gr-almost noetherian*.

Theorem 1.1 *M is gr-almost noetherian if and only if M is almost noetherian.*

Proof: Let N be a cofinite submodule of M. We may take a finite dimensional vector space V such that $N + V = M$. There exists $r_0 \in I\!N$ such that $V \subseteq M_0 \oplus M_1 \oplus ... \oplus M_{r_0}$. Set $L = M_1 \oplus ... \oplus M_{r_0}$. Any element $n \in N$ may be written as $n = n_{k_1} + ... + n_{k_s}$ where n_{k_i} is a non zero element in N_{k_i} and $k_1 < ... < k_s$. Let $\bar{N}$ denote the submodule of N generated by n_{k_s} for all $n \in N$, i.e., $\bar{N}$ is generated by the homogeneous components of highest degree appearing in elements of N. From [14, page 83], $\bar{N}$ is a graded submodule of M. We check that $\bar{N} + L = M$. Let $m \in M$ with $deg(m) = t > r_0$. We write $m = x + y$ with $x \in N, y \in V$. Assume that x_{max} is the component of highest degree of x. Since $y \in L$, $m = x_{max}$ and so $m \in \bar{N}$.

On the other hand, $M/\bar{N} \cong L/(\bar{N} \cap L)$. Since $N \cap M_i \subset \bar{N}$ for any $i \geq 0$, we have $\oplus_{i=0}^{r_0}(N \cap M_i) \subseteq \bar{N} \cap L$. There is an epimorphism

$$\oplus_{i=0}^{r_0}(M_i/N \cap M_i) = L/(\oplus_{i=0}^{n_0}(N \cap M_i)) \to L/\bar{N} \cap L.$$

Each $M_i/(N \cap M_i)$ is finite dimensional because it is a submodule of M/N. Hence $\bar{N}$ is cofinite in M. By the hypothesis, it is finitely generated. Set $\bar{N} = Rx_1 + ... + Rx_q$ where $x_1, ..., x_q$ are homogeneous. We may find elements $y_i \in N$ such that x_i is the highest component of y_i. Setting $P = Ry_1 + ... + Ry_n$, we have that $P \subseteq N$ and clearly $\bar{P} = \bar{N}$. By [14, Corollary II.2.4, page 85], $P = N$ and thus N is finitely generated. The other implication is obvious. ∎

Proposition 1.2 *If M is gr-almost noetherian, then M_0 is a left almost noetherian R_0-module. In particular, if R is left gr-almost noetherian, then R_0 is left almost noetherian.*

Proof: Let N_0 be a cofinite R_0-submodule of M_0. The space $N = N_0 \oplus (\sum_{i\geq 1} M_i)$ is a cofinite graded R-submodule of M because $M/N \cong M_0/N_0$. By the hypothesis, N is finitely generated as R-module. Let $N = Rx_1 + ... + Rx_t$ where $x_1, ..., x_t$ are homogeneous elements. There exists $x_1, ..., x_p$ ($p \leq t$) such that $N_0 = R_0x_1 + ... + R_0x_p$ (R is positively graded). Hence N_0 is finitely generated as R_0-module. ∎

Corollary 1.3 *Let $R = \oplus_{n\geq 0} R_n$ be a positively graded algebra such that $R_n R_1 = R_{n+1}$ for any $n \geq 0$. Assume that R_0 is left almost noetherian and R_1 is finitely generated as R_0-module. Then R is left almost noetherian.*

Proof: From the hypothesis, $R_n = R_1^n$ for any $n \geq 0$. Let $I = \oplus_{i\geq 0} I_i$ be a cofinite left graded ideal of R. There exists $n_0 \in I\!\!N$ such that $I_n = R_n$ for all $n > n_0$. As $R/I = \oplus_{i=0}^{n_0}(R_i/I_i)$, I_i is cofinite in R_i. For any $n \geq 2$, $R_n = R_1^n$ is almost noetherian since R_1 is finitely generated and R_0 is left almost noetherian. Then I_i is finitely generated as a left R_0-module. From $RR_{n_0+1} = R_{n_0+1} \oplus ... \oplus R_n \oplus ...$, we get that I is finitely generated as a left R-module. Therefore R is gr-almost noetherian and by Theorem 1.1, R is left almost noetherian. ∎

Corollary 1.4 *Let $R = R_0\{x_1, ..., x_n\}$ be the free algebra on n variables with R_0 being left almost noetherian. Then R is left almost noetherian. In particular, a finitely generated algebra is left and right almost noetherian.*

Proof: Just consider R with the natural grading and apply the foregoing result. Note that quotients of left almost noetherian algebras are such. ∎

For an ideal I of an algebra R, the Rees ring associated to I is the graded subring of $R[x]$ defined as $R(I) = \oplus_{n\geq 0} I^n x^n$ where $I^0 = R$.

Corollary 1.5 *If R is left almost noetherian and I is a finitely generated left ideal, then the Rees ring $R(I)$ is left almost noetherian.*

Proof: Straightforward from Corollary 1.3. ∎

Let R be a filtered k-algebra, i.e., it contains a descending chain of k-subspaces

$$R = F_0R \supset F_1R \supset ... \supset F_nR \supset ....$$

such that $(F_nR)(F_mR) \subseteq F_{n+m}R$ for any $n, m \geq 0$. One may associate to this filtration the completion:

$$\hat{R} = \lim_{\substack{\longleftarrow \\ i\geq 0}} R/F_iR,$$

and the graded k-algebra $G(R) = \oplus_{i\geq 0} F_iR/F_{i+1}R$, (see [14, Chapter D]).

Proposition 1.6 *If $G(R)$ is left almost noetherian, then $\hat{R}$ is left almost noetherian.*

Proof: For any $p \geq 0$ we define

$$F_p\hat{R} = \varinjlim_{n \geq p} F_pR/F_nR.$$

The family $\{F_i\hat{R}\}_{i\geq 0}$ is a filtration on $\hat{R}$ and $\hat{R}$ is complete respect to this filtration. Moreover, $G(\hat{R}) \cong G(R)$ as graded rings. These facts allow us to assume that R is complete. Let $R - \mathcal{F}ilt$ denote the category of left filtered R-modules. Let I be a cofinite left ideal of R, and consider the exact sequence $0 \to I \to R \to R/I \to 0$. This sequence is strict exact in $R - \mathcal{F}ilt$ when considering on I and R/I the induced filtrations $\{I \cap F_iR\}_{i\geq 0}$ and $\{(I + F_iR)/I\}_{i\geq 0}$ respectively. From [14, Chapter D] we have that the following sequence is exact in $G(R) - gr$;

$$0 \to G(I) \to G(R) \to G(R/I) \to 0.$$

Since I is cofinite in R, there is an $i_0 \in \mathbb{N}$ such that $I + F_iR = I + F_{i+1}R$ for any $i \geq i_0$. Thus $G(R/I)_i = \{0\}$ for every $i \geq i_0$. On the other hand, for any $i \geq i_0$, $F_i(R/I)/F_{i+1}(R/I) \cong (I + F_iR)/(I + F_{i+1}R) = \{0\}$. Hence $G(R/I)$ is of finite dimension and so $G(I)$ is cofinite in $G(R)$. By hypothesis, $G(I)$ is finitely generated as a left $G(R)$-module. The completeness of R combined with [14, Proposition IV.3, Chapter D] yield that I is finitely generated as a left R-module. ∎

Corollary 1.7 *Let R be a k-algebra and I a two sided ideal such that I is finitely generated as left ideal and R/I is left almost noetherian. Consider on R the I-adic filtration. Then $\hat{R}$ and $G(R)$ are left almost noetherian.*

Proof: Follows from Corollary 1.3 and Proposition 1.6. ∎

2 Coalgebras having a torsion rat functor

For general facts on coalgebras and comodules we refer to [1], [4], or [20]. For a coalgebra C its dual algebra C^* is a topological vector space with the weak-* topology. The closed subspaces of C^* are the annihilators $W^{\perp(C^*)}$ of subspaces W of C. The closure of a subspace U of C^* in this topology, denoted by $\bar{U}$, is $U^{\perp(C)\perp(C^*)}$. Finitely generated left (or right) ideals of C^* are closed, [8, Proposition 1.3.1 b)].

Let ${}_{C^*}\mathcal{M}$ denote the category of left C^*-modules. It is well-known that the category of right C-comodules $\mathcal{M}^C$ is isomorphic to $Rat({}_{C^*}\mathcal{M})$, the subcategory of ${}_{C^*}\mathcal{M}$

consisting of all rational left C^*-modules. $Rat({}_{C^*}\mathcal{M})$ is closed under submodules, quotients and arbitrary direct sums. In the sense of [19], it is an hereditary pretorsion class in ${}_{C^*}\mathcal{M}$. See loc. cit. for further detail on torsion classes. Recall from [19, Proposition 4.2] that there is a bijective correspondence between hereditary pretorsion classes (resp. torsion classes), left exact preradicals (resp. radicals) and left linear topologies (resp. Gabriel filters). The left exact preradical associated to $Rat({}_{C^*}\mathcal{M})$ is the *rational functor* $Rat_C(-) : {}_{C^*}\mathcal{M} \to {}_{C^*}\mathcal{M}$. Given $M \in {}_{C^*}\mathcal{M}$, $Rat_C(M)$ is the sum of all rational modules contained in M. The left linear topology $\mathcal{F}_C$ on C^* corresponding to $Rat({}_{C^*}\mathcal{M})$ is the family of all closed (in the weak-* topology) and cofinite left ideals of C^*. If $I \in \mathcal{F}_C$ there is a finite dimensional left coideal W of C such that $I = W^{\perp(C^*)}$. From the Fundamental Theorem of coalgebras is derived that $\mathcal{F}_C$ is a *symmetric topology*. This means that it contains a basis of two-sided ideals, i.e., for every $I \in \mathcal{F}_C$ there is a two-sided ideal J of C^* such that $J \subseteq I$ and $J \in \mathcal{F}_C$. The category

$$Rat({}_{C^*}\mathcal{M}) = \{M \in {}_{C^*}\mathcal{M} : Ann_{C^*}(m) \in \mathcal{F}_C \ \forall m \in M\}.$$

In this section we study when $Rat({}_{C^*}\mathcal{M})$ is a torsion theory (also called *localizing subcategory*) in ${}_{C^*}\mathcal{M}$, that is, when $Rat({}_{C^*}\mathcal{M})$ is closed under extensions. Equivalently, $\mathcal{F}_C$ is a Gabriel filter, or $Rat_C(-)$ is a radical, i.e., $Rat(M/Rat(M)) = \{0\}$ for all $M \in {}_{C^*}\mathcal{M}$.

Definition 2.1 *A coalgebra C is said to have a torsion rat functor if $Rat_C(-)$ is a radical.*

Some results concerning coalgebras with a torsion rat functor are contained in [9], [11], [18], and [16]. Let us just recollect some properties:

Remark 2.2 *i) If C is finite dimensional, then C has a torsion rat functor.*

ii) If C has a torsion rat functor, then every subcoalgebra also has that property.

iii) The direct sum of a family of coalgebras has a torsion rat functor if and only if each term has it.

As a consequence, a direct sum of finite dimensional coalgebras has a torsion rat functor. In particular, *cosemisimple coalgebras have a torsion rat functor.* The following sufficient condition to have a torsion rat functor was given in [16, 2.5, page 521].

Proposition 2.3 *If every left ideal in $\mathcal{F}_C$ is finitely generated, then C has a torsion rat functor.*

Definition 2.4 *A coalgebra C satisfying the hypothesis of Proposition 2.3 is called $\mathcal{F}$-noetherian.*

Recall from [8] that a coalgebra C is called *left strongly reflexive* if C^* is left almost noetherian. A left strongly coreflexive coalgebra is clearly $\mathcal{F}$-noetherian. By [8, Theorem 3.3] the finite dual of a left almost noetherian algebra is left strongly coreflexive. Then, *the finite dual of a left almost noetherian algebra has a torsion rat functor.* In particular, since a finitely generated algebra A is almost noetherian (Corollary 1.4), A^o has a torsion rat functor.

It is known that C being left strongly coreflexive implies that C is coreflexive, that is, the canonical embedding $\lambda_C : C \to C^{*o}$ is surjective. The notion of $\mathcal{F}$-noetherian is just the additional hypothesis needed to have a converse.

Proposition 2.5 *The following assertions about a coalgebra C are equivalent:*

i) C is left strongly coreflexive.

ii) C is coreflexive and $\mathcal{F}$-noetherian.

iii) C_0 is coreflexive and C is $\mathcal{F}$-noetherian.

Proof: *$i \Rightarrow ii) \Rightarrow iii)$* Obvious.

$iii) \Rightarrow i)$ By [8, 1.1.8], it suffices to show that every cofinite maximal two-sided ideal is finitely generated as left ideal. Let I be such an ideal. Since C_0 is coreflexive, in light of [8, Proposition 3.5.3], I is closed. By hypothesis, I is finitely generated as a left ideal. ∎

The following result establishes some basic properties of $\mathcal{F}$-noetherian coalgebras.

Proposition 2.6 *i)Subcoalgebras of $\mathcal{F}$-noetherian coalgebras are $\mathcal{F}$-noetherian.*

ii) Let $\{C_i\}_{i\in I}$ be a family of coalgebras and $C = \oplus_{i\in I} C_i$. Then, C is $\mathcal{F}$-noetherian if and only if C_i is $\mathcal{F}$-noetherian for all $i \in I$.

Proof: *i)* Let C be a $\mathcal{F}$-noetherian coalgebra and let D be a subcoalgebra. The inclusion map $i : D \to C$ induces a projection $i^* : C^* \to D^*$. Given $I \in \mathcal{F}_D$, the left ideal $i^{*-1}(I) = i(I^{\perp(C)})^{\perp(C^*)}$ is closed and cofinite. By hypothesis, $i^{*-1}(I)$ is finitely generated, and from $I = i^* i^{*-1}(I)$ it follows that I is finitely generated as a left ideal.

ii) See [18, Corollary 4.9]. If C is $\mathcal{F}$-noetherian, then it follows from i) that C_i is $\mathcal{F}$-noetherian for all $i \in I$.

Conversely, let I be a closed cofinite left ideal of C^* and W a finite dimensional subspace of C such that $I = W^{\perp(C^*)}$. There are finitely many indexes $i_1, ..., i_n$ such that $W \subset C_{i_1} \oplus ... \oplus C_{i_n}$. Let $j : C_{i_1} \oplus ... \oplus C_{i_n} \to C$ be the inclusion map. Then we have an exact sequence,

$$0 \longrightarrow \prod_{i \neq i_1,...,i_n} C_i^* \longrightarrow C^* \xrightarrow{j^*} (\oplus_{j=1}^n C_{i_j})^* \longrightarrow 0.$$

Since, the image $i^*(I)$ and $\prod_{i \neq i_1,...,i_n} C_i^*$ are finitely generated as C^*-modules, I is finitely generated. ∎

A coalgebra C is called *almost connected* if its coradical C_0 is finite dimensional. As an application of the results obtained in the first section, we next characterize almost connected coalgebras having a torsion rat functor. We first need to recall several facts on graded coalgebras.

For any coalgebra C let $\{C_n\}_{n \in \mathbb{N}}$ its coradical filtration. It is well-known that $C = \cup_{n \in \mathbb{N}} C_n$ and $\Delta(C_n) \subset \sum_{i=0}^n C_i \otimes C_{n-i}$. We may consider the graded coalgebra associated to C,

$$gr(C) = C_0 \oplus (C_1/C_0) \oplus ... \oplus (C_{n+1}/C_n) \oplus ...$$

Recall that a coalgebra D is *graded* if $D = \oplus_{n \in \mathbb{N}} D_{(n)}$ where $\{D_{(n)}\}_{n \geq 0}$ is a family of subspaces of D verifying that $\Delta(D_{(n)}) \subseteq \sum_{i+j=n} D_{(i)} \otimes D_{(j)}$ and $\varepsilon(D_{(n)}) = \{0\}$ for all $n \geq 0$. Using the graded dual of D, we may associate a graded ring $R = \oplus_{n \geq 0} R_n$ to D where $R_n = \{f \in D^* : f(D_i) = 0,\ i \neq n\}$. Clearly R is a subring of D^* and $R_n \cong (\sum_{i \neq n} D_i)^{\perp(D^*)} \cong D_{(n)}^*$.

The coradical filtration on C induces a filtration on C^*,

$$C^* \supset C_0^{\perp(C^*)} \supset C_1^{\perp(C^*)} \supset ... \supset .... \qquad (1)$$

The two-sided ideal $J = C_0^{\perp(C^*)}$ is the Jacobson radical of C^*.

Proposition 2.7 *Let C be a coalgebra and R the graded dual ring of $gr(C)$. Denote by $gr(C^*)$ the graded ring associated to the filtration (1). Then,*

i) $gr(C^*) \cong R$ as graded rings.

ii) C^* is complete with respect to (1).

Proof: *i)* This may easily be checked.

ii) Since $C = \varinjlim_{n \geq 0} C_n$,

$$C^* = Hom_k(C, k) \cong \varprojlim_{n \geq 0} C_n^* \cong \varprojlim_{n \geq 0} C^*/C_n^{\perp(C^*)}. \qquad ∎$$

Theorem 2.8 *Let C be an almost connected coalgebra. The following assertions are equivalent.*

i) C has a torsion rat functor.

ii) J is finitely generated as a left (or right) ideal.

iii) C_1 is finite dimensional.

iv) Each term C_n of the coradical filtration is finite dimensional.

v) C^* is left (or right) almost noetherian.

vi) R is left (or right) almost noetherian.

vii) $gr(C)$ has a torsion rat functor.

Proof: *i) $\Rightarrow$ iv)* By hypothesis, $J \in \mathcal{F}_C$. Then C_0^*/J is a rational C^*-module. Since $Rat({}_{C^*}\mathcal{M})$ is closed under extensions, we obtain that C^*/J^n is also a rational C^*-module. Hence $J^n \in \mathcal{F}_C$ for all $n \in I\!N$. But $C_n^{\perp(C^*)} = \overline{J^{n+1}} = J^{n+1}$ and thus $C^*/J^{n+1} \cong C_n^*$ is finite dimensional for all $n \in I\!N$.

iv) $\Rightarrow$ iii) Obvious.

iii) $\Rightarrow$ ii) From $C_1^{\perp(C^*)} = \overline{J^2}$ we obtain that $C^*/\overline{J^2} \cong C_1^*$. Since C_1 is finite dimensional, it follows that $J/\overline{J^2}$ is also finite dimensional. Thus we can find a finite dimensional vector space V such that $J = \overline{J^2} + V$. This yields $J = \overline{J^2 + V}$, that is, $J^2 + V$ is dense in J. By [17, Lemma 2.2.14], $C^*V = J$ and thus J is finitely generated as a left ideal.

ii) $\Rightarrow$ vi) From the hypothesis J^n is finitely generated for all $n \geq 1$. Consequently J^n is closed. Then $C_n^{\perp(C^*)} = \overline{J^{n+1}} = J^{n+1}$. Hence

$$gr(C^*) \cong C^*/J \oplus J/J^2 \oplus ....$$

Proposition 2.7 i) yields $R \cong gr(C^*)$, and Corollary 1.3 implies that R is almost noetherian.

vi) $\Rightarrow$ v) Follows from Proposition 2.7 ii) and Proposition 1.6.

v) $\Rightarrow$ i) Follows from Proposition 2.3.

vi) $\Rightarrow$ vii) Let $D = gr(C)$ and let R be its graded dual ring. Then $D = \oplus_{n \geq 0} D_{(n)}$ where $D_{(n)} = C_n/C_{n-1}$. This graduation induces a filtration on D,

$$D_{(0)} \subset D_{(0)} \oplus D_{(1)} \subset .... \subset D_{(0)} \oplus D_{(1)} \oplus .... \oplus D_{(n)} \subset ...$$

which in turn yields a filtration on D^* by letting $F_n D^* = (D_{(0)} \oplus D_{(1)} .... D_{(n)})^{\perp(D^*)}$ for $n \geq 1$ and $F_0 D^* = D^*$. From $D = \cup_{n \geq 0} (D_{(0)} \oplus D_{(1)} .... D_{(n)})$ we get

$$D^* \cong \varprojlim_{n \geq 0} D^*/F_n D^*,$$

and thus D^* is complete with respect to the filtration $\{F_nD^*\}_{n\geq 0}$. Let $gr(D^*)$ be the associated graded algebra. Since $F_nD^*/F_{n+1}D^* \cong D_n^* \cong (C_n/C_{n-1})^*$, $gr(D^*) \cong R$. Proposition 1.6 implies that D^* is left almost noetherian. By Proposition 2.3, D has a torsion rat functor.

vii) $\Rightarrow$ *iii)* With notation as above, note that D is almost connected. Then $J = D_0^{\perp(D^*)}$ is cofinite. Since D has a torsion rat functor, J^2 is cofinite. But $J^2 = (D_0^{\perp(D^*)})^2 \subseteq (D_0 \oplus D_1)^{\perp(D^*)}$. Hence $(D_0 \oplus D_1)^{\perp(D^*)}$ is also cofinite. This implies that $D_0^{\perp(D^*)}/(D_0 \oplus D_1)^{\perp(D^*)} \cong D_1^*$ is finite dimensional, and from the equality $D_1 = C_1/C_0$, it follows that C_1 is finite dimensional. ∎

Remark 2.9 i) Part of this result, with a different formulation, was proved in [8, Theorem 4.6]. The alternative proof offered here is an application of the graded techniques developed in the first section.

ii) The characterization *i)* $\Leftrightarrow$ *iii)* $\Leftrightarrow$ *iv)* $\Leftrightarrow$ *vi)* appears in [18, Theorem 4.6].

We provide a necessary condition for a coalgebra to have a torsion rat functor. We recall from [8] that a coalgebra C is called *locally finite* if for any two finite dimensional subspaces U, V of C, the wedge $U \wedge V$ is finite dimensional. Equivalently, for any subcoalgebra D, $D \wedge D$ is finite dimensional, [8, 2.2].

Lemma 2.10 *Let C be a coalgebra. The following assertions are equivalent:*

i) $\mathcal{F}_C$ is closed under products.

ii) C is locally finite and any two finite dimensional left coideals U, V verify that $(U \wedge V)^{\perp(C^*)} = U^{\perp(C^*)}V^{\perp(C^*)}$.

Hence any coalgebra having a torsion rat functor satisfies ii).

Proof: Let D be a finite dimensional subcoalgebra of C. By hypothesis, there is a finite dimensional subspace W of C such that $D^{\perp(C^*)}D^{\perp(C^*)} = W^{\perp(C^*)}$. Now $D \wedge D = (D^{\perp(C^*)}D^{\perp(C^*)})^{\perp(C)} = W^{\perp(C^*)\perp(C)} = W$. Hence $D \wedge D$ is of finite dimension. A similar argument shows that $(U \wedge V)^{\perp(C^*)} = U^{\perp(C^*)}V^{\perp(C^*)}$ for any finite dimensional left coideals U, V. Conversely, suppose now that $I, J \in \mathcal{F}_C$, then $I = U^{\perp(C^*)}$, $J = V^{\perp(C^*)}$ for some finite dimensional left coideals U, V of C. The product $IJ = U^{\perp(C^*)}V^{\perp(C^*)} = (U \wedge V)^{\perp(C^*)}$ and $U \wedge V$ is finite dimensional since C is locally finite. Hence $IJ \in \mathcal{F}_C$.

The other assertion follows from the fact that a Gabriel filter is closed under products, [19, Lemma 5.3]. ∎

It is not true in general that a locally finite coalgebra has a torsion rat functor. An example of locally finite coalgebra C such that $\mathcal{F}_C$ is not closed under products is given in [17, Example 3.4]. However there are cases where both notions are equivalent.

Theorem 2.11 *Let $\{C_i\}_{i\in I}$ be a family of almost connected coalgebras and $C = \oplus_{i\in I} C_i$. The following assertions are equivalent:*

i) C has a torsion rat functor.

ii) C is locally finite.

iii) C is $\mathcal{F}$-noetherian.

In particular, this characterization holds for cocommutative coalgebras.

Proof: *i) $\Rightarrow$ ii)* and *iii) $\Rightarrow$ i)* are known.

ii) $\Rightarrow$ iii) Since being locally finite is an hereditary property ([8, 2.3.2]), every C_i is locally finite. Then, the terms of the coradical filtration of C_i are finite dimensional. By Theorem 2.8, C_i is $\mathcal{F}$-noetherian for all $i \in I$. Proposition 2.6 ii) now applies. ∎

Semiperfect coalgebras are another kind of coalgebras having a torsion rat functor. We recall from [9] that a coalgebra C is *right semiperfect* if for each simple right comodule S the injective hull $E(S)$ in $\mathcal{M}^C$ is finite dimensional. It was shown in [9, Theorem 23] that any right semiperfect coalgebra has a torsion rat functor. Indeed, it was proved in [12, Theorem 3.3] that a coalgebra is right semiperfect if and only if the rational functor $Rat(-) : {}_{C^*}\mathcal{M} \to {}_{C^*}\mathcal{M}$ is exact. If it is exact, then it is certainly a radical.

For semiperfect coalgebras we also get the finiteness conditions on the Gabriel filter. This, combined with Proposition 2.3, provides an alternative proof to [9, Theorem 23]. In fact, semiperfect coalgebras may be characterized by a finiteness condition on the Gabriel filter.

Theorem 2.12 *The following assertions about a coalgebra C are equivalent:*

i) C is right semiperfect.

ii) $\mathcal{F}_C$ has a basis of principal left ideals generated by an idempotent.

If C is right semiperfect, then C is $\mathcal{F}$-noetherian and locally finite. Consequently, C has a torsion rat functor.

Proof: *i) $\Rightarrow$ ii)* Let $I \in \mathcal{F}_C$ and I' be a two-sided ideal such that $I' \subseteq I$ and $I' \in \mathcal{F}_C$. Let D be the finite dimensional subcoalgebra such that $I' = D^{\perp(C^*)}$. Now $D \subseteq E(D) = E(D_0)$ and D_0 is the direct sum of a finite number of simple right D-comodules, [5, 1.3b]. By hypothesis $E(D_0)$ is finite dimensional. From [5, 1.5f], $E(D_0)$ is a direct summand of C as right C-comodules, then $E(D_0) = eC$ for some idempotent $e \in C^*$, [3, Proposition 1.12]. The left ideal $K = E(D_0)^{\perp(C^*)}$ is closed and cofinite and $K = (eC)^{\perp(C^*)} = C^*e'$ where $e' + e = \varepsilon$.

ii) ⇒ i) Let S be a right simple C-comodule and D be a simple subcoalgebra such that S is a D-comodule. We may take S to be a simple right coideal of C, [5, 1.3b]. Since $E(S) \subseteq E(D)$, it suffices to show that $E(D)$ is finite dimensional. The proof will be complete once we prove that D is contained in a finite dimensional injective comodule. Let $I = D^{\perp(C^*)}$. By hypothesis, we may find an idempotent $e \in C^*$ such that $C^*e \subseteq I$ and $C^*e \in \mathcal{F}_C$. We have that $D = I^{\perp(C)} \subseteq (C^*e)^{\perp(C)} = eC$ and eC is finite dimensional. Since eC is a direct summand of C as a right C-comodule, eC is injective. ∎

Remark 2.13 In view of Theorems 2.8, 2.11, and 2.12, one could conjecture that a coalgebra C has a torsion rat functor if and only if C is $\mathcal{F}$-noetherian.

We show that the property of having a torsion rat functor is invariant under so-called strong equivalences (see [10] for details on strong equivalences). Assume that C and D are strongly equivalent coalgebras. By [10, Theorem 5], there is an equivalence $F : {}_{C^*}\mathcal{M} \rightleftarrows {}_{D^*}\mathcal{M} : G$ such that $F(Rat({}_{C^*}\mathcal{M})) \subseteq Rat({}_{D^*}\mathcal{M})$ and $G(Rat({}_{D^*}\mathcal{M})) \subseteq Rat({}_{C^*}\mathcal{M})$. It is clear that $Rat({}_{C^*}\mathcal{M})$ is closed under extensions if and only if $Rat({}_{D^*}\mathcal{M})$ is so. Then, C has a torsion rat functor if and only if D has too. In particular, if C has a torsion rat functor, then the comatrix coalgebra of order n over C, $M^c(n, C)$, has too.

3 The noetherian case. The Artin-Rees property

In this section we characterize when the dual algebra of a coalgebra is noetherian in terms of the stability of $Rat({}_{C^*}\mathcal{M})$.

Proposition 3.1 *Assume that C^* is a left noetherian algebra. Then $J = C_0^{\perp(C^*)} = Rad(C^*)$ has the Artin-Rees property for left ideals, that is, for every left ideal K of C^* and any $n \in I\!N$, there is $h(n) \in I\!N$ such that $J^{h(n)} \cap K \subseteq J^n K$.*

Proof: By hypothesis, J^n is finitely generated as a left ideal. Since K is left finitely generated, $J^n K$ is finitely generated as a left ideal. Hence it is closed in C^*. It is not difficult to see that C is almost connected because C^* is left noetherian. Theorem 2.8 yields that J^n is cofinite and thus $J^n K$ is cofinite in K (see [8, Lemma 1.1.1]). As $J^n K$ is closed, $J^n K = \cap_{s=1}^{\infty}(J^n K + J^s)$. Then,

$$J^n K = J^n K \cap K = \cap_{s=1}^{\infty}(J^n K + J^s) \cap K = \cap_{s=1}^{\infty}(J^n K + J^s \cap K).$$

This implies $\cap_{s=1}^{\infty}(J^nK + J^s \cap K)/J^nK = \{0\}$ in K/J^nK. Since K/J^nK is finite dimensional, there is an $h(n) \in \mathbb{N}$ such that $(J^nK + (J^{h(n)} \cap K))/J^nK = \{0\}$. Therefore $J^{h(n)} \cap K \subseteq J^nK$. ∎

Recall that $Rat({}_{C^*}\mathcal{M})$ is said to be *stable* if it is closed under injective hulls. Recall also that an injective left A-module M is called Σ-*injective* if every direct sum $M^{(\Gamma)}$ is injective for any non-empty set Γ.

Theorem 3.2 *The following assertions on a coalgebra C are equivalent:*

i) C^* is left noetherian.

ii) C is almost connected and $Rat({}_{C^*}\mathcal{M})$ is a stable localizing subcategory.

iii) C is almost connected and $Rat({}_{C^*}\mathcal{M})$ is closed under injective hulls.

Proof: *i) ⇒ ii)* We already observed that C is almost connected when C^* is left noetherian. That $Rat({}_{C^*}\mathcal{M})$ is a localizing subcategory follows from Theorem 2.8. It was proved there that $\mathcal{F}_C$ is generated by the powers J^n, $n \geq 1$. Proposition 3.1 combined with [13, Proposition 8.5.3] yields that $Rat({}_{C^*}\mathcal{M})$ is stable under injective hulls.

ii) ⇒ iii) Obvious.

iii) ⇒ i) For any non empty set Γ, ${}_{C^*}C^{(\Gamma)}$ is injective in $Rat({}_{C^*}\mathcal{M})$. By hypothesis, ${}_{C^*}C^{(\Gamma)}$ is injective in ${}_{C^*}\mathcal{M}$. Therefore ${}_{C^*}C$ is Σ-injective. As C_0 is finite dimensional and $C^*/C_0^{\perp(C^*)} \cong C_0^*$, we obtain that $C_0^* \in Rat({}_{C^*}\mathcal{M})$. The equality $J = C_0^{\perp(C^*)}$ yields that every simple C^*-module is rational. It is known that C contains all simple rational C^*-modules. Hence ${}_{C^*}C$ is a Σ-injective cogenerator in ${}_{C^*}\mathcal{M}$. Thus C^* is a left noetherian ring. ∎

Theorem 3.3 *Let C be an almost connected cocommutative coalgebra and $\{C_n\}_{n\in\mathbb{N}}$ its coradical filtration. Then:*

i) C is almost noetherian if and only if C^* is noetherian.

ii) The formal series in $k[[x]]$,

$$P_C(x) = dim(C_0) + \sum_{n=0}^{\infty}(dim(C_n)/dim(C_{n-1}))x^n,$$

is a rational function, i.e., a quotient of two polynomials.

iii) For a large enough n, $dim(C_n)$ is a polynomial function of n with degree at most s, where $s = dim(C_1/C_0)$.

Proof: *i)* It is clear that if C^* is noetherian, then C is almost noetherian. By the proof of Theorem 2.8,

$$gr(C^*) \cong C^*/J \oplus J/J^2 \oplus ...$$

Since C^*/J and J/J^2 are finite dimensional, $gr(C^*)$ is a commutative noetherian algebra. On the other hand, as C^* is complete in the J-adic topology, it follows that C^* is noetherian, see [14].

ii) and *iii)* follow by applying Proposition 2.7 and [2, Theorems 11.1 and 11.4]. ∎

We finish this paper by showing that the global dimension of a coalgebra C coincides with the left global dimension of C^* whenever the latter is left noetherian. Given $M \in \mathcal{M}^C$, the *injective dimension of* M, denoted by $inj.dim.(M)$, may be defined as the minimal length of the injective resolution

$$0 \to M \to Q_0 \to Q_1 \to ... \to Q_n \to ...$$

where Q_i are injective objects in $\mathcal{M}^C$ for all $i \geq 0$. The right global dimension of C is defined as $r.gl.dim(C) = sup\{inj.dim(M) : M \in \mathcal{M}^C\}$. The left global dimension, l.gl.dim(C), may be similarly defined. It was proved in [15] that both dimensions are equal. We will simply write $gl.dim(C)$.

Proposition 3.4 *Assume that C^* is left noetherian. Then $gl.dim(C) = l.gl.dim(C^*)$.*

Proof: From [13, Corollary 8.3.24], $l.gl.dim(C^*) = inj.dim(C^*/J)$. Since C is almost connected, $C^*/J \in Rat({}_{C^*}\mathcal{M})$. Theorem 3.2 now applies. ∎

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