When Does the Rational Submodule Split Off?

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Abstract

A coalgebra C is said to have the splitting property if the maximal rational submodule $Rat(M)$ of any left C^*-module M is a direct summand of it. In this paper we prove that a coalgebra C satisfying this property is finite dimensional. Cocommutative coalgebras such that $Rat(M)$ is a direct summand for any finitely generated left C^*-module M are explicitly described.

Introduction

One of the most interesting problems in Torsion Theory is the sppliting problem, that may be formulated as follows: given a torsion theory $(\mathcal{T}, \mathcal{F})$ in $R{-}Mod$ and a certain class $\mathcal{C} \subseteq R{-}Mod$, when is the torsion submodule $\tau(M)$ a direct summand of M for all $M \in \mathcal{C}$? This problem has its roots in the splitting of the torsion submodule of any finitely generated module over a Dedekind domain, see [10], [11], [13]. For general rings and torsion theories the splitting problem has captivated the attention of many mathematicians and an extensive literature on it exists. An intense effort was dedicated to the splitting problem for the Goldie torsion theory (see [5] and [21]) and the simple torsion theory (see [17], [18], [19] and [20]).

In Coalgebra and Comodule Theory an important example of torsion theory appears when considering the notion of rational module. Given a

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coalgebra C it has been well-known through the years that the subcategory $Rat(C^*{-}Mod)$ of $C^*{-}Mod$ consisting of rational left modules over the dual algebra C^* is closed under subobjects, quotients, and direct sums. In terms of Torsion Theory ([15]), $Rat(C^*{-}Mod)$ is an hereditary pretorsion class in $C^*{-}Mod$. The first references where $Rat(C^*{-}Mod)$ is treated from a torsion theoretic point of view are [9], [14], [8] and [23]. In these works the problem of when $Rat(C^*-Mod)$ is closed under extensions (i.e. $Rat(C^*-Mod)$ is an hereditary torsion class) is discussed. In [2] we studied when the linear topology on C^* associated to $Rat(C^*{-}Mod)$ is of finite type, i.e., contains a basis of finitely generated ideals, and related this property to some finiteness conditions on C. This study was continued in [4] where new examples of coalgebras enjoying this property were constructed. Some partial results about the stability of $Rat(C^*{-}Mod)$ were given in [2].

Recently, Năstăsescu and Torrecillas studied in [12] for which coalgebras C the maximal rational submodule $Rat(M)$ is a direct summand of every left C^*-module M. These coalgebras are said to have the splitting property. They proved that if C has the splitting property, then C is necessarily finite dimensional. In proving this result they use numerous results of different nature. In a first step, using a result of Sandomierski on the endomorphism ring of an injective module, C^* is shown to be left noetherian. In a second step, invoking a theorem of Teply on the splitting of the simple torsion theory, the result is proved for colocal coalgebras. In particular, it holds for the localization of a coalgebra in a primitive idempotent, see [22] and [6]. Finally, the general case is proved using the previous steps and the localization theory of coalgebras.

The goal of this short paper is to give a direct proof of the above result and to describe cocommutative coalgebras C such that $Rat(M)$ is a direct summand for any finitely generated left C^*-module M. If a coalgebra C has the splitting property, then $Rat(C^*{-}Mod)$ coincides with the simple torsion theory in $C^*{-}Mod$. In his series of beautiful papers, Teply deeply investigated the splitting property for the simple torsion theory. We derive, as a consequence of one of his results, that C is finite dimensional. The main result of this paper states that for a cocommutative coalgebra C the maximal rational submodule $Rat(M)$ is a direct summand of any finitely generated left C^*-module M if and only if C is a finite direct sum of finite dimensional coalgebras and infinite dimensional serial coalgebras. When C is either pointed or the ground field is algebraically closed, the infinite dimensional serial coalgebras occurring in this decomposition are isomorphic to the divided power coalgebra. This result is proved using another result of Teply on the splitting of the simple torsion theory and the structure theorems of serial coalgebras obtained in [3].

1 Preliminaries

For the reader's convenience and trying to make our exposition self-contained, we include some general facts about Torsion Theory and Coalgebra Theory. The concepts and results on Torsion Theory presented here may be consulted in [15, Chapter VI]. Our main references for Coalgebra and Comodule Theory are texts [1] and [16].

Torsion Theory: Let R be a ring (associative and unital) and let $R\text{–}Mod$ denote its category of left R-modules. A functor $\tau : R\text{–}Mod \rightarrow R\text{–}Mod$ is a *left exact preradical* if it is a subfunctor of the identity functor, it is left exact, and $\tau \circ \tau = \tau$. Such a functor determines two classes of modules:

$$\mathcal{T}_\tau = \{M \in R\text{–}Mod : \tau(M) = M\}, \quad \mathcal{F}_\tau = \{M \in R\text{–}Mod : \tau(M) = \{0\}\}.$$

The class $\mathcal{T}_\tau$ is closed under subobjects, quotients, and direct sums. A class of left R-modules enjoying these properties is said to be an *hereditary pretorsion class*. The class $\mathcal{T}_\tau$ is closed under extensions if and only if $\tau(M/\tau(M)) = \{0\}$ for all $M \in R\text{–}Mod$. In this case, τ is called a *radical* and $\mathcal{T}_\tau$ is called an *hereditary torsion class*. An relevant example of torsion class is the *simple torsion class* $\mathcal{S}$ which is defined by $\mathcal{S} = \{M \in R\text{–}Mod$: every non-zero homomorphic image of M has non-zero socle$\}$. A pair of classes of R-modules $(\mathcal{T}, \mathcal{F})$ is an *hereditary torsion theory* in $R\text{–}Mod$ if there is a radical $\tau : R\text{–}Mod \rightarrow R\text{–}Mod$ such that $\mathcal{T} = \mathcal{T}_\tau$ and $\mathcal{F} = \mathcal{F}_\tau$. When $(\mathcal{T}, \mathcal{F})$ satisfies that $\tau(M)$ is a direct summand of M for all $M \in R\text{–}Mod$, it is said that $(\mathcal{T}, \mathcal{F})$ has the *splitting property* (SP property for short). If the above condition is only satisfied for finitely generated R-modules, then $(\mathcal{T}, \mathcal{F})$ is said to have the *finitely generated splitting property* (FGSP property).

Associated to a left exact preradical $\tau : R\text{–}Mod \rightarrow R\text{–}Mod$ there is a family $\mathcal{G}$ of left ideals of R defined by $\mathcal{G} = \{I \leq {}_RR : R/I \in \mathcal{T}_\tau\}$. This family is called the *linear topology associated to* τ. The pretorsion class $\mathcal{T}_\tau$ may be recovered from $\mathcal{G}$ as $\mathcal{T}_\tau = \{M \in R\text{–}Mod : Ann_R(m) \in \mathcal{G}\ \forall m \in M\}$. The reader is referred to [15, Theorem 5.1] for more details on the bijective correspondence between hereditary pretorsion classes (resp. torsion classes), linear topologies (resp. Gabriel topologies), and left exact preradicals (resp. radicals).

Coalgebras and comodules: Throughout all vector spaces, algebras, coalgebras, tensor products, etc are considered over a fixed ground field k. Given a coalgebra C, we regard the weak-* topology on the dual algebra C^*. The closed (resp. open) subspaces of C^* are the annihilators $W^{\perp(C^*)}$ of arbitrary (resp. finite dimensional) subspaces W of C. A subspace U of C^* is called *cofinite* if C^*/U is of finite dimension. For additional details on this topology and some of its main properties, [7, Section 1.2] is recommended. A left ideal J of C^* is *closed and cofinite* if and only if there is a finite dimensional left coideal W of C such that $J = W^{\perp(C^*)}$. Finitely generated left (right) ideals are always closed.

The category ${}^C\mathcal{M}$ of left C-comodules is isomorphic to $Rat(C^*{-}Mod)$ the full subcategory of $C^*{-}Mod$ consisting of all rational left C^*-modules. For $M \in C^*{-}Mod$, an element $m \in M$ is called *rational* if there is $\rho_m = \sum_{i=1}^n m_i \otimes c_i \in M \otimes C$ such that $c^* \cdot m = \sum_{i=1}^n \langle c^*, c_i\rangle m_i$ for all $c^* \in C^*$. The set $Rat(M)$ of rational elements of M is a C^*-submodule of M. When $M = Rat(M)$, M is called a *rational* module. The assignment $Rat_C(-) : C^*{-}Mod \to C^*{-}Mod, M \mapsto Rat(M)$, called the *rational functor*, is a left exact preradical. The associated hereditary pretorsion class is $Rat(C^*{-}Mod)$. The corresponding linear topology is the family of all closed and cofinite left ideals of C^*.

Considering C^* as a left C^*-module, $Rat(C^*)$ is a two-sided ideal and it is the sum of all finite dimensional left ideals of C^*. The equality $Rat(C^*) = C^*$ holds if and only if C is of finite dimension. To prove that C is finite dimensional, the latter equality will often be used. If D is a subcoalgebra of C, then D^* is a quotient algebra of C^* and any left D^*-module M is a left C^*-module by restriction of scalars. Then $Rat_C(M) = Rat_D(M)$.

2 The full splitting property

Definition 2.1 *A coalgebra C is said to have the splitting property (SP property) if $Rat(M)$ is a direct summand of M for any left C^*-module M.*

Any finite dimensional coalgebra C has the SP property since every left C^*-module is rational. The purpose of this section is to prove that:

Theorem 2.2 *Any coalgebra C having the SP property is finite dimensional.*

Our proof is based on the fact that the coradical of a coalgebra with the SP property is finite dimensional and on a theorem of Teply quoted below.

Lemma 2.3 *If C has the SP property, then C_0 is finite dimensional.*

Proof: We first show that the SP property is inherited to subcoalgebras. For a subcoalgebra D of C, the torsion theory induced on D^*–Mod by $Rat(C^*$–$Mod)$ coincides with $Rat(D^*-Mod)$. Apply now [17, Lemma 0.2 (1)] stating that the SP property passes to the induced torsion theory on quotient algebras.

By the above it suffices to show that any cosemisimple coalgebra D having the SP property is finite dimensional. Note that $Rat(D^*) \neq \{0\}$. By hypothesis, ${}_{D^*}D^* = Rat(D^*) \oplus X$ with $Rat(X) = \{0\}$. Thus $Rat(D^*)$ is closed. Let E be a subcoalgebra of D such that $Rat(D^*) = E^{\perp(D^*)}$. From $E^* \cong D^*/E^{\perp(D^*)} \cong X$ we get that $Rat(E^*) = \{0\}$. Since E is a cosemisimple coalgebra, $E^* = \{0\}$ which implies $D^* = Rat(D^*)$. ∎

For a simple right R-module S let $\mathcal{S}_S$ denote the torsion class consisting of those $M \in R-Mod$ such that every non-zero homomorphic image of M contains an isomorphic copy of S. Let $F(\mathcal{S}_S)$ denote the associated Gabriel topology. Teply proved in [20] the following result:

Theorem 2.4 *Suppose that for each simple left R-module S, $F(\mathcal{S}_S)$ contains two-sided ideals L_S, K_S satisfying that $K_S \subseteq L_S \neq R$, L_S is finitely generated as a left ideal and K_S is finitely generated as a right ideal. Then, the simple torsion class $\mathcal{S}$ has the splitting property if and only if $R \in \mathcal{S}$.*

Proof of Theorem 2.2: By Lemma 2.3, C_0 is finite dimensional. If C has the SP property, then $Rat(C^*-Mod)$ is a torsion class. Applying [14, Theorem 4.6], we get that every closed and cofinite left (resp. right) ideal of C^* is finitely generated. Since the Jacobson radical $J = C_0^{\perp(C^*)}$ is closed and cofinite, every simple right C^*-module is rational (consequently finite dimensional). Hence $Rat(C^*-Mod)$ coincides with the simple torsion class in C^*-Mod. If S is a simple left C^*-module, then $Ann(S) \in F(\mathcal{S}_S)$. Now, $Ann(S)$ is a closed and cofinite two-sided ideal. Then it is finitely generated as a right and left ideal. From Theorem 2.4, $C^* = Rat(C^*)$. ∎

If the class of modules for which its rational submodule splits off is a proper class of C^*-Mod, then C is not necessarily of finite dimension as the following example shows:

Example 2.5 Let $C(\infty) = k\{c_n : n \in \mathbb{N}\}$ be the divided power coalgebra whose comultiplication and counit is given by

$$\Delta(c_n) = \sum_{i=0}^{n} c_i \otimes c_{n-i}, \quad \varepsilon(c_n) = \delta_{n,0},$$

where $\delta_{n,0}$ denotes the Kronecker symbol. The dual algebra $C(\infty)^*$ is isomorphic to the power series ring $k[[x]]$ and the subcategory of rational modules coincides with the class of torsion modules in the classical sense. Since $k[[x]]$ is a Dedekind domain, the torsion submodule splits off for any finitely generated left $k[[x]]$-module.

In the next section we describe cocommutative coalgebras satisfying that the rational submodule splits off for any finitely generated left module over the dual algebra. The preceding example provides a clue of how these coalgebras could look like. Our description is based on the theory of serial coalgebras developed in [3] and on another theorem of Teply.

3 The splitting property for finitely generated modules

Definition 3.1 *A coalgebra C is said to have the finitely generated splitting property (FGSP property) if $Rat(M)$ is a direct summand of M for any finitely generated left C^*-module M.*

Lemma 3.2 *The FGSP property is stable under subcoalgebras.*

Proof: In [17, Lemma 0.2 (1)] it is shown that the FGSP property passes to the induced torsion theory on quotient algebras. Using this, the statement is proved in a similar way to Lemma 2.3. ∎

Lemma 3.3 *If C has the FGSP property, then C_0 is finite dimensional.*

Proof: Notice that Lemma 2.3 remains valid replacing the SP property by the FGSP property since the C^*-module ${}_{C^*}C^*$ is finitely generated. ∎

In order to describe cocommutative coalgebras having the FGSP property we need the notion of serial coalgebra and the following deep result due to Teply, see [18, Theorem 4.7].

Theorem 3.4 *Let R be a commutative ring and consider $(\mathcal{S}, \mathcal{F})$ the simple torsion theory in $R-Mod$. Assume that $R \in \mathcal{F}$ and each maximal ideal of R is finitely generated. Then $(\mathcal{S}, \mathcal{F})$ has the FGSP property if and only if $R = \oplus_{i=1}^{n} R_i$ where R_i is a Dedekind domain for every $i = 1, 2, .., n$.*

We recall from [3] that a coalgebra is called *right serial* if the injective hull of each simple right comodule is uniserial, i.e., its lattice of subcomodules is a chain. A *serial coalgebra* is a coalgebra which is right and left serial. With this notion available we may formulate the main result of this paper.

Theorem 3.5 *Let C be a cocommutative coalgebra and let $C = \oplus_{i \in I} C_i$ be a decomposition into irreducible components. Then C has the FGSP property if and only if I is finite and either C_i is finite dimensional or C_i is an infinite dimensional serial coalgebra.*

Proof: If C has the FGSP property, then C_0 is finite dimensional by Lemma 3.3, and hence I must be a finite set. On the other hand, each C_i has the FGSP property by Lemma 3.2. As C_i^* is a local algebra, either $C_i^* = Rat(C_i^*)$ or $Rat(C_i^*) = \{0\}$. In the first case, C_i is finite dimensional. Assume that $Rat(C_i^*) = \{0\}$. Since C_i is irreducible, the Jacobson radical $J_i = (C_i)_0^{\perp(C_i^*)}$ is closed and cofinite and so $Rat(C_i^*-Mod)$ coincides with the simple torsion theory in C_i^*-Mod. Moreover, from [14, Theorem 4.6], J_i is finitely generated. Applying Theorem 3.4, C_i^* is a Dedekind domain. In [3, Corollary 1.11] we proved that the finite dual coalgebra of a Dedekind domain is a serial coalgebra. Hence C_i^{*0} is a serial coalgebra. Since C_i^* is noetherian, C_i is coreflexive and then $C_i \cong C_i^{*0}$. Therefore C_i is serial.

To prove the converse, first notice that a finite direct sum of coalgebras has the FGSP property if and only if each summand has it. Then we are reduce to proving that an infinite dimensional cocommutative irreducible serial coalgebra D has the FGSP property. In [3, Theorem 3.2] it was shown that the only ideals of D^* are the powers of the Jacobson radical J, which is principal. Since J is finitely generated and cofinite, J^n is closed and cofinite for all $n \in \mathbb{N}$. Applying [14, Theorem 4.6], $Rat(D^*-Mod)$ is a torsion class in D^*-Mod. Then $Rat(D^*-Mod)$ coincides with the simple torsion class. On the other hand, since D is infinite dimensional and J^n is cofinite for all $n \in \mathbb{N}$, it follows that $Rat(D^*) = \{0\}$ and that $J^n \neq \{0\}$ for all $n \in \mathbb{N}$. From the latter, D^* is a domain and consequently a Dedekind domain. Applying Theorem 3.4, we get the statement. ∎

Notice that from the arguments used in the proof it follows that an infinite dimensional cocommutative irreducible coalgebra C is serial if and only if C^* is a Dedekind domain. It was proved in [3, Theorem 3.2] that if C is pointed irreducible and serial, then C is isomorphic to a subcoalgebra of the divided power coalgebra $C(\infty) = k\{c_0, c_1, c_2, ...\}$. Since the only proper subcoalgebras of $C(\infty)$ are of the form $C(n) = k\{c_0, c_1, ..., c_n\}$ for $n \in \mathbb{N}$, if C is infinite dimensional, then C is isomorphic to $C(\infty)$. Over an algebraically closed field any cocommutative coalgebra is pointed. In this case we have an explicit description of cocommutative coalgebras having the FGSP property.

Corollary 3.6 *A pointed cocommutative coalgebra has the FGSP property if and only if it is a finite direct sum of finite dimensional coalgebras and divided power coalgebras.*

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