OUTER AUTOMORPHISMS AND PICARD GROUPS OF COALGEBRAS

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1. INTRODUCTION

It is well-known that for a coalgebra C, every autoequivalence of the category of comodules $\mathbf{M}^C$ is given by an invertible (C,C)-bicomodule. The Picard group of C, denoted by $Pic(C)$, is the set of all isomorphism classes of invertible (C,C)-bicomodules. It becomes a group with the multiplication induced by the cotensor product.

If $Aut(C)$ and $Inn(C)$ are the groups of automorphisms and inner automorphisms of C respectively, there is an exact sequence

$$1 \longrightarrow Inn(C) \longrightarrow Aut(C) \xrightarrow{\ \omega\ } Pic(C)$$

where $\omega(f) = [\ {}_fC_1]$ for all $f \in Aut(C)$ and ${}_fC_1$ is the coalgebra C with comultiplication twisted by f. Then, ω induces a monomorphism from the group of outer automorphism $Out(C) = Aut(C)/Inn(C)$ into $Pic(C)$.

In this note we study when the map ω is surjective. Coalgebras verifying this are called coalgebras with Aut-Pic property and its Picard group is isomorphic to the group of outer automorphisms. Examples of these coalgebras are those such that every injective comodule is free, matrix coalgebras over coalgebras with Aut-Pic, the smash coproduct coalgebra associated to a graded irreducible coalgebra, etc. The main result shows that basic coalgebras (in particular, pointed coalgebras) also have Aut-Pic. As a consequence, we obtain that for any coalgebra C, $Pic(C) \cong Out(C')$ with C' Morita-Takeuchi equivalent to C. This result allows us to provide new examples of Picard groups of coalgebras as well as to reduce the study of $Pic(C)$ to computational facts on automorphisms of basic coalgebras.

We also find an exact sequence relating $Pic(C)$ and the Picard group of its coradical, $Pic(C_0)$, when C_0 is finite dimensional. From this exact sequence, we deduce an interpretation of $Picent(C)$ in terms of cocentral automorphisms.

Finally, we give some applications to graded coalgebras. Graded crossed coproducts were introduced in [DNR] in connection with the cohomological interpretation of the Brauer group of a coalgebra (see [VZ]). These coalgebras are examples of strongly graded coalgebras. We prove that for a graded coalgebra $C = \oplus_{\sigma \in G} C_\sigma$ with C_e having the Aut-Pic property, the graded dual algebra $R = \oplus_{\sigma \in G} R_\sigma$ is a strongly graded ring if and only if C is a graded crossed coproduct. If C_e is, in addition, finite dimensional, then C is a strongly graded coalgebra precisely when C is a graded crossed coproduct.

We next fix our notation and present some preliminaries. Throughout k is a fixed ground field and $\mathbf{M}_k$ denotes the category of k-vector spaces. All coalgebras, vector spaces and unadorned $\otimes$, Hom, etc., are over k.

Coalgebras and Comodules (See [A], [SW]). For a coalgebra C, let Δ and ε be the comultiplication and the counit, respectively. The category of right C-comodules is denoted by $\mathbf{M}^C$; for X in $\mathbf{M}^C$, ρ_X is the comodule structure map. For $X, Y \in \mathbf{M}^C$, $Com_{-C}(X, Y)$ is the space of right C-comodule maps from X to Y. Similarly, ${}^C\mathbf{M}$ denotes the category of left C-comodules. If D is another coalgebra, then X is a (D, C)-bicomodule if $X \in \mathbf{M}^C$ via ρ_X, $X \in {}^D\mathbf{M}$ via ${}_X\rho$ and $(1 \otimes \rho_X)\ {}_X\rho = (\ {}_X\rho \otimes 1)\rho_X$.

A comodule $X \in \mathbf{M}^C$ is said to be:
- a cogenerator if $C \hookrightarrow W \otimes X$ for some $W \in \mathbf{M}_k$,
- injective if the functor $Com_{-C}(-, X)$ is exact, and
- simple if it has no proper subcomodules.

Every simple comodule is of finite dimension. Since the category $\mathbf{M}^C$ is locally finite, then it is locally noetherian. It is well-known that an injective object $X \in \mathbf{M}^C$ has a unique decomposition $X = \oplus_{\alpha \in \Gamma} E(S_\alpha)$ where S_α are simple right comodules and $E(S_\alpha)$ is the injective hull of S_α.

Every simple right comodule is isomorphic to a simple right coideal of C. Let I be a simple right coideal and J its isotypic component, i.e., the sum of all right coideal isomorphic to I. Then J is a simple subcoalgebra; that is, it has no proper subcoalgebras. All these facts can be found in [G].

Morita-Takeuchi Theory (See [T]). A comodule $X \in \mathbf{M}^C$ is called quasi-finite if $Com_{-C}(Y, X)$ is finite dimensional for every finite dimensional comodule $Y \in \mathbf{M}^C$. For a quasi-finite comodule $X \in \mathbf{M}^C$ and any $Y \in \mathbf{M}^C$, the co-hom functor is defined by

$$h_{-C}(X, Y) = \lim_{\overrightarrow{\lambda}} Com_{-C}(Y_\lambda, X)^*,$$

where $\{Y_\lambda\}$ is a directed system of finite dimensional subcomodules of Y such that $Y = \lim_{\overrightarrow{\lambda}} Y_\lambda$. When $Y = X$, then $h_{-C}(X, X)$ is denoted by $e_{-C}(X)$ and it becomes a coalgebra, called the co-endomorphism coalgebra. The following result, due to Takeuchi, characterizes the equivalences between two categories of comodules, [T, Prop. 2.5, Th. 3.5]:

THEOREM 1.1. *Let C, D be coalgebras.*

a) If $F : \mathbf{M}^C \to \mathbf{M}^D$ is a left exact linear functor that preserves direct sums, then there exists a (C, D)-bicomodule M such that $F(-) \cong -\Box_C M$.

b) Let M be a (C, D)-bicomodule. The following assertions are equivalent:

i) The functor $-\Box_C M : \mathbf{M}^C \to \mathbf{M}^D$ is an equivalence.

ii) M is a quasi-finite injective cogenerator as D-comodule and $e_{-D}(M) \cong C$ as coalgebras.

When the conditions hold, $N = h_{-D}(M, D)$ is a (D, C)-bicomodule and the inverse equivalence is given by $-\Box_D N : \mathbf{M}^D \to \mathbf{M}^C$. The coalgebras C and D are called Morita-Takeuchi equivalent coalgebras.

The cocenter (See [TVZ]). Let C be a coalgebra. If we view C as a right C^e-comodule ($C^e = C^{op} \otimes C$), then C is quasi-finite. The co-endomorphism coalgebra has the following universal property:

i) $e_{-C^e}(C)$ is a cocommutative coalgebra with a surjective coalgebra map $\epsilon : C \to e_{-C^e}(C)$ which is cocentral, i.e., for all $c \in C$,

$$\textstyle\sum_{(c)} \epsilon(c_1) \otimes c_2 = \sum_{(c)} \epsilon(c_2) \otimes c_1.$$

ii) For any cocentral coalgebra map $f : C \to D$, there exists a unique coalgebra map $g : e_{-C^e}(C) \to D$ such that $f = g\epsilon$. In particular, an injective coalgebra map induces a coalgebra map from $e_{-C^e}(C)$ to $e_{-D^e}(D)$.

$e_{-C^e}(C)$ is denoted by $Z(C)$ and it is called the cocenter of C. Let C be a cocommutative coalgebra, a coalgebra D is said to be a C-coalgebra if D is a coalgebra together a cocentral coalgebra map $\eta : D \to C$ called C-counit. A map of C-coalgebras is a coalgebra map which respects the C-counits.

The Picard group (See [TZ]). A (C, C)-bicomodule M is called invertible if the functor $-\Box_C M : \mathbf{M}^C \to \mathbf{M}^C$ defines a Morita-Takeuchi equivalence. This is equivalent to the existence of a (C, C)-bicomodule N and two bicomodule isomorphisms $M\Box_C N \cong C$ and $N\Box_C M \cong C$. The Picard group of C, denoted by $Pic(C)$, was introduced in [TZ] and it is defined as the set of all bicomodule isomorphism classes $[M]$ of invertible (C, C)-bicomodules. $Pic(C)$ becomes a group with the multiplication induced by the cotensor product.

Let M be a (C, C)-bicomodule with right and left structure maps ρ_M and ${}_M\rho$ respectively, $\epsilon : C \to Z(C)$ the universal cocentral map from C to its cocenter and $\tau : M \otimes C \to C \otimes M$ the twist map. The set

$$Picent(C) = \{[M] \in Pic(C) : \tau(\epsilon \otimes 1)\rho_M = (1 \otimes \epsilon)\ {}_M\rho\}$$

is a subgroup of $Pic(C)$ called Picent of C.

The following three results were proved in [TZ]:

PROPOSITION 1.2. *Let C and D be Morita-Takeuchi equivalent coalgebras, then $Pic(C) \cong Pic(D)$.*

Let $Aut(C)$ be the group of automorphisms of the coalgebra C. An automorphism $f \in Aut(C)$ is said to be inner if there is a unit $u \in C^*$ such that $f(c) = (u \otimes 1 \otimes u^{-1})(\Delta \otimes 1)\Delta(c)$ for all $c \in C$. The group of inner automorphisms of C, $Inn(C)$, is a normal subgroup of $Aut(C)$ and the factor group $Out(C) = Aut(C)/Inn(C)$ is called the group of outer automorphisms of C.

Let M be a (C, C)-bicomodule and $f, g \in Aut(C)$. We denote by ${}_fM_g$ the bicomodule constructed in the following way: as a vector space ${}_fM_g = M$ and $\rho_{{}_fM_g} = (1 \otimes g)\rho_M$, ${}_{{}_fM_g}\rho = (f \otimes 1)\ {}_M\rho$.

THEOREM 1.3. *There is an exact sequence*

$$1 \longrightarrow Inn(C) \longrightarrow Aut(C) \xrightarrow{\ \omega\ } Pic(C)$$

where $\omega(f) = [\ {}_fC_1]$ for all $f \in Aut(C)$. Hence, ω induces a monomorphism from $Out(C)$ to $Pic(C)$.

PROPOSITION 1.4. *Let $[M], [N] \in Pic(C)$. Then, $M \cong N$ as right comodules if and only if there exists $f \in Aut(C)$ such that $N \cong\ {}_fM_1$ as bicomodules.*

2. THE AUT-PIC PROPERTY

Definition 2.1. *A coalgebra C has the Aut-Pic property if ω of Theorem* 1.3 *is surjective. In this case $Pic(C) \cong Out(C)$.*

If a coalgebra C has the Aut-Pic property, the Picent of C is also described in terms of automorphisms. Let $Aut_{Z(C)}(C)$ and $Inn_{Z(C)}(C)$ be the groups of $Z(C)$-automorphisms and $Z(C)$-inner automorphisms, respectively and $Out_{Z(C)}(C) = Aut_{Z(C)}(C)/Inn_{Z(C)}(C)$.

PROPOSITION 2.2. *Suppose C has the Aut-Pic property, then $Picent(C) \cong Out_{Z(C)}(C)$.*

Proof. Let $[M] \in Picent(C) \subseteq Pic(C)$. By hypothesis, there exists $f \in Aut(C)$ such that $M \cong {}_fC_1$ as (C,C)-bicomodules. Since $M \in Picent(C)$, $\tau(\epsilon \otimes 1)(f \otimes 1)\Delta = (1 \otimes \epsilon)\Delta$, that is, for $c \in C$, we have:

$$\sum_{(c)} c_1 \otimes \epsilon(c_2) = \sum_{(c)} c_2 \otimes \epsilon f(c_1).$$

Applying $\varepsilon \otimes 1$, $\epsilon f(c) = \epsilon(c)$ and this just means that f is a map of $Z(C)$-coalgebras. Hence $Picent(C) \cong Out_{Z(C)}(C)$. ∎

The first example of coalgebra which has the Aut-Pic property was given in [TZ, Th. 2.10]:

PROPOSITION 2.3. *Let C be a cocommutative coalgebra, then $Pic(C) \cong Aut(C)$.*

Next, we introduce another family of coalgebras with the Aut-Pic property.

PROPOSITION 2.4. *Let C be a coalgebra verifying that every right injective comodule is free. Then C has Aut-Pic.*

Proof. Let M be an invertible (C,C)-bicomodule, then M is a quasi-finite injective cogenerator as a right comodule. By hypothesis, $M \cong C^{(n)}$ for some $n \geq 1$ as right comodules. Let N be the inverse of M, then

$$C \cong M \Box_C N \cong C^{(n)} \Box_C N \cong N^{(n)}$$

as right comodules. Again by hypothesis, we have that $N \cong C^{(m)}$ for some $m \geq 1$ as right comodules, therefore $C \cong C^{(nm)}$. Since C has the IBN property, cf. [NTV, Prop. 4.1], then $nm = 1$ and so $M \cong C$ as right comodules. From Proposition 1.4, there exists $f \in Aut(C)$ such that $M \cong {}_fM_1$ as bicomodules. ∎

More examples of coalgebras with Aut-Pic are obtained in the following proposition via matrix coalgebras. We remember that the matrix coalgebra of order n over k, denoted by k_n^*, is given by the vector space generated by the set $\{x_{ij} : 1 \leq i,j \leq n\}$ with comultiplication and counit

$$\Delta(x_{ij}) = \sum_{u=1}^{n} x_{iu} \otimes x_{uj}, \qquad \varepsilon(x_{ij}) = \delta_{ij},$$

for all $i,j = 1,...,n$, where δ_{ij} is the Kronecker symbol. For a coalgebra C, the matrix coalgebra of order n over C is defined as the coalgebra $C \otimes k_n^*$.

PROPOSITION 2.5. *Let C be a coalgebra with Aut-Pic. Then the matrix coalgebra $D = C \otimes k_n^*$ has Aut-Pic for all $n \geq 1$.*

Proof. Let W be a vector space of dimension n with basis $\{w_1, ..., w_n\}$. The left C-comodule $C \otimes W$ has co-endomorphism coalgebra isomorphic to the comatrix coalgebra D, cf. [DNRV, Prop. 3.1]. Also $C \otimes W$ is a (D, C)-bicomodule with the structure maps:

$$\begin{array}{ll} \rho^l : C \otimes W \to (C \otimes W) \otimes C, & c \otimes w_i \mapsto \sum_{(c)} c_1 \otimes w_i \otimes c_2 \\ \rho^r : C \otimes W \to D \otimes (C \otimes W), & c \otimes w_i \mapsto \sum_{(c)} \sum_{u=1}^{n} c_1 \otimes x_{iu} \otimes c_2 \otimes w_u. \end{array}$$

Analogously, $W \otimes C$ is a (C, D)-bicomodule and by [DNRV, Prop. 3.1] both are inverse to each other. From Proposition 1.2, there is a group isomorphism $Pic(C) \to Pic(D),\ [X] \mapsto [(C \otimes W)\Box_C X \Box_C (W \otimes C)]$.

We also have a group monomorphism $\phi : Aut(C) \to Aut(D),\ f \mapsto f \otimes 1$. We claim that the below diagram is commutative:

$$\begin{array}{ccc} Aut(C) & \xrightarrow{\omega_C} & Pic(C) \\ \downarrow & & \downarrow \\ Aut(D) & \xrightarrow{\omega_D} & Pic(D) \end{array}$$

Let $f \in Aut(C)$, we have to show that $(C \otimes W)\Box_C\ {}_fC_1 \Box_C (W \otimes C) \cong\ {}_{\phi(f)}D_1$ as (D, D)-bicomodules. First, it is not hard to check that the maps

$$\begin{array}{ll} \Phi : (C \otimes W)\Box_C\ {}_fC_1 \to\ {}_{\phi(f)}(C \otimes W), & \sum_j c_j \otimes w_j \otimes d_j \mapsto \sum_j \varepsilon(d_j) f^{-1}(c_j) \otimes w_j \\ \Psi :\ {}_{\phi(f)}(C \otimes W) \to (C \otimes W)\Box_C\ {}_fC_1, & c \otimes w_i \mapsto \sum_{(c)} f(c_1) \otimes w_i \otimes c_2 \end{array}$$

are (D, C)-bicomodule maps inverse to each other. Then,

$$(C \otimes W)\Box_C\ {}_fC_1 \Box_C (W \otimes C) \cong\ {}_{\phi(f)}((C \otimes W)\Box_C (W \otimes C))$$

as (D, D)-bicomodules. But the map

$$\Theta :\ {}_{\phi(f)}((C \otimes W)\Box_C (W \otimes C)) \to\ {}_{\phi(f)}D_1, \quad c \otimes w_i \otimes w_j \otimes d \mapsto c\varepsilon(d) \otimes x_{ij}$$

is an isomorphism of (D, D)-bicomodules. Thus, the commutativity is proved. Since, by hypothesis, ω_C is onto, we obtain that ω_D is also onto. ∎

We recall from [CM] that a coalgebra is basic if every simple subcoalgebra is the dual of a division algebra over k. The following theorem shows that basic coalgebras also have Aut-Pic.

THEOREM 2.6. *Let C be a basic coalgebra, then C has Aut-Pic.*

Proof. First, we prove that basic coalgebras verify that the isotypic component of a simple right coideal only contains that coideal. Let I be a simple right coideal of C and let J be the isotypic component of I. Then J is a simple subcoalgebra, cf [MTW] or [G], therefore there exists a finite dimensional division algebra D over k such that $J \cong D^*$ as coalgebras. Since I is a right coideal of J, $I^\perp$ is a right ideal of D and then $I^\perp = \{0\}$. This gives that $I = J$.

Secondly, by Proposition 1.4, we are going to show that every invertible bicomodule is isomorphic to C as right comodules. Let $[M] \in Pic(C)$, then M is a right quasi-finite injective cogenerator. Since M is quasi-finite, by [T, Prop. 4.5] $soc(M) \cong \oplus_{\alpha\in\Gamma} S_\alpha^{(n_\alpha)}$ where $\{S_\alpha\}$ is a representative family of simples and n_α is a finite cardinal number. As M is an injective cogenerator, then $M \cong \oplus_{\alpha\in\Gamma} E(S_\alpha)^{(n_\alpha)}$ where $n_\alpha \geq 1$ for all $\alpha \in \Gamma$. Every S_α is isomorphic to a simple right coideal I_α of C and $C \cong \oplus_{\alpha\in\Gamma} E(I_\alpha)$ as right comodules.

Set $P = \oplus_{\alpha\in\Gamma} E(S_\alpha)^{(n_\alpha - 1)}$, then $M \cong C \oplus P$ as right comodules. Let N be the inverse of M, then $C \cong M\square_C N \cong (C\square_C N) \oplus (P\square_C N)$ as right comodules and so $soc(C) \cong soc(N) \oplus soc(P\square_C N)$. If we write $N \cong \oplus_{\alpha\in\Gamma} E(S_\alpha)^{(t_\alpha)}$, it follows that $t_\alpha \leq 1$ for all $\alpha \in \Gamma$ from the above isomorphism. But, since N is a cogenerator, $t_\alpha \geq 1$, it must be $t_\alpha = 1$ for all $\alpha \in \Gamma$. Thus, $N \cong C$ as right comodules and from $C \cong N\square_C M$, we have that $M \cong C$ as right comodules. ∎

In [CM, Cor. 2.2] it was proved that any coalgebra is Morita-Takeuchi equivalent to a basic one. If we combine this with Proposition 1.3 and the above result, we obtain that computing the Picard group of any coalgebra reduces to compute automorphisms of a basic coalgebra. This is the content of the following corollary:

COROLLARY 2.7. *Let C be a coalgebra and D its Morita-Takeuchi equivalent basic coalgebra, then $Pic(C) \cong Out(D)$.*

Remark. 1.- In the ring case, this property happens if the ring is semiperfect, cf. [B, Prop. 3.8]. However, in the coalgebra case this result is possible because finiteness conditions automatically appear in the category of comodules.

2.- In light of Proposition 2.4 and Corollary 2.7, it is natural to ask whether the Aut-Pic property is a Morita-Takeuchi invariant. For the ring case the answer is false as it was shown in [B, Ex. 1.7]. In this example the rings are finite dimensional k-algebras, hence if we consider the finite dual coalgebras of them, we have an example where the Aut-Pic property is not a Morita-Takeuchi invariant.

Noting that pointed coalgebras are basic coalgebras we have:

Corollary 2.8. *Every pointed coalgebra has Aut-Pic.*

Example. Let C be the Sweedler coalgebra, i.e., the coalgebra generated by the set $\{g_n, s_n : n \in \mathbb{N}\}$ with comultiplication and counit given by

$$\begin{array}{ll} \Delta(g_n) = g_n \otimes g_n & \varepsilon(g_n) = 1 \\ \Delta(s_n) = g_n \otimes s_n + s_n \otimes g_{n+1} & \varepsilon(s_n) = 0 \end{array}$$

C is a pointed coalgebra, and by the above corollary, $Pic(C) \cong Out(C)$. We show that $Pic(C)$ is trivial. Before computing $Aut(C)$, it is easy to observe:

i) For $i, j \in \mathbb{N}$ with $j \neq i+1$ the (g_i, g_j)-primitive elements are $\lambda(g_i - g_j)$ for $\lambda \in k$.

ii) For all $n \in \mathbb{N}$, the (g_n, g_{n+1})-primitive elements are $\alpha_n g_n + \beta_n s_n - \alpha_n g_{n+1}$ with $\alpha_n, \beta_n \in k$.

Let $f \in Aut(C)$, noting that f takes group-like elements into such elements, (g_i, g_j)-primitives into $(f(g_i), f(g_j))$-primitives, and that f is bijective, we conclude that f is of the form:

$$f(g_i) = g_i \qquad\qquad f(s_i) = \alpha_i g_i + \beta_i s_i - \alpha_i g_{i+1}$$

with $\alpha_i \in k$ and $\beta_i \in k^*$ for all $i \in \mathbb{N}$. Now, we check that every automorphism is inner and hence $Pic(C)$ is trivial. Let f as above, we define $u, v : C \to k$ by:

$$\begin{array}{ll} u(g_1) = 1 & v(g_1) = 1 \\ u(g_{i+1}) = \beta_i^{-1} u(g_i) & v(g_{i+1}) = \beta_i u(g_i)^{-1} \\ u(s_i) = -\alpha_i u(g_{i+1}) & v(s_i) = u(g_i)^{-1}\alpha_i \end{array}$$

It is not hard to prove by induction that u is a unit with inverse v and that $f = (u \otimes 1 \otimes u^{-1})(\Delta \otimes 1)\Delta$.

If C is a coalgebra with finite dimensional coradical C_0, in [L, Th. 5] it is shown that every autoequivalence in $\mathbf{M}^C$ induces an equivalence in the category of left C^*-modules ${}_{C^*}\mathbf{M}$. In fact, if M is an invertible (C, C)-bicomodule, the dual M^* is an invertible (C^*, C^*)-bimodule. The map $(-)^* : Pic(C) \to Pic(C^*)$, $[M] \mapsto [M^*]$ is a group monomorphism. When C is finite dimensional, the above map is a group isomorphism. The next theorem relates the Picard groups of C and C_0 when the latter is of finite dimension.

We denote by $Aut(C, C_0)$ the set of automorphisms of C that are the identity on C_0 and analogously for $Inn(C, C_0)$. $J = C_0^{\perp}$ is the Jacobson radical of the dual algebra C^* and $Aut(C^*, J)$ is the set of automorphism of C^* which induce the identity on C^*/J. Similarly for $Inn(C^*, J)$. We have inclusions from $Aut(C, C_0)$ to $Aut(C^*, J)$ and from $Inn(C, C_0)$ to $Inn(C^*, J)$ via the map $f \mapsto f^*$.

THEOREM 2.9. *Let C be a coalgebra with finite dimensional coradical C_0. The following diagram has exact rows and commutative squares,*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & Inn(C, C_0) & \longrightarrow & Aut(C, C_0) & \xrightarrow{\omega} & Pic(C) & \xrightarrow{\Phi} & Pic(C_0) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & Inn(C^*, J) & \longrightarrow & Aut(C^*, J) & \xrightarrow{\omega'} & Pic(C^*) & \xrightarrow{\Phi'} & Pic(C^*/J)
\end{array}$$

where $\Phi([M]) = [M\square_C C_0]$ for all $[M] \in Pic(C)$.

Proof. We first prove that if $[M] \in Pic(C)$, then $M\square_C C_0 = C_0\square_C M$ and hence $M\square_C C_0$ is a (C_0, C_0)-bicomodule. Identifying $M\square_C C$ and $C\square_C M$ with M via the maps $1 \otimes \varepsilon$ and $\varepsilon \otimes 1$ we have that

$$\begin{array}{l} M\square_C C_0 = \{m \in M : \rho_M(m) \in M \otimes C_0\}, \\ C_0\square_C M = \{m \in M : \ {}_M\rho(m) \in C_0 \otimes M\}. \end{array}$$

Since C_0 is of finite dimension, by [L, Th. 5] M^* is an invertible (C^*, C^*)-bimodule. On the other hand, $C^*/J \cong C_0^*$ and so C^* is a semilocal ring. From [CR, Ex. 55.7], $JM^* = M^*J$. We check that $M\square_C C_0 = C_0\square_C M$. Let $m \in M\square_C C_0$ then $\rho_M(m) \in M \otimes C_0$ and suppose that ${}_M\rho(m)$ is not in $C_0 \otimes M$. Then, there exist non zero elements $m_1, ..., m_r, m'_1, ..., m'_t \in M, c_1, ..., c_r \in C_0$ and $c'_1, ..., c'_t \in C - C_0$ such that

$${}_M\rho(m) = \sum_{i=1}^r c_i \otimes m_i + \sum_{j=1}^t c'_j \otimes m'_j.$$

We can suppose $\{c_i, c'_j\}$ linearly independent and choose $c^* \in C^*$, $m^* \in M^*$ with $< c^*, c'_1 >= 1$, $< m^*, m'_1 >= 1$ and zero elsewhere. Thus,

$$c^*m^*(m) = \sum_{i=1}^r < c^*, c_i >< m^*, m_i > + \sum_{j=1}^t < c^*, c'_j >< m^*, m'_j > \neq 0.$$

As $JM^* = M^*J$ there are $n_1^*, ..., n_s^* \in M^*$ and $d_1^*, ..., d_s^* \in C_0^{\perp}$ such that $c^*m^* = \sum_{i=1}^s n_i^* d_i^*$. But $(\sum_{i=1}^s n_i^* d_i^*)(m) = 0$ since $\rho_M(m) \in M \otimes C_0$ which is a contradiction. A similar argument proves the converse.

Thus we can define $\Phi : Pic(C) \to Pic(C_0)$ by $\Phi([M]) = [M\square_C C_0]$ that is a group homomorphism. We show that $ker(\Phi) = \omega(Aut(C, C_0))$. It is clear that $\omega(Aut(C, C_0)) \subseteq ker(\Phi)$. Conversely, let $[M] \in Pic(C)$ such that $M\square_C C_0 \cong C_0$ as (C_0, C_0)-bicomodules and let ψ denote this isomorphism as right C_0-comodules.

Since $soc(M) \cong M\square_C C_0 \cong soc(C)$ and M is injective, we can lift ψ to an isomorphism of right C-comodules from M to C. Let g be the automorphism of C given by Proposition 1.3 such that $M \cong {}_gC_1$ as bicomodules. Hence ${}_{g|_{C_0}}C_0 \cong C_0$ as bicomodules and by [TZ, Lem. 2.6], $g \in Inn(C_0)$, that is, there is a unit $u \in C_0^*$ such that $g(d) = \sum_{(d)} u(d_1)d_2u^{-1}(d_3)$ for all $d \in C_0$. If $i : C_0 \to C$ denotes the inclusion map, then $i^* : C^* \to C^*/J$ is the canonical projection. There is a unit $v \in C^*$ such that $i^*(v) = u$. Let f be the inner automorphism of C given by the unit v^{-1}, then $fg \in Aut(C, C_0)$ and ${}_gC_1 \cong {}_{fg}C_1$ as bicomodules. Thus, $[M] = [\,{}_{fg}C_1]$ and $[M] \in \omega(Aut(C, C_0))$.

The maps $\omega' : Pic(C^*) \to Pic(C^*/J)$ and $\Phi' : Aut(C^*, J) \to Pic(C^*/J)$ are defined by (see [CR, Th. 55.41]) $\omega'(f) = [\,{}_fC_1^*]$ and $\Phi'([M]) = [M \otimes_{C^*} C^*/J]$ for all $f \in Aut(C^*, J), [M] \in Pic(C^*)$. It is not hard to check that the squares are commutative. ∎

If we put $Aut_{Z(C)}(C, C_0)$ for the set of cocentral automorphisms of C that are the identity on C_0 and similarly for $Inn_{Z(C)}(C, C_0)$ we obtain:

Corollary 2.10. *Let C be a coalgebra with finite dimensional coradical C_0, and suppose that the induced map in the cocenters by the inclusion is injective, then $Picent(C) \cong Out_{Z(C)}(C, C_0)$.*

Proof. If we look at Picent in the diagram of the above theorem, the hypothesis yields that $\Phi(Picent(C)) \subseteq Picent(C_0)$ and the following sequence is exact.

$$1 \longrightarrow Inn_{Z(C)}(C, C_0) \to Aut_{Z(C)}(C, C_0) \longrightarrow Picent(C) \longrightarrow Picent(C_0).$$

By [TZ, Th. 2.13], the Picent of a cosemisimple coalgebra is trivial, thus $Picent(C_0)$ is trivial and the desired result is proved. ∎

3. APPLICATIONS TO GRADED COALGEBRAS

The theory of graded coalgebras also provides us some examples of coalgebras with Aut-Pic. We previously need some definitions.

A coalgebra C is graded by a group G (see [NT]) if it admits a decomposition as a direct sum of subspaces $C = \oplus_{g\in G} C_g$ verifying:

1) $\Delta(C_g) \subseteq \sum_{ab=g} C_a \otimes C_b$.

2) $\varepsilon(C_g) = 0$ for all $g \neq e$, with e the identity element in G.

If M is a right C-comodule, then M is called a G-graded comodule over C if $M = \oplus_{g\in G} M_g$, with M_g subspaces of M and $\rho_M(M_g) \subseteq \sum_{ab=g} M_a \otimes C_b$ for any $g \in G$. If $m \in M$, then $m = \sum_{g\in G} m_g$, with $m_g \in M_g$ (the sum has only a finite number of nonzero elements). Each m_g, $g \in G$, is called the homogeneous components of degree g of m and we write $deg(m_g) = g$. $\mathbf{gr}^C$ denotes the category of right graded C-comodules. For $M, N \in \mathbf{gr}^C$ a morphism $f : M \to N$ is a C-colinear map with $f(M_g) \subseteq N_g$ for all $g \in G$. This map is called graded C-colinear map and the set of these maps is denoted by $Com_{gr-C}(M, N)$. Let $M = \oplus_{g\in G} M_g$ be an object in $\mathbf{gr}^C$ and $h \in G$. Then, the h-suspension of M, $M(h)$, coincides with M as a vector space but with the grading $M(h)_g = M_{hg}$ for all $g \in G$. $M(h)$ is again an object in $\mathbf{gr}^C$. The map $M \mapsto M(\sigma)$ defines an isomorphism of categories from $\mathbf{gr}^C$ to $\mathbf{gr}^C$.

Definition 3.1. *Let $C = \oplus_{\sigma\in G} C_\sigma$ be a graded coalgebra and $X \in \mathbf{gr}^C$.*

i) A graded subcoalgebra $D \subseteq C$ is called gr-simple if it has no proper graded subcoalgebras.

ii) C is said to be gr-irreducible if it has an unique graded gr-simple subcoalgebra.

iii) X is called gr-free if $X \cong^{gr} \oplus_{g\in G} C(g)^{(I_g)}$ for some indexed sets I_g for all $g \in G$ where $\cong^{gr}$ denotes a graded isomorphism.

Lemma 3.2. *Let $C = \oplus_{\sigma\in G} C_\sigma$ be a cocommutative gr-irreducible graded coalgebra. Then every gr-injective C-comodule is gr-free.*

Proof. Let $X \in \mathbf{gr}^C$ gr-injective. We know that $X = \oplus_{i\in I} E^{gr}(S_i)$ where $E^{gr}(S_i)$ denotes the graded injective hull of the gr-simple comodule S_i for all $i \in I$. Since C is gr-irreducible then it has an unique gr-simple subcoalgebra S and the only gr-simple comodules are of the form $S(g)$ with $g \in G$. Thus, we can set $X \cong \oplus_{g\in G} E^{gr}(S(g))^{(I_g)}$. But $E^{gr}(S(g)) = C(g)$ for all $g \in G$. Hence $X \cong \oplus_{g\in G} C(g)^{(I_g)}$ and so X is gr-free. ∎

Every graded coalgebra C has associated another coalgebra $C \rtimes kG$, called smash coproduct, constructed in the following way: as a vector space $C \rtimes kG = C \otimes kG$, for any homogenous element $c \in C$ and $g \in G$ the comultiplication and the counit are:

$$\Delta(c \rtimes g) = \sum_{(c)} (c_{(1)} \rtimes deg(c_{(2)})g) \otimes (c_{(2)} \rtimes g),$$
$$\varepsilon(c \rtimes g) = \varepsilon(c).$$

For $M \in \mathbf{gr}^C$, M becomes a right $C \rtimes kG$-comodule via

$$\rho : M \to M \otimes (C \rtimes kG), \quad m \mapsto \sum_{(m)} m_{(0)} \otimes m_{(1)} \rtimes deg(m)^{-1}$$

for homogeneous $m \in M$. Any morphism $f : M \to N$ of graded comodules is also a morphism of $C \rtimes kG$-comodules. Thus, we have a functor $A : \mathbf{gr}^C \to \mathbf{M}^{C\rtimes kG}$ which verifies that $A(\oplus_{g\in G}C(g)) \cong C \rtimes kG$ as right $C \rtimes kG$-comodules. In [DNRV, Th. 1.6] it was proved that A defines an isomorphism between the categories $\mathbf{gr}^C$ and $\mathbf{M}^{C\rtimes kG}$. Hence, the category $\mathbf{gr}^C$ is a locally finite category.

PROPOSITION 3.3. *Let $C = \oplus_{\sigma\in G}C_\sigma$ be a cocommutative gr-irreducible graded coalgebra. Then the smash coproduct coalgebra $C \rtimes kG$ has Aut-Pic.*

Proof. Let $A : \mathbf{gr}^C \to \mathbf{M}^{C\rtimes kG}$ and $B : \mathbf{M}^{C\rtimes kG} \to \mathbf{gr}^C$ be the isomorphisms between both categories. Suppose that $[M] \in Pic(C \rtimes kG)$, then M is a quasi-finite injective cogenerator as right $C \rtimes kG$-comodule. Hence $B(M)$ is a quasi-finite injective cogenerator as graded right C-comodule. Since C is gr-irreducible, by the above lemma, $B(M)$ is gr-free, that is, $B(M) \cong^{gr} \oplus_{g\in G} C(g)^{(I_g)}$. As $B(M)$ is a quasi-finite cogenerator, from [T, Prop. 4.5], I_g is a finite index set and $|I_g| \geq 1$ for all $g \in G$. We can write $B(M) \cong^{gr} \oplus_{g\in G} C(g) \oplus V$ with $V \in \mathbf{gr}^C$, then $M \cong AB(M) \cong A(\oplus_{g\in G}C(g)) \oplus A(V)$. But $A(\oplus_{g\in G}C(g)) \cong C \rtimes kG$ as right $C \rtimes kG$-comodules. Thus $M \cong (C \rtimes kG) \oplus W$ with $W = A(V)$ as right $C \rtimes kG$-comodules.

On the other hand, since M is invertible, there is an invertible $C \rtimes kG$-bicomodule N such that $M \square_{C\rtimes kG} N \cong C \rtimes kG$ as right $C \rtimes kG$-comodules. Then, $(C \rtimes kG \square_{C\rtimes kG} N) \oplus (W \square_{C\rtimes kG} N) \cong C \rtimes kG$. Setting $Z = W \square_{C\rtimes kG} N$ we have that $N \oplus Z \cong C \rtimes kG$, and so $soc(N) \oplus soc(Z) \cong soc(C \rtimes kG)$. The argument of the above paragraph applied to N gives that $N \cong (C \rtimes kG) \oplus W'$ and then $soc(N) \cong soc(C \rtimes kG) \oplus soc(W')$. Combining this with the fact that N is a quasi-finite injective cogenerator and $C \rtimes kG$ contains all simples of the category, we conclude that $soc(Z) = \{0\}$ and therefore $Z = \{0\}$. Using that $-\square_{C\rtimes kG} N$ is an equivalence, it follows that $W = \{0\}$. Thus, $M \cong C \rtimes kG$ as right $C \rtimes kG$-comodules. Finally, from Proposition 1.4 we have the desired result. ∎

In [DNR] graded crossed coproducts were defined and it was proved that every graded crossed coproduct is a strongly graded coalgebra. As application of the above results, we give a converse of this fact for strongly graded coalgebras with the component of degree e having the Aut-Pic property. We first have to remember some definitions and properties on strongly graded coalgebras (see [NT] and [DNR]).

PROPOSITION 3.4. [NT, Prop. 3.1]. *Let $C = \oplus_{\sigma\in G}C_\sigma$ be a graded coalgebra and $M = \oplus_{\sigma\in G}M_\sigma$ a right graded C-comodule.*

i) If $\sigma,\tau\in G$ there exists a unique linear map $u^M_{\sigma,\tau}:M_{\sigma\tau}\to M_\sigma\otimes C_\tau$ such that $u^M_{\sigma,\tau}\pi^M_{\sigma\tau}=(\pi^M_\sigma\otimes\pi^C_\tau)\rho_M$ where $\pi_g:C\to C_g$ denotes the canonical projection for all $g\in G$.

ii) For any σ, τ, $\lambda\in G$: $(u^M_{\sigma,\tau}\otimes 1)u^M_{\sigma\tau,\lambda}=(1\otimes u^C_{\tau,\lambda})u^M_{\sigma,\tau\lambda}$.

iii) If $\sigma\in G$, $(1\otimes\epsilon)u^M_{\sigma,e}=1$.

iv) If we write $\Delta_e=u^C_{e,e}:C_e\to C_e\otimes C_e$, then (C_e,Δ_e,ϵ) is a coalgebra and $\pi_e:C\to C_e$ is a coalgebra map. Moreover, if $M=\oplus_{\sigma\in G}M_\sigma$ is a right C-comodule, then for any $\sigma\in G$, M_σ is a right C_e-comodule via the canonical map $u^M_{\sigma,e}:M_\sigma\to M_\sigma\otimes C_e$.

Definition 3.5. *Let $C=\oplus_{\sigma\in G}C_\sigma$ be a graded coalgebra. C is said to be strongly graded coalgebra if the maps $u^C_{\sigma,\tau}:C_{\sigma\tau}\to C_\sigma\otimes C_\tau$ are injective for all $\sigma,\tau\in G$.*

For any $\sigma\in G$ we put $R_\sigma=\{f\in C^*\mid f(C_\tau)=0\ for\ all\ \tau\neq\sigma\}$, (note that $R_\sigma\cong C^*_\sigma$ as vector spaces). We define $R=\sum_{\sigma\in G}R_\sigma=\oplus_{\sigma\in G}R_\sigma$. By [NT, Prop. 6.1], R is a G-graded ring with multiplication defined as follows: for $f\in R_\sigma$, $g\in R_\tau$ and $c\in C$, $(f*g)(c)=\sum_{(c)}f(\pi_\sigma(c_1))g(\pi_\tau(c_2))$ and $\epsilon:C\to k$ as the unit. R is called the graded dual algebra of the graded coalgebra C.

The following proposition is [DNR, Cor. 2.2, 2.4] and [NT, Cor. 5.5].

Proposition 3.6. *Let $C=\oplus_{\sigma\in G}C_\sigma$ be a graded coalgebra and $R=\oplus_{\sigma\in G}R_\sigma$ the graded dual algebra.*

i) If R is a strongly graded algebra, then C is a strongly graded coalgebra. When C_e is finite dimensional, the converse is true.

ii) If C is a strongly graded coalgebra, then for each $\sigma\in G$, C_σ is an invertible (C_e,C_e)-bicomodule.

Definition 3.7. *A graded coalgebra is called a graded crossed coproduct if $C\neq 0$ and for any $\sigma\in G$ there exist linear maps $u_\sigma:C_\sigma\to k$ and $v_\sigma:C_{\sigma^{-1}}\to k$ such that*

$$\textstyle\sum_{(c)}u_\sigma(\pi_\sigma(c_1))v_\sigma(\pi_{\sigma^{-1}}(c_2))=\sum_{(c)}v_\sigma(\pi_{\sigma^{-1}}(c_1))u_\sigma(\pi_\sigma(c_2))=\varepsilon(c),\qquad\forall c\in C_e.$$

Proposition 3.8. [DNR, Cor. 2.6]. *Every graded crossed coproduct is a strongly graded coalgebra.*

We are now able to prove our result.

Proposition 3.9. *Let $C=\oplus_{\sigma\in G}C_\sigma$ be a graded coalgebra such that C_e has the Aut-Pic property. The following assertions are equivalent:*

i) The graded dual algebra is strongly graded.

ii) C is a graded crossed coproduct.

Proof. ii) ⇒ i) If C is a graded crossed coproduct then the graded dual algebra is a crossed product. Hence, it is a strongly graded ring.

i) ⇒ ii) Suppose that R is a strongly graded ring. By Proposition 3.6 C is a strongly graded coalgebra and every C_σ is an invertible (C_e, C_e)-bicomodule. From the hypothesis, there is an automorphism $f_\sigma : C_e \to C_e$ such that $C_\sigma \cong {}_{f_\sigma}C_{e1}$ as (C_e, C_e)-bicomodules. Let $\theta_\sigma : C_\sigma \to {}_{f_\sigma}C_{e1}$ be this isomorphism. Noting that $({}_{f_\sigma}C_{e1})^* \cong {}_{f_\sigma^*}R_{e1}$ where $f_\sigma^* : R_e \to R_e$ is the dual automorphism of f_σ, we have an isomorphism of (R_e, R_e)-bimodules $\theta_\sigma^* : {}_{f_\sigma^*}R_{e1} \to R_\sigma$. If we write $u_\sigma = \theta_\sigma^*(\varepsilon)$, then $R_\sigma = R_e * u_\sigma = u_\sigma * R_e$. Since R is strongly graded, then $R_\sigma * R_{\sigma^{-1}} = R_e$ and thus $u_\sigma * R_e * u_{\sigma^{-1}} = R_e$ and $u_{\sigma^{-1}} * R_e * u_\sigma = R_e$. Let $\phi, \psi \in R_e$ such that $u_\sigma * \phi * u_{\sigma^{-1}} = \varepsilon$ and $u_{\sigma^{-1}} * \psi * u_\sigma = \varepsilon$. We write $v_\sigma = \phi * u_{\sigma^{-1}} = u_{\sigma^{-1}} * \psi$, then for $c \in C_e$ we have that:

$$\textstyle\sum_{(c)} u_\sigma(\pi_\sigma(c_1))v_\sigma(\pi_{\sigma^{-1}}(c_2)) = \sum_{(c)} v_\sigma(\pi_{\sigma^{-1}}(c_1))u_\sigma(\pi_\sigma(c_2)) = \varepsilon(c)$$

and so C is a graded crossed coproduct. ∎

We do not know if this result is also true when the coalgebra is only strongly graded although if we add that C_e is of finite dimension the following corollary gives an affirmative answer.

COROLLARY 3.10. *Let $C = \oplus_{\sigma\in G}C_\sigma$ be a graded coalgebra such that C_e is finite-dimensional and it has the Aut-Pic property. Then, C is a strongly graded coalgebra if and only if C is a graded crossed coproduct.*

Proof. It is just to combine Proposition 3.6 i) and the foregoing proposition. ∎

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REFERENCES

[A] E. Abe, *Hopf Algebras.* Cambridge University Press, 1977.

[B] M. Bolla, *Isomorphism between Endomorphism Rings of Progenerators.* J. Algebra **87** (1984), 261-284.

[CM] W. Chin and S. Montgomery, *Basic Coalgebras.* Collection Modular interfaces, Riverside, CA, 1995, 41-47.

[CR] C.W. Curtis and I. Reiner, *Methods of Representation Theory with Applications to Finite Groups and Orders*, Vol. II. Wiley, 1987.

[DNR] S. Dăscălescu, C. Năstăsescu and S. Raianu, *Strongly Graded Coalgebras and Crossed Coproducts*. Proceedings dedicated to A. Orsatti, Lecture Notes in Applied Math. **201**, Marcel Dekker.

[DNRV] S. Dăscălescu, C. Năstăsescu, S. Raianu and F. Van Oystaeyen, *Graded Coalgebras and Morita-Takeuchi Contexts*. Tsukuba J. Math. **20** (1995), 395-407.

[G] J.A. Green, *Locally Finite Representations*. J. Algebra **41** (1) (1976), 137-171.

[MTW] C. Menini, B. Torrecillas and R. Wisbauer, *Strongly rational Comodules and Semiperfect Coalgebras*. To appear in J. Pure Appl. Algebra.

[NT] C. Năstăsescu and B. Torrecillas, *Graded Coalgebras*. Tsukuba J. Math. **17** (2) (1993), 461-479.

[NTV] C. Năstăsescu, B. Torrecillas and F. Van Oystaeyen, *IBN for graded Rings*. To appear in Comm. Algebra.

[L] B. I-Peng Lin, *Morita's Theorem for Coalgebras*. Comm. Algebra **1** (4) (1974), 311-344.

[SW] M. E. Sweedler, *Hopf Algebras*. Benjamin, New York, 1969.

[T] M. Takeuchi, *Morita Theorems for Categories of Comodules*. J. Fac. Sci. Univ. Tokyo **24** (1977), 629-644.

[TZ] B. Torrecillas and Y.H. Zhang, *The Picard Groups of Coalgebras*. Comm. Algebra **24** (7) (1996), 2235-2247.

[TVZ] B. Torrecillas, F. Van Oystaeyen and Y.H. Zhang, *The Brauer Group of a Cocommutative Coalgebra*. J. Algebra **177** (1995), 536-568.

[VZ] F. Van Oystaeyen and Y.H. Zhang, *Crossed Coproduct Theorem and Galois Cohomology*. Israel J. Math. **96** (1996), 579-607.

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