Subgroups of the Brauer Group of a Cocommutative Coalgebra.

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To the memory of M.M. Marqués Escámez

1 Introduction.

In the classical theory of the Brauer group of a commutative ring R important subgroups related to group theory appear, i.e, the Schur and projective Schur subgroups relative to a class $\mathcal{H}$ of finite groups closed by finite products. The $\mathcal{H}$-projective Schur subgroup represents classes of Azumaya R-algebras which are epimorphic image of a twisted group algebra $R *_\alpha G$ for some $G \in \mathcal{H}$. Restricting to the case where only the trivial cocycle appears we obtain the $\mathcal{H}$-Schur subgroup.

On the other hand, the Brauer group of a cocommutative coalgebra C was introduced in [15]. In this paper we introduce an $\mathcal{H}$-Schur and an $\mathcal{H}$-projective Schur subgroup of the Brauer group of a cocommutative coalgebra. For this, we consider the twisted cogroup coalgebra which is a particular case of crossed coproduct $C \rtimes_\alpha H$ where $H = (kG)^*$ for some $G \in \mathcal{H}$, $(kG)^*$ coacts trivially on C and the cocycle α is convolution invertible. Then, we define the $\mathcal{H}$-projective Schur subgroup of C consisting of the classes of C-Azumaya coalgebras which are subcoalgebras of some twisted cogroup coalgebra $C \rtimes_\alpha (kG)^*$ with $G \in \mathcal{H}$. If we impose α to be trivial we obtain the definition of the $\mathcal{H}$-Schur group. The interest of C-Azumaya coalgebras represented in this way comes from the good structure of the twisted cogroup coalgebras. For example, in general the Brauer group of C is not a torsion group, however the $\mathcal{H}$-Schur and $\mathcal{H}$-projective Schur subgroups are torsion when the ground field of C is of characteristic zero.

The paper is organized as follows: In Section 3 we study some properties of the twisted cogroup coalgebra. In Section 4 we introduce the $\mathcal{H}$-Schur and $\mathcal{H}$-projective Schur subgroups of the Brauer group of C. When C is finite dimensional these subgroups coincide with the $\mathcal{H}$-Schur respectively $\mathcal{H}$-projective Schur subgroup of C^*. This also is true if the coalgebra C is coreflexive and irreducible. In this case, if C_0 denotes the coradical of C then $S^{\mathcal{H}}(C) \cong S^{\mathcal{H}}(C_0)$ and $S^{\mathcal{H}}(C_0)$ is the classical $\mathcal{H}$-Schur subgroup of the Brauer group of some finite field extension. In general, for $PS^{\mathcal{H}}(C)$ we cannot claim this property though if the ground field of C is of characteristic zero then $PS^{\mathcal{H}}(C) \cong PS^{\mathcal{H}}(C_0)$. With this isomorphism we can deduce some results for $PS^{\mathcal{H}}(C)$ using the classical results for $PS^{\mathcal{H}}(C_0)$. Finally, in Section 5 we provide examples of $\mathcal{H}$-Schur, $\mathcal{H}$-projective Schur and Brauer group of several cocommutative coalgebras.

2 Notation and preliminaries.

Throughout k is a fixed field. All algebras, coalgebras, vector spaces and unadorned $\otimes$ are over k. We use the usual sigma notation for coalgebras and comodules. $\mathcal{M}^C$ denotes the category of right C-comodules and for right C-comodules X,Y, $Com_{-C}(X,Y)$ denotes the vector space of all C-colinear maps from X to Y.

Let $\alpha : C \to D$ be a coalgebra map. Every right C-comodule X may be viewed as a right D-comodule with the structure map:

$$(1 \otimes \alpha)\rho_X : X \to X \otimes C \to X \otimes D$$

In this case, we will say X_D is induced by X_C via α. A $(C{-}D)$-bicomodule is a left C-comodule and a right D-comodule X, denoted by ${}_CX_D$, such that the C-comodule structure map $\rho_X : X \to X \otimes C$ is D-colinear.

Cotensor product : Let M a right C-comodule and N a left C-comodule with structure maps ρ_M and ρ_N. Then the cotensor product $M \square_C N$ is the kernel of the map

$$\rho_M \otimes 1 - 1 \otimes \rho_N : M \bigotimes N \to M \bigotimes C \bigotimes N$$

The functors $M\Box_C-$ and $-\Box_C N$ are left exact and preserve direct sums. If ${}_CM_D$ and ${}_DN_E$ are bicomodules, then $M\Box_C N$ is a $(C-E)$-bicomodule with comodule structures induced by those of X and Y.

Co-hom functor: A right C-comodule X is quasi-finite if $Com_{-C}(Y, X)$ is finite dimensional for any finite dimensional comodule Y. A comodule $X \in \mathcal{M}^C$ is said to be a cogenerator if for any comodule $M \in \mathcal{M}^C$ there is a vector space W such that $M \hookrightarrow W \otimes X$ as comodules. X is injective if the functor $Com_{-C}(-, X)$ is exact, or equivalently, the functor $X\Box_C-$ is exact.

Now, we recall from [14] the definition of the co-hom functor:

Lemma 2.1 *Let ${}_CX_D$ be a bicomodule. Then X_D is quasi-finite if and only if the functor $-\Box_C X : \mathcal{M}^C \to \mathcal{M}^D$ has a left adjoint functor, denoted by $h_{-D}(X, -)$. That is, for comodules Y_D and Z_C,*

$$Com_{-C}(h_{-D}(X, Y), Z) \cong Com_{-D}(Y, Z\Box_C X) \qquad (1)$$

where,

$$h_{-D}(X, Y) = lim_{\to_\mu} Com_{-D}(Y_\mu, X)^*$$

is a right C-comodule and $\{Y_\mu\}$ is a directed family of finite dimensional subcomodules of Y_D such that $Y_D = \cup_\mu Y_\mu$. We denote by θ the canonical D-colinear map $Y \to h_{-D}(X, Y)\Box_C X$ which corresponds to the identity map $h_{-D}(X, Y) \to h_{-D}(X, Y)$ in (1).

If we assume that X_D is a quasi-finite comodule, then $e_{-D}(X) = h_{-D}(X, X)$ is a coalgebra, called the co-endomorphism coalgebra of X. The comultiplication of $e_{-D}(X)$ corresponds to $(1 \otimes \theta)\theta : X \to e_{-D}(X) \otimes e_{-D}(X) \otimes X$ in (1) when $C = k$ and the counit of $e_{-D}(X)$ corresponds to the identity map 1_X.

Crossed coproduct (See [3]): Let C be a coalgebra and H a Hopf algebra which weakly coacts on C to the left, i.e., there is a linear map $C \to H \otimes C$, $c \mapsto \sum_{(c)} c_{(-1)} \otimes c_{(0)}$, such that the following conditions hold:

(W1) $\sum c_{(-1)} \otimes c_{(0)1} \otimes c_{(0)2} = \sum c_{1(-1)} c_{2(-1)} \otimes c_{1(0)} \otimes c_{2(0)}$,

(W2) $\sum_{(c)} c_{(-1)} \varepsilon(c_{(0)}) = \varepsilon(c) 1_H$,

(W3) $\sum_{(c)} \varepsilon(c_{(0)}) c_{(-1)} = c$, $c \in C$.

Let $\alpha : C \to H\otimes H$ be a linear map, $\alpha(c) = \sum \alpha_1(c)\otimes\alpha_2(c)$. Let $C \rtimes_\alpha H$ be the vector space $C \otimes H$ with comultiplication given by:

$$\Delta_\alpha(c \rtimes h) = \sum c_1 \rtimes c_{2(-1)}\alpha_1(c_3)h_1 \otimes c_{2(0)} \rtimes \alpha_2(c_3)h_2$$

$C \rtimes_\alpha H$ is said to be a crossed coproduct if Δ_α is coassociative and $\varepsilon_C \otimes \varepsilon_H$ is a counit.

Lemma 2.2 *$C \rtimes_\alpha H$ is a crossed coproduct if and only if the following conditions hold:*

(CU) (normal cocycle condition);

$$\sum \varepsilon(\alpha_1(c))\alpha_2(c) = \sum \varepsilon(\alpha_2(c))\alpha_1(c) = \varepsilon(c)1_H$$

(C) (cocycle condition);

$$\sum c_{1(-1)}\alpha_1(c_2) \otimes \alpha_1(c_{1(0)})\alpha_2(c_2)_1 \otimes \alpha_2(c_{1(0)})\alpha_2(c_2)_2 =$$
$$= \sum \alpha_1(c_1)\alpha_1(c_2)_1 \otimes \alpha_2(c_1)\alpha_1(c_2)_2 \otimes \alpha_2(c_2)$$

(TC) (Twisted comodule condition);

$$\sum c_{1(-1)}\alpha_1(c_2) \otimes c_{1(0)(-1)}\alpha_2(c_2) \otimes c_{1(0)(0)} =$$
$$= \sum \alpha_1(c_1)c_{2(-1)1} \otimes \alpha_2(c_1)c_{2(-1)2} \otimes c_{2(0)}$$

Proof: cf. [3, Lem. 2.2, 2.3]

Brauer group (See [15]): We recall the notion of Azumaya coalgebra, the Brauer group of a cocommutative coalgebra and some of its properties. A coalgebra map $f : D \to E$ is said to be cocentral if

$$\sum_{(c)} f(c_1) \otimes c_2 = \sum_{(c)} f(c_2) \otimes c_1$$

For a coalgebra D, there exists a cocommutative coalgebra $Z(D)$ with a surjective, cocentral coalgebra map $1^d : D \to Z(D)$ which satisfies the universal property: for any cocentral coalgebra map $f : D \to E$ there is a unique coalgebra map $g : Z(D) \to E$ such that $f = g1^d$. $(Z(D), 1^d)$ is called the cocenter of D. In fact, $Z(D) = h_{-D^e}(D, D) = e_{-D^e}(D)$ where $D^e = D \otimes D^{op}$.

Let C be a cocommutative coalgebra. A C-coalgebra D is a k-coalgebra with a cocentral coalgebra map $\epsilon_D : D \to C$, called the C-counit. A k-coalgebra map $f : D \to E$ is a C-coalgebra map if $\epsilon_E f = \epsilon_D$. A C-coalgebra D is said to be cocentral if $Z(D) \cong C$ and D is said to be C-coseparable if there is a D-bicomodule map $\pi : D\Box_C D \to D$ such that $\pi\Delta = 1_D$. An Azumaya C-coalgebra is defined to be a C-cocentral and C-coseparable coalgebra. If P is an injective quasi-finite cogenerator then $e_{-C}(P)$ is an Azumaya coalgebra. Denote by $B(C)$ the set of the isomorphism classes of Azumaya C-coalgebras. In [15], an equivalence relation (indeed a Morita-Takeuchi equivalence relation) was introduced in $B(C)$ as follows: if $E, F \in B(C)$, then E is equivalent to F, denoted by $E \sim F$, if there exist two quasi-finite injective cogenerators M, N in $\mathcal{M}^C$ such that

$$E\Box e_{-C}(M) \cong F\Box e_{-C}(N).$$

The quotient set $B(C)/\sim$, denoted by $Br(C)$, is an abelian group with the multiplication $[E][F] = [E\Box F]$, unit element $[C]$ and for $[E]$ the inverse is $[E^{op}]$. The group $Br(C)$ is called the *Brauer group* of the cocommutative coalgebra C.

Let $\eta : D \to C$ a cocommutative coalgebra map, then η induces a group homomorphism $\eta_* : Br(C) \to Br(D)$ given by $\eta_*([E]) = [E\Box D]$ for all $[E] \in Br(C)$. If C is of finite dimension, then the Brauer group of C is isomorphic to the Brauer group of the commutative algebra C^*. To compute the Brauer group of C it is enough to compute the Brauer group of an irreducible coalgebra and if C is irreducible, the map $(-)^* : Br(C) \to Br(C^*), [D] \mapsto [D^*]$ is a group homomorphism. Moreover, if C is coreflexive then $Br(C) \cong Br(C^*)$ and $Br(C) \cong Br(C_0)$ where C_0 is the coradical of C and $Br(C_0)$ is isomorphic to the classical Brauer group of some finite field extension.

3 Twisted cogroup coalgebra.

Let G be a finite group and we regard the Hopf algebra $H = (kG)^*$ with basis $\{p_g : g \in G\}$ dual to the basis of kG; that is $p_g(h) = \delta_{g,h} \quad \forall g, h \in G$. The comultiplication and counit are defined by:

$$\Delta(p_g) = \textstyle\sum_{hk=g} p_h \otimes p_k \qquad \varepsilon(p_g) = \delta_{g,e}$$

with e the identity element in G. We consider a linear map $\alpha : C \to (kG)^* \otimes (kG)^*$ expressed in the following way:

$$\alpha(c) = \textstyle\sum_{x,y\in G} \alpha_{x,y}(c) p_x \otimes p_y \qquad \forall c \in C$$

with $\alpha_{x,y} \in C^*$ for all $x, y \in G$ and the trivial coaction $C \to (kG)^* \otimes C$, i.e., $c \mapsto 1_{(kG)^*} \otimes c = \sum_{g\in G} p_g \otimes c$. In this case Lemma 2.2 transforms to:

Lemma 3.1 *$C \rtimes_\alpha (kG)^*$ is a crossed coproduct if and only if the following conditions hold:*

(CU) $\alpha_{g,e}(c) = \alpha_{e,g}(c) = \varepsilon(c) \quad \forall g \in G, c \in C.$

(C) $\sum_{(c)} \alpha_{s,rq}(c_2)\alpha_{r,q}(c_1) = \sum_{(c)} \alpha_{s,r}(c_1)\alpha_{sr,q}(c_2) \quad \forall s, r, q \in G, c \in C.$

(TC) $\sum_{(c)} \alpha_{s,t}(c_2)c_1 = \sum_{(c)} \alpha_{s,t}(c_1)c_2 \qquad \forall s, t \in G, c \in C.$

Proof: It is only apply the Lemma 2.2 to this situation and note that $\{p_g : g \in G\}$ is a basis of orthogonal idempotents and $\varepsilon(p_g) = \delta_{g,e}$ for all $g \in G$.

We note that if C is cocommutative, (TC) is trivial. In the sequel we suppose that C is cocommutative and we may forget this condition. The following two propositions can be proved straightforward.

Proposition 3.2 *Let $C \rtimes_\alpha (kG)^*$ be a crossed coproduct where $(kG)^*$ coacts trivially on C. The map $\epsilon_\alpha = (1 \otimes \varepsilon) : C \rtimes_\alpha (kG)^* \to C$, $\sum_{g\in G} c_g \rtimes p_g \mapsto c_e$ is a cocentral coalgebra map. Hence $C \rtimes_\alpha (kG)^*$ is a C-coalgebra.*

Proposition 3.3 *Let $\alpha : C \to (kG)^* \otimes (kG)^*$ be a linear map expressed as $\alpha(c) = \sum_{x,y\in G} \alpha_{x,y}(c) p_x \otimes p_y$. Then α is convolution invertible if and only if $\alpha_{x,y} \in U(C^*)$, the set of units of C^* for all $x, y \in G$.*

Definition 3.4 *A convolution invertible map $\alpha : C \to (kG)^* \otimes (kG)^*$ satisfying (CU) and (C) is said to be a convolution invertible cocycle.*

Definition 3.5 *We say that $C \rtimes_\alpha (kG)^*$ is a twisted cogroup coalgebra if $C \rtimes_\alpha (kG)^*$ is a crossed coproduct with the trivial coaction of $(kG)^*$ on C and α is a convolution invertible cocycle.*

Proposition 3.6 *Let $\alpha : C \to (kG)^* \bigotimes (kG)^*$ be a convolution invertible cocycle. Then,*

i) $\alpha^{op} : C \to (kG)^{*op} \bigotimes (kG)^{*op} \cong (kG^{op})^* \bigotimes (kG^{op})^*$ *defined by* $\alpha^{op}(c) = \sum_{x,y\in G} \alpha_{y,x}(c) p_x \otimes p_y$ *is a convolution invertible cocycle.*

ii) $(C \bowtie_\alpha (kG)^*)^{op} \cong C \bowtie_{\alpha^{op}} (kG^{op})^*$ *as C-coalgebras.*

Proof: We write $\alpha^{op}(c) = \sum_{x,y\in G} \alpha^{op}_{x,y}(c) p_x \otimes p_y$ with $\alpha^{op}_{x,y}(c) = \alpha_{y,x}(c)$ for all $x, y \in G$. Using (CU), (C) of α, the definition of product in kG^{op} and the above proposition it follows the first claim. For the second one, the isomorphism is given by the identity map.

Proposition 3.7 *Let G, H be finite groups and $\alpha : C \to (kG)^* \bigotimes (kG)^*$, $\beta : C \to (kH)^* \bigotimes (kH)^*$ convolution invertible cocycles. Then,*

i) The map $\alpha \times \beta : C \to (kG \times H)^* \otimes (kG \times H)^*$ defined by $\alpha \times \beta = (1 \otimes \tau \otimes 1)(\alpha \otimes \beta)\Delta$, where τ is the twist map, is a convolution invertible cocycle.

ii) $(C \bowtie_\alpha (kG)^*)\square_C(C \bowtie_\beta (kH)^*) \cong C \bowtie_{\alpha\times\beta} (kG\times H)^*$ *as C-coalgebras.*

Proof: *i)* For all $c \in C$, we express α, β and $\rho = \alpha \times \beta$ as

$$\alpha(c) = \sum_{x,y\in G} \alpha_{x,y}(c) p_x \otimes p_y, \quad \beta(c) = \sum_{z,t\in H} \beta_{z,t}(c) p_z \otimes p_t$$

$$\rho(c) = (\alpha \times \beta)(c) = \sum_{(x,z),(y,t)\in G\times H} \rho_{(x,z),(y,t)}(c) p_{(x,z)} \otimes p_{(y,t)}$$

where $\rho_{(x,z),(y,t)}(c) = \sum_{(c)} \alpha_{x,y}(c_1)\beta_{z,t}(c_2)$.

(CU) Using that α and β satisfy (CU), we obtain:

$\rho_{(x,z),(e,e)}(c) = \sum_{(c)} \alpha_{x,e}(c_1)\beta_{z,e}(c_2) = \sum_{(c)} \varepsilon(c_1)\varepsilon(c_2) = \varepsilon(c)$.

Similarly, $\rho_{(e,e),(y,t)}(c) = \varepsilon(c)$.

(C) Using the cocommutativity of C and (C) of α, β we have:

$\sum_{(c)} \rho_{(a,b),(x,z)(y,t)}(c_1)\rho_{(x,z),(y,t)}(c_2) = \sum_{(c)} \rho_{(a,b),(xy,zt)}(c_1)\rho_{(x,z),(y,t)}(c_2)$
$= \sum_{(c)}(\sum_{(c_1)} \alpha_{a,xy}(c_{11})\beta_{b,zt}(c_{12}))(\sum_{(c_2)} \alpha_{x,y}(c_{21})\beta_{z,t}(c_{22}))$
$= \sum_{(c)}(\sum_{(c_2)} \alpha_{x,y}(c_{21})\alpha_{a,xy}(c_{22}))(\sum_{(c_1)} \beta_{z,t}(c_{11})\beta_{b,zt}(c_{12}))$

$= \sum_{(c)} (\sum_{(c_2)} \alpha_{a,x}(c_{21})\alpha_{ax,y}(c_{22}))(\sum_{(c_1)} \beta_{b,z}(c_{11})\beta_{bz,t}(c_{12}))$
$= \sum_{(c)} (\sum_{(c_1)} \alpha_{a,x}(c_{11})\beta_{b,z}(c_{12}))(\sum_{(c_2)} \alpha_{ax,y}(c_{21})\beta_{bz,t}(c_{22}))$
$= \sum_{(c)} \rho_{(a,b),(x,z)}(c_1)\rho_{(a,b)(x,z),(y,t)}(c_2)$.

The inverse of ρ is given by $\rho^{-1}(c) = \sum_{(x,z),(y,t)\in G\times H} \rho^{-1}_{(x,z),(y,t)}(c)p_{(x,z)} \otimes p_{(y,t)}$, with $\rho^{-1}_{(x,z),(y,t)}(c) = \sum_{(c)} \alpha^{-1}_{x,y}(c_1)\beta^{-1}_{z,t}(c_2)$.

ii) $C \rtimes_\alpha (kG)^*, C \rtimes_\beta (kH)^*$ are right and left C-comodules respectively with the maps $\omega_\alpha = (1 \otimes \epsilon_\alpha)\Delta_\alpha$ and $\omega_\beta = (\epsilon_\beta \otimes 1)\Delta_\beta$. We have that $\sum_{g\in G,h\in H} c_g \rtimes p_g \otimes d_h \rtimes p_h \in (C \rtimes_\alpha (kG)^*)\Box_C(C \rtimes_\beta (kH)^*)$ if and only if $c_g \otimes d_h \in C\Box_C C$ for all $g \in G, h \in H$. (2)

We define

$$\begin{aligned}&\psi : (C \rtimes_\alpha (kG)^*)\Box_C(C \rtimes_\beta (kH)^*) \to C \rtimes_{\alpha\times\beta} (kG \times H)^* \\ &c \rtimes p_g \otimes d \rtimes p_h \mapsto c\varepsilon(d)p_{(g,h)}.\end{aligned}$$

ψ is well-defined by (2) since if $c \otimes d \in C\Box_C C$ then $c\varepsilon(d) = \varepsilon(c)d$. We check that ψ is a C-coalgebra map:

$$\begin{aligned}&\varepsilon\psi(c_g \rtimes p_g \otimes d_h \rtimes p_h) = \varepsilon(c_g\varepsilon(d_h)p_{(g,h)}) = \varepsilon(c_g)\varepsilon(d_h)\delta_{(g,h),(e,e)} \\ &\quad = \varepsilon(c_g \rtimes p_g \otimes d_h \rtimes p_h).\end{aligned}$$

$$\begin{aligned}&\Delta\psi(c_g \rtimes p_g \otimes d_h \rtimes p_h) = \Delta(c_g\varepsilon(d_h) \rtimes p_{(g,h)}) \\ &= \sum_{(a,u)(b,v)=(g,h)} \sum \rho_{(a,u),(b,v)}(c_{g3})\varepsilon(d_h)c_{g1} \rtimes p_{(a,u)} \otimes c_{g2} \rtimes p_{(b,v)} \\ &= \sum_{(a,u)(b,v)=(g,h)} \sum(\sum \alpha_{a,b}(c_{g31})\beta_{u,v}(c_{g32})\varepsilon(d_h))c_{g1} \rtimes p_{(a,u)} \otimes c_{g2} \rtimes p_{(b,v)}\end{aligned}$$

Since $\delta : C \bigotimes C \to C$, $c \otimes d \mapsto \varepsilon(d)c$ is a coalgebra map we have $\sum_{(c_{g3})} c_{g31} \otimes c_{g32}\varepsilon(d_h) = c_{g3} \otimes d_h = \sum_{(d_h)} c_{g3} \otimes d_{h3}\varepsilon(d_{h2})\varepsilon(d_{h1})$. Applying this to the last equality we obtain,

$$\begin{aligned}&= \sum_{(a,u)(b,v)=(g,h)} \sum \alpha_{a,b}(c_{g3})\beta_{u,v}(d_{h3})\varepsilon(d_{h1})c_{g1} \rtimes p_{(a,u)} \otimes c_{g2}\varepsilon(d_{h2}) \rtimes p_{(b,v)} \\ &= (\psi \otimes \psi)\Delta(c_g \rtimes p_g \otimes d_h \rtimes p_h).\end{aligned}$$

We remember that, cf [15, Prop. 2.9], $(C \rtimes_\alpha (kG)^*)\Box_C(C \rtimes_\beta (kH)^*)$ is a C-coalgebra with the map ϵ, $\sum_{g\in G,h\in H} c_g \rtimes p_g \otimes d_h \rtimes p_h \mapsto c_e\varepsilon(d_e)$,

$$\epsilon_\rho\psi(\sum_{g\in G,h\in H} c_g \bowtie p_g\otimes d_h \bowtie p_h) = \epsilon_\rho(\sum_{g\in G,h\in H} c_g\varepsilon(d_h) \bowtie p_{(g,h)})$$
$$= c_e\varepsilon(d_e) = \epsilon(\sum_{g\in G,h\in H} c_g \bowtie p_g\otimes d_h \bowtie p_h)$$

Hence ψ is a C-coalgebra map. Indeed, ψ is an isomorphism of C-coalgebras. We suppose that

$$\psi(\textstyle\sum_{g\in G,h\in H} c_g \bowtie p_g\otimes d_h \bowtie p_h) = \sum_{g\in G,h\in H} c_g\varepsilon(d_h) \bowtie p_{(g,h)} = 0.$$

Then $c_g\varepsilon(d_h) = 0$ for all $g\in G, h\in H$. But, by (2), $c_g\otimes d_h \in C\Box_C C$ thus

$$c_g\otimes d_h = \textstyle\sum_{(c_g)} c_{g1}\otimes c_{g2}\varepsilon(d_h) = \Delta(c_g\varepsilon(d_h)) = 0.$$

Therefore, $\sum_{g\in G,h\in H} c_g \bowtie p_g\otimes d_h \bowtie p_h = 0$.

To see that φ is surjective let $\sum_{(g,h)\in G\times H} c_{(g,h)} \bowtie p_{(g,h)} \in C \bowtie_{\alpha\times\beta} (kG\times H)^*$. Since $c_{(g,h)} = \sum_{(c_{(g,h)})} c_{(g,h)1}\varepsilon(c_{(g,h)2})$, we obtain:

$$\psi(\sum_{g\in G,h\in H}\sum_{(c_{(g,h)})} c_{(g,h)1} \bowtie p_g\otimes c_{(g,h)2} \bowtie p_h) = \sum_{(g,h)\in G\times H} c_{(g,h)} \bowtie p_{(g,h)}$$

and $\sum_{g\in G,h\in H}\sum_{(c_{(g,h)})} c_{(g,h)1} \bowtie p_g\otimes c_{(g,h)2} \bowtie p_h \in (C\bowtie_\alpha (kG)^*)\Box_C(C\bowtie_\beta (kH)^*)$ by (2).

Proposition 3.8 *Let $f: D\to C$ be a map of cocommutative coalgebras and $\alpha: C\to (kG)^*\bigotimes(kG)^*$ a convolution invertible cocycle. Then,*

i) The map $\overline{\alpha}: D\to (kG)^*\bigotimes(kG)^*$ given by $\overline{\alpha}(d) = \alpha f(d)$ is a convolution invertible cocycle.

ii) $(C\bowtie_\alpha (kG)^*)\Box_C D \cong D\bowtie_{\overline{\alpha}} (kG)^*$ as D-coalgebras.

Proof: *i)* We express $\overline{\alpha}$ as $\overline{\alpha}(d) = \sum_{x,y\in G}\overline{\alpha_{x,y}}(d)p_x\otimes p_y$ with $\overline{\alpha_{x,y}}(d) = \alpha_{x,y}(f(d))$. Using (CU) and (C) of α and the fact that f is a map of cocommutative coalgebras we obtain:

(CU) $\overline{\alpha_{g,e}}(d) = \alpha_{g,e}(f(d)) = \varepsilon(f(d)) = \varepsilon(d)$. Similarly, $\overline{\alpha_{e,g}}(d) = \varepsilon(d)$.

(C) $\sum_{(d)}\overline{\alpha_{s,rq}}(d_2)\overline{\alpha_{r,q}}(d_1) = \sum_{(d)}\alpha_{s,rq}(f(d_2))\alpha_{r,q}(f(d_1))$
$= \sum_{(f(d))}\alpha_{s,rq}(f(d)_2)\alpha_{r,q}(f(d)_1) = \sum_{(f(d))}\alpha_{s,r}(f(d)_1)\alpha_{sr,q}(f(d)_2)$
$= \sum_{(d)}\alpha_{s,r}(f(d_1))\alpha_{sr,q}(f(d_2)) = \sum_{(d)}\overline{\alpha_{s,r}}(d_1)\overline{\alpha_{sr,q}}(d_2)$.

Since α is convolution invertible, $\alpha_{x,y} \in U(C^*)$ for all $x, y \in G$. The map $\beta(d) = \sum_{x,y\in G} \alpha_{x,y}^{-1}(f(d))p_x \otimes p_y$ is the inverse of $\overline{\alpha}$.

ii) This can be proved in a similar way to *ii)* of the above proof.

Proposition 3.9 *Let $\alpha : C \to kG^* \bigotimes kG^*$ be a convolution invertible cocycle. Then,*

i) *$\alpha^* : kG \bigotimes kG \to C^*$ is a normalized cocycle and $\alpha^*(kG \bigotimes kG) \subseteq U(C^*)$.*

ii) *$(C \rtimes_\alpha (kG)^*)^* \cong C^* *_{\alpha^*} kG$ as C^*-algebras.*

Proof: *i)* If we express $\alpha(c) = \sum_{x,y\in G} \alpha_{x,y}(c)p_x \otimes p_y$ then $\langle \alpha^*(x, y), c\rangle = \alpha_{x,y}(c)$. By Proposition 3.3, $\alpha_{x,y} \in U(C^*)$ for all $x, y \in G$ and from (C) and (CU) of α, α^* is a normalized cocycle.

ii) It follows from *i)* and [3, Remark 2.6].

Proposition 3.10 *Let R be a commutative k-algebra, G a finite group and $\alpha : kG \bigotimes kG \to U(R)$ a normalized cocycle.*

i) *If R^0 is the finite dual of R, then $\alpha^o : R^0 \to kG^* \bigotimes kG^*$ is a convolution invertible cocycle.*

ii) *The finite dual of the twisted group algebra $R *_\alpha kG$, $(R *_\alpha kG)^0$, is isomorphic to $R^0 \rtimes_{\alpha^0} (kG)^*$ as a R^0-coalgebra.*

Proof: *i)* We write $\alpha^o(a^*) = \sum_{x,y\in G} {\alpha^o}_{x,y}(a^*)p_x \otimes p_y$ with ${\alpha^o}_{x,y} \in R^{**}$. Then, if $a^* \in R^0$ we have $\langle a^*, \alpha(g \otimes h)\rangle = \langle \alpha^o(a^*), g \otimes h\rangle = {\alpha^o}_{(g,h)}(a^*)$. Using this fact and that α is a normalized cocycle we obtain the properties (C) and (CU) for α^0. Since α is convolution invertible, given $g, h \in G$ we can find $\beta(g \otimes h) \in R$ such that $\alpha(g \otimes h)\beta(g \otimes h) = 1$. For $a^* \in R^0$ we define $\beta_{g,h}(a^*) = \langle a^*, \beta(g \otimes h)\rangle$. Then $\beta(a^*) = \sum_{g,h\in G} \beta_{g,h}(a^*)p_g \otimes p_h$ is the inverse of α^0.

ii) By [11, Lem. 6.0.1] we have that $(R\otimes kG)^0 \cong R^0\otimes kG^*$ as vector spaces and it is not difficult to check that this is an isomorphism of R^0-coalgebras from $(R *_\alpha kG)^0$ to $R^0 \rtimes_{\alpha^0} kG^*$.

In the following proposition we characterize when the twisted cogroup coalgebra is coseparable and we obtain a dual result to the classic one. We remember from [15, Prop. 3.2] the notion of coseparability idempotent. Let D be a coseparable C-coalgebra, a C-colinear map $\rho : D\Box_C D \to C$ is a coseparibility idempotent of D if $\rho\Delta = \epsilon$ and $(1\otimes\rho)(\Delta\otimes 1) = (\rho\otimes 1)(1\otimes\Delta)$.

Proposition 3.11 *$C \rtimes_\alpha (kG)^*$ is C-coseparable if and only if $|G|^{-1} \in k$.*

Proof: We suppose that $|G|^{-1} \in k$, we define $\rho : C \rtimes_\alpha (kG)^* \Box_C C \rtimes_\alpha (kG)^* \to C$, $\sum_{a,b\in G} c_a \rtimes p_a \otimes d_b \rtimes p_b \mapsto \frac{1}{|G|}\sum_{a\in G} \alpha^{-1}_{a,a^{-1}}(c_a) d_{a^{-1}}$. It is routine to check that ρ is a coseparability idempotent of $C \rtimes_\alpha (kG)^*$. From [15, Prop. 3.2] we have that $C \rtimes_\alpha (kG)^*$ is C-coseparable.

Conversely, we suppose that $C \rtimes_\alpha (kG)^*$ is C-coseparable. Let D an irreducible subcoalgebra of C and D_0 its coradical. From [15, Prop. 3.6] and Proposition 3.8 $D_0 \rtimes_{\bar\alpha} (kG)^*$ is D_0-coseparable. Since D_0 is cosemisimple, from [15, Prop. 3.4], $D_0 \rtimes_{\bar\alpha} (kG)^*$ is cosemisimple. Using Proposition 3.9 $D_0^* *_{\bar\alpha^*} G$ is semisimple because D_0 is finite dimensional and hence $|G|^{-1} \in k$.

4 Subgroups of the Brauer group.

From now on, unless otherwise stated, $\mathcal{H}$ is a class of groups closed by finite product and opposite.

Definition 4.1 *Let A be a C-coalgebra. We say that A is a projective Schur C-coalgebra relative to $\mathcal{H}$ ($\mathcal{H}$-PSC) if A is C-Azumaya and there exists a twisted cogroup coalgebra $C \rtimes_\alpha (kG)^*$ with $G \in \mathcal{H}$ and an injective C-coalgebra map $i : A \to C \rtimes_\alpha (kG)^*$. When α is trivial, i.e., $\alpha(c) = \sum_{x,y\in G} \varepsilon(c) p_x \otimes p_y$, A is called Schur C-coalgebra relative to $\mathcal{H}$ ($\mathcal{H}$-SC).*

Proposition 4.2 *The set $PS^{\mathcal{H}}(C) = \{[A] \in Br(C) : A \quad is \quad \mathcal{H} - PSC\}$ is a subgroup of $Br(C)$. This subgroup is called $\mathcal{H}$-proyective Schur subgroup of C.*

Proof: Let $[A], [B] \in PS^{\mathcal{H}}(C)$, then there are twisted cogroup coalgebras $C \rtimes_\alpha (kG)^*, C \rtimes_\beta (kH)^*$ with $G, H \in \mathcal{H}$ and injective C-coalgebra maps $i : A \to C \rtimes_\alpha (kG)^*$ and $j : B \to C \rtimes_\beta (kH)^*$. The map $j^{op} : B^{op} \to$

$(C \rtimes_{\beta} (kH)^*)^{op}$ is an injective C-coalgebra map and $(C \rtimes_{\beta} (kH)^*)^{op} \cong C \rtimes_{\beta^{op}} (kH^{op})^*$ by Proposition 3.6. Using the left exactness of $-\Box B^{op}$ and $C \rtimes_{\alpha} (kG)^*\Box-$ it follows that $i\Box 1 : A\Box B^{op} \to C \rtimes_{\alpha} (kG)^*\Box B^{op}$ and $1\Box j^{op} : C \rtimes_{\alpha} (kG)^*\Box B^{op} \to C \rtimes_{\alpha} (kG)^*\Box C \rtimes_{\beta^{op}} (kH^{op})^*$ are injective C-coalgebra maps. From Proposition 3.7, $C \rtimes_{\alpha} (kG)^*\Box C \rtimes_{\beta^{op}} (kH^{op})^* \cong C \rtimes_{\alpha\times\beta^{op}} (kG \times H^{op})^*$ and the composition $(1\Box j^{op})(i\Box 1)$ is an injective C-coalgebra map from $A\Box B^{op}$ to $C \rtimes_{\alpha\times\beta^{op}} (kG \times H^{op})^*$. By hypothesis on $\mathcal{H}$, $G \times H^{op} \in \mathcal{H}$ and hence $[A]\Box[B]^{op} = [A\Box B^{op}] \in PS^{\mathcal{H}}(C)$.

Corollary 4.3 *The set $S^{\mathcal{H}}(C) = \{[A] \in Br(C) : A \;\; is \;\; \mathcal{H} - SC\}$ is a subgroup of $Br(C)$ called $\mathcal{H}$-Schur subgroup of C.*

Examples 4.4 *We restrict our attention to the following classes of finite groups.*

(1) If $\mathcal{H}$ is the class of finite groups then we denote $PS^{\mathcal{H}}(C)$ and $S^{\mathcal{H}}(C)$ simply by $PS(C)$ and $S(C)$. $PS(C)$ is called the projective Schur subgroup of $Br(C)$ and $S(C)$ the Schur subgroup.

(2) If $\mathcal{H}$ is the class of finite abelian groups, finite nilpotent groups or p-groups, we write $PS^{ab}(C), PS^{nil}(C), PS^{p}(C)$ respectively for $PS^{\mathcal{H}}(C)$.

(3) Let $[D] \in PS^{\mathcal{H}}(C)$ with injective C-coalgebra map $i : D \to C \rtimes_{\alpha} (kG)^*$ where $C \rtimes_{\alpha} (kG)^*$ is C-coseparable. The set of this classes, denoted by $PS^{\mathcal{H}}_{*}(C)$, is a subgroup of $PS(C)$ since the cotensor product of two C-coseparable coalgebras is again C-coseparable.

(4) All the above definition hold for $S^{\mathcal{H}}(C)$

Proposition 4.5 *Let $\eta : D \to C$ be a map of cocommutative coalgebras, then η induces group homomorphisms $\eta_* : PS^{\mathcal{H}}(C) \to PS^{\mathcal{H}}(D)$ and $\eta_{**} : S^{\mathcal{H}}(C) \to S^{\mathcal{H}}(D)$. Hence, $PS^{\mathcal{H}}(-)$ and $S^{\mathcal{H}}(-)$ are contravariant functors from the category of cocommutative coalgebras to abelian groups.*

Proof: By [15, page 563] the map $\eta_* : Br(C) \to Br(D)$, $[A] \mapsto [A\Box D]$ is a group homomorphism. We consider the restriction of η_* to $PS^{\mathcal{H}}(C)$ and we show its image is contained in $PS^{\mathcal{H}}(D)$. Let $[A] \in PS^{\mathcal{H}}(C)$ with injective C-coalgebra map $i : A \to C \rtimes_{\alpha} (kG)^*$ for some $G \in \mathcal{H}$. By the left exactness of $-\Box D$, we have an injective D-coalgebra map $i\Box 1 : A\Box D \to C \rtimes_{\alpha} (kG)^*\Box D$. Since $C \rtimes_{\alpha} (kG)^*\Box D \cong D \rtimes_{\overline{\alpha}} (kG)^*$ by Proposition 3.8, it follows that $[A\Box D] \in PS^{\mathcal{H}}(D)$.

Proposition 4.6 *Suppose that C is a finite dimensional cocommutative coalgebra and let A be a C-coalgebra. Then*

i) A is a projective Schur (resp. Schur) C-coalgebra relative to $\mathcal{H}$ if and only if A^* is a projective Schur (resp. Schur) C^*-algebra relative to $\mathcal{H}$.

ii) $PS^{\mathcal{H}}(C) \cong PS^{\mathcal{H}}(C^*)$ (resp. $S^{\mathcal{H}}(C) \cong S^{\mathcal{H}}(C^*)$) mapping $[A] \mapsto [A^*]$.

Proof: First, we recall from [15, Prop. 4.6] that A is C-Azumaya if and only if A^* is C^*-Azumaya, and A is finite dimensional. Moreover, the map $(-)^* : Br(C) \to Br(C^*)$, $[A] \mapsto [A^*]$ is a group isomorphism.

i) If A is a projective Schur C-coalgebra relative to $\mathcal{H}$ then A is C-Azumaya and there is a twisted cogroup coalgebra $C \rtimes_\alpha (kG)^*$ with $G \in \mathcal{H}$ and an injective C-coalgebra map $i : A \to C \rtimes_\alpha (kG)^*$. Dualizing, we have A^* is C^*-Azumaya and $i^* : (C \rtimes_\alpha (kG)^*)^* \to A^*$ is a surjective C^*-algebra map. Since $(C \rtimes_\alpha (kG)^*)^* \cong C^* *_{\alpha^*} kG$ by Proposition 3.9 and $G \in \mathcal{H}$ it follows that A^* is a projective Schur C^*-algebra relative to $\mathcal{H}$. The converse is similar using Proposition 3.11 since $C^{**} \cong C$ and $A^{**} \cong A$.

When α is trivial, $C \rtimes_\alpha (kG)^* = C \otimes (kG)^*$ and $(C \otimes (kG)^*)^* \cong C^*[G]$. Reasoning as above combined with the fact that $C^*[G]^* \cong C \otimes (kG)^*$ prove the claim for Schur C-coalgebras.

ii) The restriction of $(-)^*$ to $PS^{\mathcal{H}}(C)$ (resp. $S^{\mathcal{H}}(C)$) is an injective group homomorphism. *i)* show that the image is $PS^{\mathcal{H}}(C^*)$ (resp. $S^{\mathcal{H}}(C^*)$).

Next, we study the behaviour of the functors $PS^{\mathcal{H}}(-)$, $S^{\mathcal{H}}(-)$ with respect to direct sums.

Proposition 4.7 *Let $\{C_i\}_{i\in I}$ be a family of subcoalgebras of C such that $C = \oplus_{i\in I} C_i$. Then $PS^{\mathcal{H}}(C) \hookrightarrow \prod_{i\in I} PS^{\mathcal{H}}(C_i)$.*

Proof: By [15, Lem. 4.8] we have that every C-coalgebra D is of the form $\oplus_{i\in I} D_i$ with $D_i = D \square C_i$ and conversely, if $D_i's$ are C_i-coalgebras then, $\oplus_{i\in I} D_i$ is a C-coalgebra and $D_i \cong D \square C_i$. Moreover, D is C-Azumaya if and only if D_i is C_i-Azumaya for all $i \in I$. By [15, Th. 4.9] the map $\eta : Br(C) \to \prod_{i\in I} Br(C_i)$, $[D] \mapsto \prod_{i\in I}[D_i]$ is a group isomorphism. Regarding η restricted to $PS^{\mathcal{H}}(C)$ we obtain a group monomorphism. We show its image is contained in $\prod_{i\in I} PS^{\mathcal{H}}(C_i)$. Let D be a $\mathcal{H}$-projective Schur coalgebra, reasonig as in Proposition 4.5 for each C_i we obtain that $D_i = D \square C_i$

is a proyective Schur C_i-coalgebra relative to $\mathcal{H}$ for all $i \in I$. Therefore $\prod_{i\in I}[D_i] \in \prod_{i\in I} PS^{\mathcal{H}}(C_i)$.

Corollary 4.8 *With C as before, $S^{\mathcal{H}}(C) \hookrightarrow \prod_{i\in I} S^{\mathcal{H}}(C_i)$.*

In general, we cannot prove that foregoing monomorphisms are surjective. However, if C is the group-like coalgebra or the set I is finite then it holds for the Schur subgroup.

Proposition 4.9 *Let $\{C_i\}_{i=1}^n$ subcoalgebras of C with $C = \oplus_{i=1}^n C_i$. Then $S^{\mathcal{H}}(C) \cong \prod_{i=1}^n S^{\mathcal{H}}(C_i)$.*

Proof: In view of Corollary 4.8 we only have to show that the map $\eta : S^{\mathcal{H}}(C) \to \prod_{i=1}^n S^{\mathcal{H}}(C_i)$, $[D] \mapsto \prod_{i=1}^n [D_i]$ with $D_i = D\Box C_i$ is surjective. Let $\prod_{i=1}^n [D_i] \in \prod_{i=1}^n S^{\mathcal{H}}(C_i)$, then D_i is C_i-Azumaya and there exists an injective C_i-coalgebra map $j_i : D_i \to C_i \otimes (kG_i)^*$ with $G_i \in \mathcal{H}$ for $i = 1, ..., n$. Thus, $D = \oplus_{i=1}^n D_i$ is C-Azumaya and we have an injective C-coalgebra map $j = \oplus_{i=1}^n j_i : D \to \oplus_{i=1}^n C_i \otimes (kG_i)^*$. Put $G = \prod_{i=1}^n G_i$, then $(kG_i)^* \hookrightarrow \prod_{i=1}^n (kG_i)^* \cong (kG)^*$ and $\oplus_{i=1}^n C_i \otimes (kG_i)^* \hookrightarrow C \otimes (kG)^*$. So there is an injective C-coalgebra map from D to $C \otimes (kG)^*$ and $G \in \mathcal{H}$, i.e., D is a C-Schur coalgebra relative to $\mathcal{H}$, and $\eta([D]) = \prod_{i=1}^n [D_i]$.

Proposition 4.10 *Let C be the group-like coalgebra over a set T. Then $S^{\mathcal{H}}(C) \cong \prod_T S^{\mathcal{H}}(k)$.*

Proof: With η as before, it is enough to prove that η is surjective. Let $C = \oplus_{t\in T} kt$ and $\prod_{t\in T}[D_t] \in \prod_T S^{\mathcal{H}}(kt)$ then D_t is kt-Azumaya and there exist a finite group $G_t \in \mathcal{H}$ and an injective kt-coalgebra map $j_t : D_t \to (kt \otimes kG_t)^*$ for all $t \in T$. We regard $D = \oplus_{t\in T} D_t$ which is C-Azumaya with $D_t = D\Box kt$. For a fixed $t \in T$ we write $G = G_t$ and let $s \in T$ arbitrary, then $D_s = D\Box ks \cong D\Box kt = D_t$. This induced an injective ks-coalgebra map $i_s : D_s \to kt \otimes (kG)^*$ for all $s \in S$. From this we have an injective C-coalgebra map $i : D = \oplus_{t\in T} D_t \to \oplus_{t\in T} kt \otimes (kG)^* \cong C \otimes (kG)^*$. It follows that D is a C-Schur coalgebra relative to $\mathcal{H}$ and $\eta([D]) = \prod_{t\in T}[D_t]$. Because kt is one dimensional, by Proposition 4.6 $S^{\mathcal{H}}(kt) \cong S^{\mathcal{H}}(k)$ for all $t \in T$.

This result has an interesting consequence. Let $C = \oplus_{n\in \mathbb{N}} \mathbb{Q}$, in [15] was proved that $Br(C)$ is not a torsion group. On the other hand, it is well-known that the Schur group of $\mathbb{Q}$ consists of central simple algebras with exponent less or equal than 2. Hence $\prod_{\mathbb{N}} S(\mathbb{Q})$ is torsion and by Corollary 4.8 $S(C)$ is torsion. This remark is a particular case of the following proposition.

Proposition 4.11 *Let C be a cocommutative coalgebra over a field k, then $S_*^{\mathcal{H}}(C)$ and $PS_*^{\mathcal{H}}(C)$ are torsion groups. If $char(k) = 0$ then $PS^{\mathcal{H}}(C)$ and $S^{\mathcal{H}}(C)$ are torsion.*

Proof: Let $[D] \in S_*^{\mathcal{H}}(C)$, then there exists an injective C-coalgebra map $i : D \to C \otimes (kG)^*$ with $G \in \mathcal{H}$ and $|G|^{-1} \in k$. By the universal property of the cocenter it is not hard to prove that the cocenter of $C \otimes (kG)^*$ is $C \otimes Z(kG)^*$. Since $|G|^{-1} \in k$, kG is Azumaya over $Z(kG)$ then there is $n \in \mathbb{N}$ such that $(kG)^n \cong M_n(Z(kG))$. If D^n denote the cotensor product of D n times, by Proposition 3.7, $D^n \hookrightarrow C \otimes (kG^n)^* \cong C \otimes M_n(Z(kG))^* \cong C \otimes Z(kG)^* \otimes M_n(k)^*$. The map i induces a map between the cocenters, cf [15, page 544], $Z(i) : C \to C \otimes Z(kG)$ and D becomes a $C \otimes Z(kG)^*$-subcoalgebra of $C \otimes Z(kG)^* \otimes M_n(k)^*$. Since $C \otimes Z(kG)^* \otimes M_n(k)^*$ is Azumaya over $C \otimes Z(kG)^*$, from [15, Cor. 3.17] there exists a subcoalgebra C' of $C \otimes Z(kG)^*$ such that $D^n \cong C' \otimes M_n(k)^*$. Hence their cocenters are isomorphic, i.e, $C \cong C'$ and $D^n \cong C \otimes M_n(k)^*$. Thus, $[D]^n$ is trivial in $Br(C)$.

We note that for a large $n \in \mathbb{N}$, α^n is trivial and $C \rtimes_{\alpha^n} (kG)^* \cong C \otimes (kG)^*$. The Proposition 3.7 and the above argument prove the claim for $PS^{\mathcal{H}}(C)$. If $char(k) = 0$ then the twisted cogroup coalgebra is always C-coseparable by Proposition 3.11 and $PS^{\mathcal{H}}(C) = PS_*^{\mathcal{H}}(C)$, $S^{\mathcal{H}}(C) = S_*^{\mathcal{H}}(C)$.

We conjecture that this result is also valid in characteristic non zero and we leave for a future work the final answer. Next, we study $PS^{\mathcal{H}}(C)$ and $S^{\mathcal{H}}(C)$ where C is a cocommutative, irreducible coalgebra. We relate $PS^{\mathcal{H}}(C)$ and $S^{\mathcal{H}}(C)$ with $PS^{\mathcal{H}}(C^*)$ and $S^{\mathcal{H}}(C^*)$ respectively, via the group homomorphism $(-)^* : Br(C) \to Br(C^*)$, cf [15, page 566]. In general, we only have a group homomorphism from $PS^{\mathcal{H}}(C)$ (resp. $S^{\mathcal{H}}(C)$) to $PS^{\mathcal{H}}(C^*)$ (resp. $S^{\mathcal{H}}(C^*)$). But, when C is coreflexive and irreducible this homomorphism is an isomorphim.

Proposition 4.12 *Let C be a cocommutative and irreducible coalgebra then there are group homomorphisms $PS^{\mathcal{H}}(C) \to PS^{\mathcal{H}}(C^*)$ and $S^{\mathcal{H}}(C) \to S^{\mathcal{H}}(C^*)$.*

Proof: From [15, Prop. 4.10], if A is C-Azumaya, then A^* is C^*-Azumaya and there is a group homomorphism $(-)^* : Br(C) \to Br(C^*)$, $[A] \mapsto [A^*]$.

The restriction of this map to $PS^{\mathcal{H}}(C)$ is a group homomorphism. Reasoning as in Proposition 4.6 i), we prove that the image of $(-)^*$ is contained in $PS^{\mathcal{H}}(C^*)$. For $S^{\mathcal{H}}(C)$ we use the same argument combined with the fact that α trivial implies α^* trivial.

Proposition 4.13 *If C is a cocommutative, coreflexive and irreducible coalgebra then $PS^{\mathcal{H}}(C) \cong PS^{\mathcal{H}}(C^*)$ and $S^{\mathcal{H}}(C) \cong S^{\mathcal{H}}(C^*)$. Also, $PS^{\mathcal{H}}_*(C) \cong PS^{\mathcal{H}}_*(C^*)$ and $S^{\mathcal{H}}_*(C) \cong S^{\mathcal{H}}_*(C^*)$*

Proof: If C is coreflexive and irreducible then the map $(-)^* : Br(C) \to Br(C^*)$ is a group isomorphism, cf [15, Th. 3.10]. The restriction to $PS^{\mathcal{H}}(C)$ yields an injective group homomorphism by the above proposition. We prove that this map is also surjective. Let $[D] \in PS^{\mathcal{H}}(C^*)$ then D is C^*-Azumaya and there exists a surjective C^*-algebra map $p : C *_\alpha kG \to D$ with $G \in \mathcal{H}$. Since $(-)^* : Br(C) \to Br(C^*)$ is surjective we can find a C-Azumaya coalgebra E such that $E^* \cong D$. Hence, we have a surjective C^*-algebra map from $C^* *_\alpha kG$ to E^*. Dualizing, we obtain an injective C-coalgebra map from E^{*0} to $(C *_\alpha (kG)^*)^0$. But, by Proposition 3.11, $(C^* *_\alpha (kG))^0 \cong C^{*0} \rtimes_{\alpha^0} (kG)^*$ and as C is coreflexive, $C^{*0} \cong C$ and $E^{*0} \cong E$, cf [15, Prop. 4.12]. Hence, E is a projective Schur C-coalgebra relative to $\mathcal{H}$ and we have proved the surjectivity.

By Proposition 3.11 we have that $C \rtimes_\alpha (kG)^*$ is C-coseparable if and only if $C^* *_{\alpha^*} kG$ is C^*-separable. This fact and the above prove the second claim.

With this proposition we can use the classical results for $PS^{\mathcal{H}}(C^*)$ and $S^{\mathcal{H}}(C^*)$ and we obtain similar results to $PS^{\mathcal{H}}(C)$ and $S^{\mathcal{H}}(C)$.

Corollary 4.14 *Let C be a cocommutative and coreflexive coalgebra over a field k of characteristic non zero. Then $S(C) = \{0\}$.*

Proof: We know that $C = \oplus_{i\in I} C_i$ where each C_i is irreducible, cf. [11, Th. 8.0.5]. By [6, Prop. 3.1.4] subcoalgebras of coreflexive coalgebras are coreflexive, so C_i is coreflexive and irreducible for all $i \in I$. Since k is a field of characteristic non zero, C_i^* is a k-algebra of characteristic non zero and by [4, Prop. 1] $S(C_i^*) = \{0\}$. From the above proposition we have $S(C_i) = \{0\}$ and since $S(C) \hookrightarrow \prod_{i\in I} S(C_i)$, we deduce $S(C) = \{0\}$.

REMARK: We conjecture that if C is a cocommutative coalgebra over a field of positive characteristic then $S(C) = \{0\}$.

Proposition 4.15 *Let C be a cocommutative, irreducible and coreflexive coalgebra.*

i) If D is a projective Schur C-coalgebra relative to p-groups with injective C-coalgebra map $i : D \to C \rtimes_\alpha (kG)^*$, then $C \rtimes_\alpha (kG)^*$ is Azumaya over C.

ii) As above, if $|G|^{-1} \in k$, then D is a direct summand of $C \rtimes_\alpha (kG)^*$.

iii) Let $\mathcal{H}$ be the class of abelian groups, p-groups or nilpotent groups. Then, $PS_*^{\mathcal{H}}(C) = PS^{\mathcal{H}}(C)$. Moreover, $S^{nil}(C) = \{0\}$.

Proof: *i)* From Proposition 4.12 we know that D^* is a projective Schur C^*-coalgebra relative to p-groups. Since D^* is C^*-Azumaya, then $C^* *_{\alpha^*} kG$ is C^*-Azumaya by [8, Lem. 2.6]. By the surjectivity of the map $(-)^* : PS^p(C) \to PS^p(C^*)$ we obtain that $C \rtimes_\alpha (kG)^*$ is C-Azumaya.

ii) Using [8, Th. 2.9] we find that D^* is a direct summand of $C^* *_{\alpha^*} kG$. Since the finite dual preserve finite direct sums, cf [13, Prop. 6.0.5], D^{*0} is a direct summand of $(C^* *_{\alpha^*} kG)^0$. But $D \cong D^{*0}$ and $(C^* *_{\alpha^*} kG)^0 \cong C \rtimes_\alpha (kG)^*$. Hence D is a direct summand of $C \rtimes_\alpha (kG)^*$.

iii) From [8, Th. 2.8] we know that $PS_*^{\mathcal{H}}(C^*) \cong PS^{\mathcal{H}}(C^*)$. By Proposition 4.12 $PS_*^{\mathcal{H}}(C) \cong PS_*^{\mathcal{H}}(C^*)$ and $PS^{\mathcal{H}}(C) \cong PS^{\mathcal{H}}(C^*)$. From this, we obtain the first claim. From [10, Th. 3.11] we find that $S^{nil}(C^*)$ is trivial and by Proposition 4.12 $S^{nil}(C) \cong S^{nil}(C^*)$. Thus, $S^{nil}(C) = \{0\}$.

The following theorem improves the Proposition 4.12 and we can relate the $\mathcal{H}$-Schur and $\mathcal{H}$-projective Schur group of an irreducible and coreflexive coalgebra with the $\mathcal{H}$-Schur and $\mathcal{H}$-projective Schur group of a finite field extension.

Theorem 4.16 *Let C be a cocommutative, coreflexive and irreducible coalgebra with coradical C_0. Then,*

i) $S^{\mathcal{H}}(C) \cong S^{\mathcal{H}}(C_0)$

ii) $PS^{\mathcal{H}}(C) \hookrightarrow PS^{\mathcal{H}}(C_0)$ and if the ground field of C is of characteristic zero $PS^{\mathcal{H}}(C) \cong PS^{\mathcal{H}}(C_0)$.

Proof: From [15, Th. 4.13] we retain that C^* is a complete local algebra and by [2, Th. 6.5] it follows that $Br(C^*) \cong Br(C^*/J)$ where J is the Jacobson radical of C^*. By [5, Th. 2.2] we obtain $S^{\mathcal{H}}(C^*) \cong S^{\mathcal{H}}(C^*/J)$. Since $J = C_0^{\perp}$, and $C^*/J \cong C_0^*$ we have $S^{\mathcal{H}}(C^*/J) \cong S^{\mathcal{H}}(C_0^*)$. Moreover, because C_0 is finite dimensional, $S^{\mathcal{H}}(C_0^*) \cong S^{\mathcal{H}}(C_0)$ from Proposition 4.6 and we obtain the following commutative diagram:

$$\begin{tikzcd}
S^{\mathcal{H}}(C) \arrow[r, "(-)^*"] \arrow[d, "i_*"'] & S^{\mathcal{H}}(C^*) \arrow[d] \\
S^{\mathcal{H}}(C_0) \arrow[r, "(-)^*"] & S^{\mathcal{H}}({C_0}^*) \cong S^{\mathcal{H}}(C^*/J)
\end{tikzcd}$$

As the horizontal arrows represent isomorphisms (Propositions 4.6 and 4.12) and the right arrow is an isomorphism, the commutativity of the diagram yields i_* is an isomorphism. This proves the first assertion. For the second one, we note that $Br(C^*) \cong Br(C^*/J)$ implies $PS^{\mathcal{H}}(C^*) \hookrightarrow PS^{\mathcal{H}}(C^*/J)$ and applying the above argument we have $PS^{\mathcal{H}}(C) \hookrightarrow PS^{\mathcal{H}}(C_0)$. If the ground field of C is of characteristic zero then the same proof of [5, Th. 2.2] yields $PS^{\mathcal{H}}(C^*) \cong PS^{\mathcal{H}}(C^*/J)$ and by an argument again similar to the above we obtain $PS^{\mathcal{H}}(C) \cong PS^{\mathcal{H}}(C_0)$.

5 Examples.

From [6, 2.2] we recall that a coalgebra C over a field k is connected if it is pointed and irreducible. In this case, $C_0 \cong k$.

Corollary 5.1 *Let C be a cocommutative, coreflexive and connected coalgebra. Then $Br(C) \cong Br(k)$.*

Proof: By [15, Th. 3.10], $Br(C) \cong Br(C^*) \cong Br(C^*/J) \cong Br(C_0)$.

Corollary 5.2 *If C is a cocommutative, coreflexive and connected coalgebra then $PS^{\mathcal{H}}(C) \cong PS^{\mathcal{H}}(k)$ and $S^{\mathcal{H}}(C) \cong S^{\mathcal{H}}(k)$. If k is a number field then $PS(C) \cong Br(C)$ and if k is a field containing enough roots of unity we have $Br(C) \cong PS^{ab}(C)$.*

Proof: The claim for $S(C)$ is directly from the above theorem. By the foregoing theorem we have $PS^{\mathcal{H}}(C) \subseteq PS^{\mathcal{H}}(k)$. On the other hand, k is a subcoalgebra of C and the inclusion map i satisfies $\varepsilon i = 1_k$. The induced maps ε_* and i_* on $PS^{\mathcal{H}}(C)$ and $PS^{\mathcal{H}}(k)$ verify $i_* \varepsilon_* = 1_{PS^{\mathcal{H}}(k)}$. Hence $PS^{\mathcal{H}}(C) = PS^{\mathcal{H}}(k) \oplus Ker(i_*)$. Since we have $PS^{\mathcal{H}}(C) \subseteq PS^{\mathcal{H}}(k)$, it follows that $PS^{\mathcal{H}}(C) = PS^{\mathcal{H}}(k)$.

If k is a number field, from [8, Th. 4.6] we have $Br(k) \cong PS(k)$ and by the above results $Br(C) \cong Br(k)$ and $PS(C) \cong PS(k)$. Hence $Br(C) \cong PS(C)$. If k contains enough roots of unity, from [9, page 278] we obtain $Br(k) \cong PS^{ab}(k)$. Since $Br(C) \cong Br(k)$ and $PS^{ab}(C) \cong PS^{ab}(k)$, we conclude that $Br(C) \cong PS^{ab}(C)$.

Now, we prove that the Brauer group of a cosemisimple or semiperfect coalgebra is the product of classical Brauer groups of some finite dimensional field extensions.

Proposition 5.3 *Let C be a cocommutative coalgebra. If C is cosemisimple or semiperfect then $Br(C) \cong \prod_{i\in I} Br({C_i}^*)$ where $\{C_i\}_{i\in I}$ is a family of finite dimensional subcoalgebras of C.*

Proof: If C is cosemisimple then $C = \oplus_{i\in I} C_i$ where C_i's are simple subcoalgebras of C. Applying [15, Prop. 4.6, Th 4.9] we obtain $Br(C) \cong \prod_{i\in I} Br({C_i}^*)$. If C is semiperfect, as a particular case of [18, 15.2] we have that the Rat_C functor is exact and by [12, Th. 3.10] $C = \oplus_{i\in I} C_i$ where C_i's are finite dimensional subcoalgebras of C. Again, applying [15, Prop. 4.6, Th. 4.9] we obtain $Br(C) \cong \prod_{i\in I} Br({C_i}^*)$.

Examples 5.4 *Let V be a finite-dimensional vector space and $C = B(V)$ the Birkhoff-Witt coalgebra of divided powers associated with V. $C^* = B(V)^* \cong k[[x_1, ..., x_n]]$ where $n = \dim_k V$, cf [11, pag. 278], and since $k[[x_1, ..., x_n]]$ is noetherian, by [6, Th. 5.2.1] C is of finite type. Because C is connected and of finite type, [6, Th. 4.2.6] yields C is coreflexive. The above results lead to: $Br(C) \cong Br(k)$, $PS^{\mathcal{H}}(C) \cong PS^{\mathcal{H}}(k)$ and $S^{\mathcal{H}}(C) \cong S^{\mathcal{H}}(k)$. We note that if k is of characteristic zero then the symmetric algebra over $V, S(V)$, is isomorphic to $B(V)$. In this case, we have that $Br(S(V)) \cong Br(k)$, $PS^{\mathcal{H}}(S(V)) \cong PS^{\mathcal{H}}(k)$ and $S^{\mathcal{H}}(S(V)) \cong S^{\mathcal{H}}(k)$.*

Depending of the ground field of C we have a curiosly fact. If k is a number field then $Br(C) \cong PS(C)$ by Corollary 5.2. If k is the rational *function field in two variables over* $\mathbb{Z}_2$*. By [1, Ex. 3.8]* $PS(k)$ *is a proper subgroup of* $Br(k)$*, so* $PS(C)$ *is a proper subgroup of* $Br(C)$ *from Corollary 5.2.*

Let L *be a finite dimensional Lie algebra over a field of characteristic zero and we consider* $U(L)$ *its universal enveloping algebra. From [11, Prop. 11.0.11] we obtain that* $U(L)$ *is connected, and since char(k)=0, the set of primitive elements is* L*. Hence* $U(L)$ *is of finite type, [6, Th. 4.2.6] yields* $U(L)$ *is coreflexive. Therefore,* $Br(U(L)) \cong Br(k)$*,* $PS^{\mathcal{H}}(U(L)) \cong PS^{\mathcal{H}}(k)$ *and* $S^{\mathcal{H}}(U(L)) \cong S^{\mathcal{H}}(k)$*.*

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References

[1] E. Aljadeff and J. Sonn, On the Projective Schur Group of a Field, J. Algebra 178 (1995), 530-540.

[2] M. Auslander and O. Goldman, The Brauer Group of a Commutative Ring, Trans. Amer. Math. Soc. 97 (1960), 367-409.

[3] S. Dascalescu, S. Raianu and Y.H. Zhang, Finite Hopf-Galois Coextensions, Crossed Coproducts and Duality, J. Algebra 178 (1995), 400-413.

[4] F. DeMeyer and R. Mollin, The Schur Subgroup of a Commutative Ring, J. Pure and Applied Algebra 35 (1985), 117-122.

[5] F. DeMeyer and R. Mollin, The Schur Subgroup, Lect. Not. Math. 1142, 205-209.

[6] R.G. Heyneman and D.E. Radford, Reflexivity and Coalgebras of Finite Type, J. Algebra 28 (1974), 215-246.

[7] F. Lorenz and H. Opolka, Einfache Algebren und projective Darstellungen endlicher Gruppen, Mathematische Zeitschrift 162 (1978), 175-186.

[8] P. Nelis and F. Van Oystaeyen, The Proyective Schur Subgroup of the Brauer Group and Root Groups of Finite Groups, J. Algebra 137 (1991), 501-518.

[9] P. Nelis and F. Van Oystaeyen, Ray Symmetry in Projective Representations and Subgroups of the Brauer Group, Israel Mathematical Conference Proceeding Vol. 1 (1989), 262-279.

[10] P. Nelis, Schur and Projective Schur Groups of Number Rings, Canadian J. Math. Vol. 43 No. 3 (1991), 540-558.

[11] M. E. Sweedler, "Hopf Algebras", Benjamin, New York, 1969.

[12] T. Shudo. A Note on Coalgebras and Rational Modules. Hiroshima Math. J. 6 (1976), 297-304.

[13] E.J. Taft, Reflexivity of Algebras and Coalgebras, Amer. J. Math. 94 (1972), 1111-1130.

[14] M. Takeuchi, Morita Theorems for Categories of Comodules. J. Fac. Sci. Univ. Tokyo 24 (1977), 629-644.

[15] B. Torrecillas, F. Van Oystaeyen and Y.H. Zhang, The Brauer Group of a Cocommutative Coalgebra, J. Algebra 177 (1995), 536-568.

[16] F. Van Oystaeyen and Y.H. Zhang, The Crossed Coproduct Theorem and Galois Cohomology. Israel J. Math. 96 (1996), 579-607.

[17] F. Van Oystaeyen, Azumaya Strongly Graded Rings and Ray Classes, J. Algebra 103 (1986), 228-240.

[18] R. Wisbauer, "Introduction to coalgebras and comodules". Preprint, 1998.

[19] T. Yamada, "The Schur Subgroup of the Brauer Group". Lecture Notes in Mathematics 397, Springer-Verlag. New York/Berlin, 1970.