PICARD GROUPS FOR GRADED COALGEBRAS

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1 Introduction

In the theory of coalgebras, many examples of graded coalgebras appear (dual of group algebras, symmetric algebras, path coalgebras, etc). Graded coalgebras were formally introduced in [NT1] and these have also been studied in [NT1], [NT2], [DNRV] and [DNR]. The aim of this paper is to introduce some group invariants for a graded coalgebra. Thus, we study the group of equivalences of the category of graded comodules. If $C = \oplus_{g\in G} C_g$ is a G-graded coalgebra, the category of graded comodules $\mathbf{gr}^C$ is isomorphic to the category of comodules over the smash coproduct $C \rtimes kG$ (see [DNRV, Th.1.6]). Using the theory of the Picard group of coalgebras developed in [TZ] the group of equivalences is described by $Pic(C \rtimes kG)$.

Moreover, in this note we introduce the notion of graded equivalence in $\mathbf{gr}^C$ and we describe this kind of equivalences by a graded Picard group $grPic(C)$. In order to do this, Morita-Takeuchi theory of equivalences of comodules is performed to the graded case. As application of this graded theory we are able to compute new examples of the usual Picard group of a coalgebra.

The paper is organized as follows: in Section 2 we fix notation and give some preliminaries. In Section 3 we construct a graded co-HOM functor. Section 4 contains the graded version of the Morita-Takeuchi theorem which describes graded equivalences and in Section 5 we introduce and study the Picard groups for graded coalgebras.

2 Preliminaries

Throughout k is a fixed ground field and G is a group with identity element e. $\mathbf{M}_k$ denotes the category of k-vector spaces. All coalgebras, vector spaces and unadorned $\otimes$, Hom, etc, are over k.

For a coalgebra C, Δ and ε denote the comultiplication and the counit, respectively. The category of right C-comodules is denoted by $\mathbf{M}^C$; for X in $\mathbf{M}^C$ we denote the comodule structure by ρ_X or ω_X. For $X, Y \in \mathbf{M}^C$, $Com_{-C}(X,Y)$ denotes the space of right C-comodule maps from X to Y. Similarly, ${}^C\mathbf{M}$ denotes the category of left C-comodules. For $X \in {}^C\mathbf{M}$ and $Y \in \mathbf{M}^C$, $X \Box_C Y$ denotes the cotensor product of X and Y (see [D], [T]). If D is also a coalgebra, then X is a (D,C)-bicomodule if $X \in \mathbf{M}^C$ via ρ_X, $X \in {}^D\mathbf{M}$ via ${}_X\rho$ and $(1 \otimes \rho_X)\ {}_X\rho = (\ {}_X\rho \otimes 1)\rho_X$ (see [D],[T]).

Graded vector spaces: Let $\mathbf{gr}_k$ be the category of graded k-vector spaces, i.e., the objects are vector spaces V which admit a decomposition as a direct sum of k-spaces $V = \oplus_{g \in G} V_g$. For $V, W \in \mathbf{gr}_k$ a morphism from V to W is a k-linear map such that $f(V_g) \subseteq W_g$ for all $g \in G$. These maps are called graded linear maps and the set of graded linear maps from V to W is denoted by $Hom_{gr}(V,W)$. If $V = \oplus_{g \in G} V_g \in \mathbf{gr}_k$ and $g \in G$, we can define another graded vector spaces $V(g)$: as vector space $V(g)$ coincides with V but the grading of $V(g)$ is $V(g)_h = V_{gh}$ for all $h \in G$. Let $U_k : \mathbf{gr}_k \to \mathbf{M}_k$ be the forgetful functor, U_k is an exact functor and it has a right adjoint functor $F_k : \mathbf{M}_k \to \mathbf{gr}_k$. For $W \in \mathbf{M}_k$, $F_k(W) = \oplus_{g \in G} W^g$ with $W^g = W$ for all $g \in G$. If the group G is finite, then F_k is also a left adjoint of U_k. Moreover, for $V \in \mathbf{gr}_k$, we have $F_k U_k(V) \cong \oplus_{g \in G} V(g)$. It is known that $F_k U_k(k)$ is a generator and a cogenerator in $\mathbf{gr}_k$.

For $V, W \in \mathbf{gr}_k$ the graded tensor is given by $V \otimes^{gr} W = \oplus_{g \in G}(V \otimes^{gr} W)_g$ where $(V \otimes^{gr} W)_g = \oplus_{ab=g} V_a \otimes W_b$ for all $g \in G$ and

$$HOM(V,W) = \oplus_{g \in G} Hom_{gr}(V, W(g)).$$

Graded coalgebras (See [NT1], [DNRV]): A coalgebra C is called G-graded coalgebra if $C \in \mathbf{gr}_k$, that is, $C = \oplus_{\sigma \in G} C_\sigma$ and verifies:

i) $\Delta(C_\sigma) \subseteq \sum_{\lambda\mu=\sigma} C_\lambda \otimes C_\mu$ for any $\sigma \in G$;

ii) $\varepsilon(C_\sigma) = 0$ for any $\sigma \neq e$.

If M is a right C-comodule then M is called a G-graded comodule over C if $M \in \mathbf{gr}_k$, $M = \oplus_{\sigma \in G} M_\sigma$, and $\rho_M(M_\sigma) \subseteq \sum_{\lambda\mu=\sigma} M_\lambda \otimes C_\mu$ for any $\sigma \in G$.

For any element $m \in M$ we have the decomposition $m = \sum_{\sigma\in G} m_\sigma$, $m_\sigma \in M_\sigma$ (the sum has only a finite number of nonzero elements). The nonzero elements m_σ, $\sigma \in G$, are called the homogeneous components of m; m_σ is called the homogeneous component of degree σ and we write $deg(m_\sigma) = \sigma$.

Let $\mathbf{gr}^C$ be the category of right graded C-comodules. For $M, N \in \mathbf{gr}^C$ a morphism $f : M \to N$ is a graded linear map which is a morphism of C-comodules. This map is called graded C-colinear map and the set of these maps is denoted by $Com_{gr-C}(M, N)$. It is easy to verify that $\mathbf{gr}^C$ is an abelian category. (In fact $\mathbf{gr}^C$ is also a Grothendieck category). Analogously, we can define ${}^C\mathbf{gr}$, the category of all left G-graded C-comodules.

Let $M = \oplus_{\sigma\in G} M_\sigma$ be an object in $\mathbf{gr}^C$ and $\sigma \in G$. Then, the σ-suspension of M, $M(\sigma)$, is again an object in $\mathbf{gr}^C$. The map $M \mapsto M(\sigma)$ defines an isomorphism of categories from $\mathbf{gr}^C$ to $\mathbf{gr}^C$.

For $M, N \in \mathbf{gr}^C$, $COM(M, N) = \oplus_{g\in G} Com_{gr-C}(M, N(g))$. When M is finite dimensional, $COM(M, N) \cong Com_{-C}(U(M), U(N))$.

We write $U : \mathbf{gr}^C \to \mathbf{M}^C$ as the forgetful functor. U is an exact functor and it has an exact right adjoint functor $F : \mathbf{M}^C \to \mathbf{gr}^C$. Moreover, if the group G is finite, then F is also a left adjoint of U. In fact, these functors are the restriction of U_k and F_k considering $\mathbf{gr}^C$ as a subcategory of $\mathbf{gr}_k$.

If $M = \oplus_{\sigma\in G} M_\sigma$ is a graded right C-comodule, for any $\sigma \in G$ we write $\pi_\sigma^M : M \to M_\sigma$ as the canonical projection. We have that:

1) If $\sigma, \tau \in G$ there exists a unique k-linear map $u_{\sigma,\tau}^M : M_{\sigma\tau} \to M_\sigma \otimes C_\tau$ such that: $u_{\sigma,\tau}^M \pi_{\sigma\tau}^M = (\pi_\sigma^M \otimes \pi_\tau^C)\rho_M$.

2) For any σ, τ, $\lambda \in G$: $(u_{\sigma,\tau}^M \otimes 1)u_{\sigma\tau,\lambda}^M = (1 \otimes u_{\tau,\lambda}^C)u_{\sigma,\tau\lambda}^M$.

3) If $\sigma \in G$, $(1 \otimes \epsilon)u_{\sigma,e}^M = 1$.

If we write $\Delta_e = u_{e,e}^C : C_e \to C_e \otimes C_e$, then $(C_e, \Delta_e, \epsilon)$ is a coalgebra and $\pi_e : C \to C_e$ is a morphism of coalgebras. $C = \oplus_{\sigma\in G} C_\sigma$ is called a *strongly graded* coalgebra if the canonical morphisms $u_{\sigma,\tau}^C : C_{\sigma\tau} \to C_\sigma \otimes C_\tau$ are monomorphism. C is a strongly graded coalgebra if and only if the coinduced functor is an equivalence of categories.

Every graded coalgebra C has associated other coalgebra $C \rtimes kG$, called smash coproduct, constructed in the following way: as vector space $C \rtimes kG = C \otimes kG$, for any homogeneus element $c \in C$ and $g \in G$ the comultiplication is $\Delta(c \rtimes g) = \sum_{(c)} (c_{(1)} \rtimes deg(c_{(2)})g) \otimes (c_{(2)} \rtimes g)$ and the counit $\varepsilon(c \rtimes g) = \varepsilon(c)$.

For $M \in \mathbf{gr}^C$ we make M into a right $C \rtimes kG$-comodule via $\rho : M \to M \otimes$

$C \bowtie kG$, $m \mapsto \sum_{(m)} m_{(0)} \otimes m_{(1)} \bowtie deg(m)^{-1}$ for homogeneous $m \in M$. Any morphism $f : M \to N$ of graded comodules is also a morphism of $C \bowtie kG$-comodules. Thus, we have defined a functor $A : \mathbf{gr}^C \to \mathbf{M}^{C \bowtie kG}$ and this functor verifies that $A(\oplus_{g\in G} C(g)) \cong C \bowtie kG$ as right $C \bowtie kG$-comodules. In [DNRV, Th. 1.6], it was proved that A defines an isomorphism between the categories $\mathbf{gr}^C$ and $\mathbf{M}^{C \bowtie kG}$. Hence, the category $\mathbf{gr}^C$ is a locally finite category (see [T, Def. 4.1]).

Clifford Theory for Graded Coalgebras (See [NT2]): $X \in \mathbf{gr}^C$ is called gr-injective if X is an injective object in the category $\mathbf{gr}^C$. $X \in \mathbf{gr}^C$ is gr-injective if and only if $U(X)$ is injective in $\mathbf{M}^C$. $S \in \mathbf{gr}^C$ is called gr-simple if it has no proper graded subcomodules. Every gr-simple comodule is of finite dimension.

Since the category $\mathbf{gr}^C$ is locally finite, then it is locally noetherian and it is well-known that an injective object $X \in \mathbf{gr}^C$ has a unique decomposition $X = \oplus_{i\in I} X_i$ with every X_i injective indecomposable object. If $Q \in \mathbf{gr}^C$ is injective indecomposable, then $Q = E^{gr}(S)$ where $E^{gr}(S)$ denotes the injective envelope of S and S is a gr-simple subcomodule of Q. S always exists because $\mathbf{gr}^C$ is locally finite. Hence every gr-injective comodule X is of the form $X = \oplus_{i\in I} E^{gr}(S_i)$ with S_i gr-simple for all $i \in I$.

For every simple right comodule $S \in \mathbf{M}^C$, there is a gr-simple comodule $S' \in \mathbf{gr}^C$ such that S is isomorphic to a C-submodule of S'.

Co-hom functor (See [T]): Let C, D be coalgebras, a comodule X_C is quasi-finite if $Com_{-C}(Y, X)$ is finite dimensional for all finite dimensional comodules Y_C. Let ${}_C X_D$ be a bicomodule, then X_D is quasi-finite if and only if the functor $-\square_C X : \mathbf{M}^C \to \mathbf{M}^D$ has a left adjoint functor, denoted by $h_{-D}(X, -)$. That is, for comodules Y_D and W_C,

$$Com_{-C}(h_{-D}(X, Y), W) \cong Com_{-D}(Y, W \square_C X). \tag{1}$$

Where, $h_{-D}(X, Y) = \varinjlim_{\mu} Com_{-D}(Y_\mu, X)^*$ is a right C-comodule, $\{Y_\mu\}$ is a directed family of any finite dimensional subcomodules of Y_D. Let θ denote the canonical D-colinear map $Y \to h_{-D}(X, Y)\square_C X$ which corresponds to the identity map $h_{-D}(X, Y) \to h_{-D}(X, Y)$ in (1). Similarly, there is a left version of the above lemma for a quasi-finite comodule ${}_C X$.

Assume that X_D is a quasi-finite comodule. The $e_{-D}(X) = h_{-D}(X, X)$ is a coalgebra, called the co-endomorphism coalgebra of X. The comultiplication of $e_{-D}(X)$ corresponds to $(1 \otimes \theta)\theta : X \to e_{-D}(X) \otimes e_{-D}(X) \otimes X$ in (1) when $C = k$, and the counit of $e_{-D}(X)$ corresponds to the identity map 1_X. X is a left $(e_{-D}(X) - D)$-bicomodule with the left comodule structure map θ, the canonical map $X \to h_{-D}(X,X) \otimes X$.

Let ${}_CX_D$ be a bicomodule such that X_D is quasi-finite. Then there exists a coalgebra map $\lambda : e_{-D}(X) \to C$ such that the left C-comodule structure equals $(\lambda \otimes 1)\theta$. Conversely, a coalgebra $\lambda : e_{-D}(X) \to C$ makes X into a $(C - D)$-bicomodule. Moreover, the $(C - C)$-bicomodule structures of $e_{-D}(X)$ through λ coincides with the induced $(C - C)$-bicomodule structures of $h_{-D}({}_CX_D, {}_CX_D)$.

Morita-Takeuchi (M-T) context (See [T]): A M-T context $(C, D, {}_CP_D, {}_DQ_C, f, g)$ consists of coalgebras C, D, bicomodules ${}_CP_D, {}_DQ_C$, and bicolinear maps $f : C \to P\Box_D Q$ and $g : D \to Q\Box_C P$ satisfying the following commutative diagrams:

$$\begin{tikzcd} P \arrow[r, "\rho_D"] \arrow[d, "{}_C\rho"'] & P\Box_D D \arrow[d, "1\Box g"] \\ C\Box_C P \arrow[r, "f\Box 1"'] & P\Box_D Q\Box_C P \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[r, "\rho_C"] \arrow[d, "{}_D\rho"'] & Q\Box C \arrow[d, "1\Box f"] \\ D\Box Q \arrow[r, "g\Box 1"'] & Q\Box_C P\Box_D Q \end{tikzcd}$$

The context is said to be strict if both f and g are injective (equivalently, isomorphism). In this case we say that C is M-T equivalent to D, denoted by $C \sim D$. Let P_D be a quasi-finite comodule and $C = e_{-D}(P)$. ${}_CP_D$ is a bicomodule. Set ${}_DQ_C = h_{-D}(P, D)$, $g = \theta : D \to Q\Box_C P$, and $f : C \cong h_{-D}(P, P\Box_D D) \to P\Box_D h_{-D}(P, D) = P\Box_D Q$. Then $(C, D, {}_CP_D, {}_DQ_C, f, g)$ is a M-T context, where f is injective if and only if P_D is injective, and g is injective if and only if P_D is a cogenerator in $\mathbf{M}^D$.

The cocenter (See [TVZ]): Let D be a coalgebra and D^e its enveloping coalgebra, that is, $D^e = D \otimes D^{op}$. View D as a right D^e-comodule in the usual way. Then D_{D^e} is quasi-finite and the co-endomorphism coalgebra $e_{-D^e}(D)$ satisfies the following universal properties:

1.- $e_{-D^e}(D)$ is a cocommutative coalgebra with a surjective coalgebra map $\eta : D \to e_{-D^e}(D)$ which cocommutes with 1_D, i.e, $\sum \eta(d_{(1)}) \otimes d_{(2)} = \sum d_{(2)} \otimes \eta(d_{(1)}),\ d \in D$.

2.- For any coalgebra E and any coalgebra map $f : D \to E$ which cocommutes with the identity map 1_D, there exists a unique coalgebra map $g : e_{-D^e}(D) \to E$ such that $f = g\eta$.

Let $(Z(D), \epsilon_D)$ denote $(e_{-D^e}(D), \epsilon)$, called the cocenter of D. From the above universal property, the cocenter of a coalgebra is unique up to isomorphism. A coalgebra map $f : D \to E$ is said to be cocentral if f cocommutes with the identity map 1_D, i.e, $\sum f(d_{(1)}) \otimes d_{(2)} = \sum f(d_{(2)}) \otimes d_{(1)},\ d \in D$. Let C be a cocommutative coalgebra, D is said to be a C-coalgebra if there is a cocentral coalgebra map $\epsilon : D \to C$. If D is cocommutative, then $Z(D) \cong D$.

Picard group (See [TZ]): Let C, D be coalgebras. A $(C - D)$-bicomodule M is said to be invertible if the functor $-\square_C : \mathbf{M}^C \to \mathbf{M}^D$ defines a Morita-Takeuchi equivalence between $\mathbf{M}^C$ and $\mathbf{M}^D$. The Picard group of C, denoted by $Pic_k(C)$ is the multiplicative group consisting of all bicomodule isomorphism classes $[M]$ of invertible comodules ${}_CM_C$. When C is cocommutative, $Pic(C) \cong Aut(C)$, the set of automorphism of the coalgebra C.

3 The co-HOM functor

For the rest of the section, C, D, E, Γ are graded coalgebras.

Definition 3.1 *Let $X \in \mathbf{gr}^C$.*

i) X is said to be quasi-finite in $\mathbf{gr}^C$ if $Com_{gr-C}(Y, X)$ is finite dimensional for all $Y \in \mathbf{gr}^C$ of finite dimension.

ii) X is called gr-quasi-finite if $COM(Y, X)$ is finite dimensional for all $Y \in \mathbf{gr}^C$ of finite dimension.

iii) We say that X is quasi-finite if $U(X)$ is quasi-finite in $\mathbf{M}^C$.

The relations among these concepts are the following:

Proposition 3.2 *Let $X \in \mathbf{gr}^C$.*

i) If X is gr-quasi-finite then X is quasi-finite in $\mathbf{gr}^C$. If G is finite, then the converse is true.

ii) If $U(X)$ is quasi-finite, then X is gr-quasi-finite.

iii) If X is quasi-finite in $\mathbf{gr}^C$ and G is finite, then $U(X)$ is quasi-finite.

Proof: Let $Y \in \mathbf{gr}^C$ of finite dimension.

i) The first claim is deduced from $Com_{gr-C}(Y, X) \subseteq COM(Y, X)$. For the converse, as X is quasi-finite in $\mathbf{gr}^C$, then $Com_{gr-C}(Y(g^{-1}), X)$ is finite dimensional for every $g \in G$. Now, since G is finite and $COM(Y, X) = \oplus_{g\in G} Com_{gr-C}(Y(g^{-1}), X)$ we deduce that X is gr-quasi-finite.

ii) It is deduced from the fact that $COM(Y, X) = Com_{-C}(U(Y), U(X))$, cf. [NT1, page 478].

iii) If G is finite then U has a left adjoint functor F, that is, $Com_{gr-C}(F(M), N) \cong Com_{-C}(M, U(N))$ for $M \in \mathbf{M}^C$ and $N \in \mathbf{gr}^C$. Let $Y \in \mathbf{M}^C$ finite dimensional. Since G is finite, $F(Y)$ is finite dimensional and by hypothesis $Com_{gr-C}(F(Y), X)$ is finite dimensional. Thus $Com_{-C}(Y, U(X))$ is finite dimensional. ∎

EXAMPLES:

1.- Let G be an infinite group and we write $M = \oplus_{g\in G} C(g)$, next we prove that M is quasi-finite in $\mathbf{gr}^C$ and however $U(M)$ is not quasi-finite in $\mathbf{M}^C$: let Y be in $\mathbf{gr}^C$ of finite dimension, then

$$Com_{gr-C}(Y, \oplus_{g\in G} C(g)) \cong Com_{gr-C}(Y, FU(C)) \cong Com_{-C}(U(Y), U(C))$$

Since $U(C)$ is quasi-finite then $Com_{gr-C}(Y, \oplus_{g\in G} C(g))$ is finite dimensional, that is, M is quasi-finite in $\mathbf{gr}^C$. However,

$$\begin{aligned} Com_{-C}(U(Y), U(M)) &\cong Com_{-C}(U(Y), \oplus_{g\in G} U(C)) \\ &\cong \oplus_{g\in G} Com_{-C}(U(Y), U(C)) \end{aligned}$$

Because G is infinite we have that $Com_{-C}(U(Y), U(M))$ is not finite dimensional.

2.- Let G be a torsionfree group and $X \in \mathbf{gr}^C$ gr-quasi-finite, then $U(X)$ is quasi-finite in $\mathbf{M}^C$: let S be in $\mathbf{M}^C$ a simple comodule, since G is a torsionfree group, by [NT2, Cor. 4.6, ii)] there is graded simple comodule $S' \in \mathbf{gr}^C$ such that $U(S') = S$. Then,

$$\begin{aligned} Com_C(S, U(X)) &\cong Com_{-C}(U(S'), U(X)) \cong Com_{gr-C}(S', \oplus_{g\in G} X(g)) \\ &\cong \oplus_{g\in G} Com_{gr-C}(S', X(g)) = COM(S', X), \end{aligned}$$

where, in the second isomorphism, we have used the right adjoint of U and, in the third one, that every gr-simple comodule is finite dimensional. $COM(S', X)$ is finite dimensional because X is gr-quasi-finite and thus $Com_{-C}(S, U(X))$ is of finite dimension. By [T, Prop. 4.5] $U(X)$ is quasi-finite.

3.- Let X be in $\mathbf{gr}^C$ and we suposse that X is gr-injective. Then, X is gr-quasi-finite if and only if $U(X)$ is quasi-finite. Let S be in $\mathbf{M}^C$ a simple comodule, from [NT2, Cor. 4.6 i)] there is a graded simple comodule S' and an injective C-comodule map $f : S \to U(S')$. As X is gr-injective, from [NT2, Cor. 3.4] $U(X)$ is injective and hence the functor $Com_{-C}(-, U(X))$ is exact. Thus we have a surjective linear map $f_* : Com_{-C}(U(S'), U(X)) \to Com_{-C}(S, U(X))$. Also, $Com_{-C}(U(S'), U(X)) \cong COM(S', X)$ because S' is finite dimensional. Since X is gr-quasi-finite, then $COM(S', X)$ is finite dimensional and hence $Com_{-C}(S, U(X))$ is finite dimensional. By [T, Prop. 4.5] $U(X)$ is quasi-finite in $\mathbf{M}^C$.

Proposition 3.3 *Let $X \in \mathbf{gr}^C$, the following assertions are equivalent:*

i) X is gr-quasi-finite.

ii) For all gr-simple $S \in \mathbf{gr}^C$, $COM(S, X)$ is finite dimensional.

iii) $U(X)$ is quasi-finite in $\mathbf{M}^C$.

Proof: *i)* $\Rightarrow$ *ii)* It is clear because every gr-simple comodule is finite dimensional.

ii) $\Rightarrow$ *iii)* First, we prove that given $X \in \mathbf{gr}^C$ such that $soc^{gr}(X)$ verifies ii), then X verifies ii). Let $S \in \mathbf{gr}^C$ gr-simple, then $COM(S, soc^{gr}(X)) \hookrightarrow COM(S, X)$. $COM(S, soc^{gr}(X)) = \oplus_{g\in G} Com_{gr-C}(S(g^{-1}), soc^{gr}(X))$ and

$$COM(S, X) = \oplus_{g\in G} Com_{gr-C}(S(g^{-1}), X).$$

Since $soc^{gr}(X)$ is quasi-finite, there is a finite number of $g \in G$ such that $Com_{gr-C}(S(g^{-1}), soc^{gr}(X)) \neq \{0\}$ and these are finite dimensional. But

$$Com_{gr-C}(S(g^{-1}), soc^{gr}(X)) = Com_{gr-C}(S(g^{-1}), X)$$

since $S(g^{-1})$ is simple. Hence $COM(S, X)$ is finite dimensional.

Suppose that X verifies ii) and we take the injective envelope $X \hookrightarrow E^{gr}(X)$. Since X verifies ii) then $soc^{gr}(X) \cong^{gr} soc^{gr}(E^{gr}(X))$ verifies ii) and so $E^{gr}(X)$ satisfies ii). As $E^{gr}(X)$ is injective and satisfies ii), Example 3 yields that $U(E^{gr}(X))$ is quasi-finite and injective. Let $S \in \mathbf{M}^C$ be simple and let $S' \in \mathbf{gr}^C$ gr-simple such that $S \hookrightarrow U(S')$, then the induced map

$$COM(S', E^{gr}(X)) \cong Com_{-C}(U(S'), U(E^{gr}(X))) \to Com_{-C}(S, U(E^{gr}(X)))$$

is surjective. Hence $Com_{-C}(S, U(E^{gr}(X)))$ is finite dimensional. Now, since $Com_{-C}(S, U(X)) \hookrightarrow Com_{-C}(S, U(E^{gr}(X)))$, we obtain that $U(X)$ is quasi-finite.

iii) $\Rightarrow$ *i)* It is ii) of the above proposition. ∎

Lemma 3.4 *Let $W \in \mathbf{gr}_k$, $X, Y \in \mathbf{gr}^C$. Then:*

i) If Y is finite dimensional, $W \otimes^{gr} COM(Y, X) \cong^{gr} COM(Y, W \otimes X)$.

ii) If $COM(Y, X)$ is finite dimensional, then

$$W \otimes^{gr} COM(Y, X) \cong^{gr} HOM(COM(Y, X)^*, W).$$

iii) With the hypothesis of the above items we have

$$Com_{gr-C}(Y, W \otimes^{gr} X) \cong Hom_{gr}(COM(Y, X)^*, W).$$

Proof: *i)* For $w \in W$, $f \in COM(Y, X)$ and $y \in Y$, the isomorphism ϕ is given by $\phi(w \otimes f)(y) = w \otimes f(y)$.

ii) For the same preceding elments and $h \in COM(Y, X)^*$ this isomorphism φ is given by $\varphi(w \otimes f)(h) = wh(f)$.

iii) It follows by the composite of the two above isomorphisms for the component of degree e.

Proposition 3.5 *Let X in $\mathbf{gr}^C$, the following assertions are equivalents:*

i) X is quasi-finite in $\mathbf{gr}^C$.

ii) The functor $\mathbf{M}_k \to \mathbf{gr}^C$, $W \to W \otimes X$ (where the grading in $W \otimes X$ is $(W \otimes X)_g = W \otimes X_g$ for all $g \in G$) has a left adjoint functor.

Proof: This proof is similar to the proof of [T, Proposition 1.3]. ∎

Definition 3.6 *Let X be in $\mathbf{gr}^C$ quasi-finite in $\mathbf{gr}^C$, the left adjoint functor of $W \to W \otimes X$ is denoted by $Y \to h_{gr-C}(X,Y)$ and it is called graded co-hom functor. We have that $Com_{gr-C}(Y, W \otimes X) \cong Hom_k(h_{gr-C}(X,Y), W)$*

Proposition 3.7 *Let X be in $\mathbf{gr}^C$, the following assertions are equivalent:*

i) X is gr-quasi-finite.

ii) The functor $\mathbf{M}_k \to \mathbf{gr}^C$ given by the composite $W \to W \otimes X \to FU(W \otimes X) = \oplus_{g \in G}(W \otimes X)(g)$ (where the grading in $W \otimes X$ is the same that in the above proposition) has a left adjoint functor.

Proof: $i) \Rightarrow ii)$ For a graded comodule of finite dimension Y note that:

$$\begin{aligned} COM(Y,X) &\cong Com_{-C}(U(Y), U(X)) \\ &\cong Com_{gr-C}(Y, FU(X)) \cong Com_{gr-C}(Y, \oplus_{g\in G} X(g)) \end{aligned}$$

Taking in acount this fact, the proof is anologous to the proof of [T, Prop. 1.3]. The adjoint is given by $\varinjlim_{\lambda} COM(Y_\lambda, X)^*$ where $\{Y_\lambda\}_{\lambda \in \Lambda}$ is a family of finite dimensional graded subcomodules of Y such that $Y = \varinjlim_{\lambda} Y_\lambda$. ∎

Proposition 3.8 *For $X \in \mathbf{gr}^C$, the following assertions are equivalent:*

i) X is gr-quasi-finite.

ii) The functor $\mathbf{gr}_k \to \mathbf{gr}^C$, $W \to W \otimes^{gr} X$ (where the grading in $W \otimes^{gr} X$ is $(W \otimes^{gr} X)_g = \oplus_{ab=g} W_a \otimes X_b$ for all $g \in G$) has a left adjoint functor.

Proof: $i) \Rightarrow ii)$ If $Y \in \mathbf{gr}^C$ is of finite dimension and $W \in \mathbf{gr}_k$, we know from Lemma 3.4 that $Com_{gr-C}(Y, W \otimes^{gr} X) \cong Hom_{gr}(COM(Y,X)^*, W)$. Let Y be in $\mathbf{gr}^C$ and let $\{Y_\lambda\}$ be a family of finite dimensional graded subcomodules of Y such that $Y = \varinjlim_{\lambda} Y_\lambda$, then we have:

$$
\begin{aligned}
&Com_{gr-C}(Y, W\otimes^{gr} X) \cong \lim_{\overleftarrow{\lambda}} Com_{gr-C}(Y_\lambda, W\otimes^{gr} X)\\
&\quad\cong \lim_{\overleftarrow{\lambda}} Hom_{gr}(COM(Y_\lambda, X)^*, W) \cong Hom_{gr}(\lim_{\overrightarrow{\lambda}} COM(Y_\lambda, X)^*, W)
\end{aligned}
$$

ii) $\Rightarrow$ *i)* For $W \in \mathbf{gr}_k$, we have that $F_k(W)\otimes^{gr} X \cong \oplus_{g\in G}(W\otimes X)(g)$. We suppose that the functor $\mathbf{gr}_k \to \mathbf{gr}^C$, $W \to W\otimes^{gr} X$ has a left adjoint functor which is denoted by $H_{-C}(X,-)$, then we prove that the functor $\mathbf{M}_k \to \mathbf{gr}^C$, $W \to \oplus_{g\in G}(W\otimes X)(g)$ has a left adjoint. Let $Y \in \mathbf{gr}^C$ and $W \in \mathbf{M}_k$, then

$$
\begin{aligned}
&Com_{gr-C}(Y, \oplus_{g\in G}(W\otimes X)(g)) \cong Com_{gr-C}(Y, F_k(W)\otimes^{gr} X)\\
&\quad\cong Hom_{gr}(H_{-C}(X,Y), F_k(W)) \cong Hom_k(U_k(H_{-C}(X,Y)), W)
\end{aligned}
$$

From the above proposition, we obtain that X is gr-quasi-finite. ∎

Definition 3.9 *For a gr-quasi-finite graded comodule X, the functor $H_{-C}(X,-) : \mathbf{gr}^C \to \mathbf{gr}_k$ is called co-HOM functor.*

Next, we study some properties of the co-HOM functor $H_{-C}(X,-)$ for a gr-quasi-finite graded comodule X:

P1. $H_{-C}(X,Y) = \oplus_{g\in G} h_{gr-C}(X(g), Y)$.

P2. (Universal Property): We denote by $\theta : Y \to H_{-C}(X,Y)\otimes^{gr} X$ the graded C-colinear map associated to the identity in $H_{-C}(X,Y)$. This map verifies the following: for every $W \in \mathbf{gr}_k$ and every graded C-colinear map $f : Y \to W\otimes^{gr} X$ there is a unique graded linear map $s : H_{-C}(X,Y) \to W$ such that $f = (s\otimes 1)\theta$.

P3. If $X \in \mathbf{gr}^C$ and $U(X)$ is quasi-finite, then $U_k(H_{-C}(X,Y)) \cong h_{-C}(U(X)$ $,U(Y))$ for all $Y \in \mathbf{gr}^C$. Moreover, the following square is commutative:

$$
\begin{array}{ccc}
Com_{gr-C}(Y, W\otimes^{gr} X) & \cong & Hom_{gr}(H_{-C}(X,Y), W)\\
\Big\downarrow {\scriptstyle U_C} & & \Big\downarrow {\scriptstyle U_k}\\
Com_{-C}(U(Y), U_k(W)\otimes U(X)) & \cong & Hom_k(h_{-C}(U(X), U(Y)), U_k(W))
\end{array}
$$

The C-colinear map associated to the identity in $h_{-C}(U(X), U(Y))$ is $U(\theta)$ and with f as in P2, $U_k(s)$ is the unique linear map such that $U(f) = (U_k(s)\otimes 1)U(\theta)$.

P4. Let $f : X' \to X$ and $g : Y \to Y'$ be graded C-colinear maps where X and X' are gr-quasi-finite. The composite

$$Y \xrightarrow{\quad g \quad} Y' \xrightarrow{\quad \theta \quad} H_{-C}(X', Y') \otimes^{gr} X' \xrightarrow{1 \otimes f} H_{-C}(X', Y') \otimes^{gr} X$$

is graded and there exists a unique graded linear map $H(f, g) : H_{-C}(X, Y) \to H_{-C}(X', Y')$ such that the above composite is of the form $(H(f, g) \otimes 1)\theta$.

P5. For a gr-quasi-finite graded comodule $X \in \mathbf{gr}^C$, the functor $H_{-C}(X, -)$ is right exact and preserves direct sums because it has a right adjoint functor.

P6. A *graded* (C, D) *-bicomodule* Y is a (C, D)-bicomodule which is graded as C-comodule and the structure map ${}_D\omega : Y \to D \otimes^{gr} Y$ is a graded C-colinear map. The composite $(1 \otimes \theta)_D\omega$ is graded C-colinear and by the universal property of P2 there is a unique graded linear map ${}_D\rho : H_{-C}(X, Y) \to D \otimes H_{-C}(X, Y)$ such that $(1 \otimes \theta)_D\omega = ({}_D\rho \otimes 1)\theta$. $H_{-C}(X, Y)$ becomes a left graded D-comodule with the map ${}_D\rho$.

If X is a graded (E,C)-bicomodule such that X as graded C-comodule is gr-quasi-finite and Y is a right graded C-comodule, then there is a unique graded linear map $\rho_E : H_{-C}(X, Y) \to H_{-C}(X, Y) \otimes E$ such that $(1 \otimes_E \omega^X)\theta = (\rho_E \otimes 1)\theta$. $H_{-C}(X, Y)$ becomes a right graded D-comodule with the map ρ_E. Moreover, if Y is a graded (D,C)-bicomodule, then $H_{-C}(X, Y)$ is a graded (D,E)-bicomodule with the maps ρ_E and ${}_D\rho$.

P7 (Graded cotensor): Let X be in $\mathbf{gr}^C$ and Y in ${}^C\mathbf{gr}$, for $g \in G$ we define $(X \Box_C^{gr} Y)_g = (\oplus_{ab=g} X_a \otimes Y_b) \cap (X \Box_C Y)$ and $X \Box_C^{gr} Y = \oplus_{g \in G} (X \Box_C^{gr} Y)_g$. The graded cotensor verifies the following:

1. $X \Box_C^{gr} C \cong X$ as graded C-comodules. The graded isomorphism is induced by the map $1 \otimes \varepsilon : X \otimes C \to X$. Similarly, $C \Box_C^{gr} X \cong X$ for $X \in {}^C\mathbf{gr}$.

2. If $f : X \to X'$ and $g : Y \to Y'$ are graded C-colinear maps, then the graded map $f \otimes^{gr} g : X \otimes^{gr} Y \to X' \otimes^{gr} Y'$ induces a graded linear map $f \Box_C^{gr} g : X \Box_C^{gr} Y \to X' \Box_C^{gr} Y'$.

3. $- \Box_C^{gr} X : {}^C\mathbf{gr} \to \mathbf{gr}_k$ is a left exact functor.

4. $- \Box_C^{gr} X$ commutes with direct sums.

5. If X and Y are graded (E,C) and (C,D)-bicomodules respectively, then $X \Box_C^{gr} Y$ is a graded (E,D)-bicomodule with the maps ${}_E\omega^X \otimes 1$ and $1 \otimes \omega_D^Y$.

6. $U(X \Box_C^{gr} Y) = U(X) \Box_C U(Y)$.

P8. In P6 we saw that if X is a graded (E,C)-bicomodule such that X is gr-quasi-finite as graded C-comodule and Y is a right graded C-comodule, $H_{-C}(X,Y)$ is a left graded C-comodule with the only map ${}_D\rho : H_{-C}(X,Y) \to D \otimes H_{-C}(X,Y)$ which verifies $(1 \otimes_D \omega^Y)\theta = ({}_D\rho \otimes 1)\theta$. This means that $Im(\theta) \subseteq H_{-C}(X,Y)\square_C^{gr} X$.

If ${}_EX_C$ and ${}_DY_C$ are graded bicomodules where X_C is gr-quasi- finite, then $H_{-C}(X,Y)$ is a graded (D,E)-bicomodule and $\theta : Y \to H_{-C}(X,Y)\square_E^{gr} X$ is graded (D,C)-bicolinear.

Proposition 3.10 *For a graded bicomodule ${}_DX_C$, the following assertions are equivalents:*

i) *X_C is gr-quasi-finite.*

ii) *The functor $\mathbf{M}^D \to \mathbf{M}^C$, $Z \mapsto Z\square_D^{gr} X$ has left adjoint functor.*

Proof: Using the above properties and Proposition 3.8, this proof is analogous to the proof of [T, Prop. 1.10]. ∎

P9. If ${}_EX_C$, ${}_DY_C$ and ${}_DZ_E$ are graded bicomodules where X_C is gr-quasi-finite and $f : Y \to Z\square_D^{gr} X$ is a graded (D,C)-bicolinear map, then the graded associated map $u : H_{-C}(X,Y) \to Z$ is (D,E)-bicolinear.

P10. Suppose that $X \in \mathbf{gr}^C$ is gr-quasi-finite and gr-injective, then by [NT2, Cor. 3.4] $U(X)$ is injective and hence the functor $Com_{-C}(-,U(X))$ is exact. Since $U : \mathbf{gr}^C \to \mathbf{M}^C$ is exact, the functor $Com_{-C}(U(-),U(X)) : \mathbf{gr}^C \to \mathbf{M}_k$ is exact. From the isomorphism

$$\begin{aligned} H_{-C}(X,Y)^* &= Hom_k(\lim_{\underset{\lambda}{\longrightarrow}} COM(Y_\lambda,X)^*,k) \\ &\cong \lim_{\underset{\lambda}{\longleftarrow}} Hom_k(COM(Y_\lambda,X)^*,k) \\ &\cong \lim_{\underset{\lambda}{\longleftarrow}} COM(Y_\lambda,X)^{**} \cong \lim_{\underset{\lambda}{\longleftarrow}} COM(Y_\lambda,X) \\ &\cong \lim_{\underset{\lambda}{\longleftarrow}} Com_{-C}(U(Y_\lambda),U(X)) \cong Com_{-C}(U(Y),U(X)) \end{aligned}$$

we obtain that $H_{-C}(X,-)$ is exact when X is gr-injective and gr-quasi-finite.

P11. For graded comodules X_C, Z_D and a graded bicomodule ${}_DY_C$ with X_C gr-quasi-finite, there exists a unique graded linear map $\delta : H_{-C}(X, Z\square_D^{gr} Y) \to Z\square_D^{gr} H_{-C}(X,Y)$ making the below triangle commutative:

$$\begin{tikzcd}
Z\square^{gr}_D Y \arrow[rr, "1\square^{gr}\theta"] \arrow[dr, "\theta"'] & & Z\square^{gr}_D H_{-C}(X,Y)\otimes X \\
& H_{-C}(X, Z\square^{gr}_D Y)\otimes X \arrow[ur, "\delta\otimes 1"'] &
\end{tikzcd}$$

If we suppose that $U(X)$ is quasi-finite and X_C is gr-inyective and hence $U(X)$ is injective, then δ is an isomorphism because the above triangle induces another commutative triangle by forgeting the maps and the objects:

$$\begin{tikzcd}
Z\square_D Y \arrow[rr, "1\square U(\theta)"] \arrow[dr, "U(\theta)"'] & & Z\square_D h_{-C}(X,Y)\otimes X \\
& h_{-C}(X, Z\square_D Y)\otimes X \arrow[ur, "U(\delta)\otimes 1"'] &
\end{tikzcd}$$

From [T, Prop. 1.14], $U(\delta)$ is an isomorphism and therefore δ is a graded isomorphism. Moreover, if X, Z are in addition graded bicomodules ${}_E X_C$, ${}_\Gamma Z_D$, then the map δ is graded (Γ, E)-bicolinear.

P12. Let X_C, W_Γ, ${}_\Gamma Z_D$ and ${}_D Y_C$ graded comodules and bicomodules where $U(X)$ is quasi-finite, then the above map δ verifies the following diagram is commmutative:

$$\begin{tikzcd}
W\square^{gr}_\Gamma H_{-C}(X, Z\square^{gr}_D Y) \arrow[r, "1\square^{gr}\delta"] & W\square^{gr}_\Gamma Z\square^{gr}_D H_{-C}(X,Y) \\
H_{-C}(X, W\square^{gr}_\Gamma(Z\square^{gr}_D Y) \arrow[u, "\delta"] \arrow[r, phantom, "\cong"] & H_{-C}(X, (W\square^{gr}_\Gamma Z)\square^{gr}_D Y) \arrow[u, "\delta"']
\end{tikzcd}$$

P13. (Graded Co-endomorphism coalgebra): Let X in $\mathbf{gr}^C$ gr-quasi-finite and we write $E_{-C}(X) = H_{-C}(X, X)$. The map $(1\otimes\theta)\theta : X \to E_{-C}(X)\otimes^{gr} E_{-C}(X)\otimes^{gr} X$ is graded and there is a unique graded linear map $\Delta : E_{-C}(X) \to E_{-C}(X)\otimes^{gr} E_{-C}(X)$ such that $(1\otimes\theta)\theta = (\Delta\otimes 1)\theta$. Δ verifies that $(\Delta\otimes 1)\Delta = (1\otimes\Delta)\Delta$.

Viewing k trivially graded (i.e. $k_e = k$ and $k_g = 0$ for all $g \neq e$) we can identify $X \cong^{gr} k \otimes^{gr} X$, then the identity map $1_X : X \to X \cong^{gr} k \otimes^{gr} X$ is of the form $(\varepsilon \otimes 1)\theta$ for a unique graded linear map $\varepsilon : E_{-C}(X) \to k$. Thus $\varepsilon(E_{-C}(X)_g) = 0$ for all $g \neq e$. The map ε satisfies that $(1 \otimes \varepsilon)\Delta = (\varepsilon \otimes 1)\Delta = 1_{E_{-C}(X)}$ and so $E_{-C}(X)$ becomes a graded coalgebra with Δ and ε.

Moreover, if X is a graded (D, C)-bicomodule, for the structure map ${}_D\omega : X \to D \otimes X$ there is a unique graded linear map $u : E_{-C}(X) \to D$ verifying ${}_D\omega = (u \otimes 1)\theta$. This map is a graded coalgebra map. If we suppose that U(X) is quasi-finite, then $U_k(E_{-C}(X)) \cong e_{-C}(U(X))$.

P14. Similarly, for a left graded gr-quasi-finite comodule we can define $H_{C-}(X, -) : {}^C\mathbf{gr} \to \mathbf{gr}_k$ and $E_{C-}(X)$.

4 Graded Equivalences for Coalgebras

Definition 4.1 *Let C, D be graded coalgebras. A linear functor $V : \mathbf{gr}^C \to \mathbf{gr}^D$ is said to be graded if V commutes with the shift, that is, $V(X(g)) \cong^{gr} V(X)(g)$ for all $g \in G$.*

Proposition 4.2 *Let ${}_DX_C$ be a graded bicomodule.*

i) The graded cotensor functor $-\square_D^{gr} X : \mathbf{gr}^D \to \mathbf{gr}^C$ is a graded functor and $U(-\square_D^{gr} X) \cong U(-)\square_D U(X)$.

ii) If $U(X)_C$ is quasi-finite the co-HOM functor $H_{-C}(X, -) : \mathbf{gr}^C \to \mathbf{gr}^D$ is a graded functor and $U \circ H_{-C}(X, -) \cong h_{-C}(U(X), -) \circ U$.

Proof: We only have to show that both functors commute with the g-shift because the other claims are known. Let $g \in G$,

i) If $h \in G$,

$$
\begin{aligned}
(Y(g)\square_D^{gr} X)_h &= (\textstyle\sum_{ab=h} Y(g)_a \otimes X_b) \cap (Y \square_C X) \\
&= (\textstyle\sum_{ab=h} Y_{ga} \otimes X_b) \cap (Y \square_C X) \\
&= \textstyle\sum_{\alpha\beta=gh} Y_\alpha \otimes X_\beta \cap (Y \square_C X) = (Y \square_D^{gr} X)_{gh} = (Y \square_D^{gr} X)(g)_h.
\end{aligned}
$$

ii)

$$
\begin{aligned}
h_{gr-C}(X,Y(g)) &= \lim_{\overrightarrow{\lambda}} Com_{gr-C}(Y(g)_\lambda, X)^* \cong \lim_{\overrightarrow{\lambda}} Com_{gr-C}(Y_\lambda(g), X)^* \\
&\cong \lim_{\overrightarrow{\lambda}} Com_{gr-C}(Y_\lambda, X(g^{-1}))^* \cong h_{gr-C}(X(g^{-1}), Y),
\end{aligned}
$$

(where in the second isomorphism we have used that the shift $Y \mapsto Y(g)$ is an isomorphism of the category $\mathbf{gr}^C$). From this, we obtain:

$$
\begin{aligned}
H_{-C}(X,Y(g)) &\cong \oplus_{h\in G}\, h_{gr-C}(X(h^{-1}),Y(g)) \cong \oplus_{h\in G}\, h_{gr-C}(X(h^{-1})(g^{-1}),Y) \\
&\cong \oplus_{h\in G}\, h_{gr-C}(X((gh)^{-1}),Y) \cong \oplus_{h\in G}\, H_{-C}(X,Y)_{gh} \cong H_{-C}(X,Y)(g) \quad \blacksquare
\end{aligned}
$$

Definition 4.3

i) We say that a graded functor $V : \mathbf{gr}^C \to \mathbf{gr}^D$ defines a graded equivalence if there exists a graded functor $V' : \mathbf{gr}^D \to \mathbf{gr}^C$ such that $V'V \cong^{gr} 1_{\mathbf{gr}^C}$ and $VV' \cong^{gr} 1_{\mathbf{gr}^D}$; in this case we say that the graded coalgebras are graded equivalent.

ii) An equivalence $L : \mathbf{M}^C \to \mathbf{M}^D$ is called a graded equivalence if there is a graded functor $V : \mathbf{gr}^C \to \mathbf{gr}^D$ such that $L \circ U \cong U \circ V$. In this case, we say that V is an associated graded functor of L.

Proposition 4.4 *Let $V : \mathbf{gr}^C \to \mathbf{gr}^D$ be a graded functor. If V is left exact and commutes with direct sums, then there is a graded bicomodule ${}_CP_D$ such that $V(-) \cong -\square_C^{gr} P$.*

Proof: We consider k trivially graded, then $k \otimes^{gr} X \cong^{gr} X$ and $k(g) \otimes^{gr} X \cong^{gr} X(g)$. Since V is a graded functor which commutes with direct sums and the above fact we have:

$$
\begin{aligned}
V(\oplus_{g\in G} k(g) \otimes^{gr} X) &\cong V(\oplus_{g\in G} X(g)) \\
&\cong \oplus_{g\in G} V(X)(g) \cong \oplus_{g\in G} k(g) \otimes^{gr} V(X)
\end{aligned}
$$

Let $W \in \mathbf{gr}_k$, since $\oplus_{g\in G} k(g)$ is a cogenerator of $\mathbf{gr}_k$ there is an exact sequence

$$
0 \longrightarrow W \longrightarrow (\oplus_{g\in G} k(g))^{(J)} \longrightarrow (\oplus_{g\in G} k(g))^{(I)}.
$$

By tensoring with X and $V(X)$ and applying the above isomorphism we obtain the below diagram with exact rows and commutative squares:

$$\begin{tikzcd}
0 \arrow[r] & V(W\otimes^{gr}X) \arrow[r] \arrow[d] & V(\underset{g\in G}{\oplus} X(g))^{(J)} \arrow[r] \arrow[d, "\cong"] & V(\underset{g\in G}{\oplus} X(g))^{(I)} \arrow[d, "\cong"] \\
0 \arrow[r] & W\otimes^{gr}V(X) \arrow[r] & \underset{g\in G}{\oplus} V(X)(g)^{(J)} \arrow[r] & \underset{g\in G}{\oplus} V(X)(g)^{(I)}
\end{tikzcd}$$

This diagram yields that $V(W\otimes^{gr}X)\cong^{gr}W\otimes^{gr}V(X)$.

Let $Z\in \mathbf{gr}^C$, since V is left exact the exact sequence

$$0 \longrightarrow Z \xrightarrow{\rho} Z\otimes^{gr}C \xrightarrow{\rho\otimes 1-1\otimes\Delta} Z\otimes^{gr}C\otimes^{gr}C$$

yields another exact sequence

$$0 \longrightarrow V(Z) \xrightarrow{V(\rho)} Z\otimes^{gr}V(C) \xrightarrow{\rho\otimes 1-1\otimes V(\Delta)} Z\otimes^{gr}C\otimes^{gr}V(C)$$

If we set $P=V(C)$, this means that P is a graded (C,D)-bicomodule with structure map $V(\Delta): P\to C\otimes^{gr}P$. Since $\rho: Z\to Z\Box_C^{gr}C$ is an isomorphism for any $Z\in\mathbf{gr}^C$, then $V(Z)\cong Z\Box_C^{gr}V(C)=Z\Box_C^{gr}P$. ▮

Definition 4.5 *A graded Morita-Takeuchi context* $(C,D,\ {}_CP_D,\ {}_DQ_C,f,g)$ *consists of graded coalgebras* C,D*, graded bicomodules* ${}_CP_D,\ {}_DQ_C$*, and graded bicolinear maps* $f:C\to P\Box_D^{gr}Q$*,* $g:D\to Q\Box_C^{gr}P$ *satisfying the following commutative diagrams:*

$$\begin{tikzcd}
P \arrow[r, "\omega_C"] \arrow[d, "{}_D\omega"'] & P\Box_C^{gr}C \arrow[d, "1\Box^{gr}g"] \\
D\Box_D^{gr}P \arrow[r, "f\Box^{gr}1"'] & P\Box_C^{gr}Q\Box_D^{gr}P
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \arrow[r, "\omega_D"] \arrow[d, "{}_C\omega"'] & Q\Box_D^{gr}D \arrow[d, "1\Box^{gr}f"] \\
C\Box_C^{gr}Q \arrow[r, "g\Box^{gr}1"'] & Q\Box_D^{gr}P\Box_C^{gr}Q
\end{tikzcd}$$

The context is said to be strict if f *and* g *are graded bicolinear isomorphisms.*

Remark: If (C,D,P,Q,f,g) is a graded Morita-Takeuchi context, then the forgotten context $(U(C),U(D),U(P),U(Q),U(f),U(g))$ is a Morita-Takeuchi context and any of them is strict if and only if the other one is so.

Theorem 4.6 *Let (C, D, P, Q, f, g) be a graded Morita-Takeuchi context and suppose that $f : C \to P\Box_D^{gr} Q$ is a monomorphism. Then:*

i) f is a graded isomorphism.

ii) The comodules $U(P)_D$ and ${}_DU(Q)$ are quasi-finite injective.

iii) The comodules ${}_CU(P)$ and $U(Q)_C$ are cogenerators.

iv) g induces isomorphisms of graded bicomodules

$$H_{-D}(P, D) \cong^{gr} Q \qquad\qquad H_{D-}(Q, D) \cong^{gr} P$$

v) The structure maps of the graded bicomodules ${}_CP_D$ and ${}_DQ_C$ induces graded coalgebra isomorphisms,

$$E_{-D}(P) \cong^{gr} C \qquad\qquad E_{D-}(Q) \cong^{gr} C.$$

Proof: *i)* The forgotten context verifies that $U(f)$ is injective, and by [T,Th. 2.5] it follows that $U(f)$ is an isomorphism. Hence f is a graded isomorphism.

ii) and iii) They follow by [T,Th. 2.5] applied to the forgotten context.

iv) The map $g : D \to Q\Box_C^{gr} P$ induces a graded (D, C)-bicomodule map $v : H_{-D}(P, D) \to Q$. Also, the map induced by $U(g) : D \to Q\Box_C P$ is $u(v) : h_{-D}(U(P), U(D)) \to U(Q)$ which is an isomorphism by [T,Th 2.5] (note that since $U(P)$ is quasi-finite by ii) , then $U(H_{-D}(P, D)) \cong h_{-D}(U(P), U(D))$). Thus, we obtain that v is a graded isomorphism. By simmetry, $H_{D-}(Q, D) \cong^{gr} P$ as graded bicomodules.

v) We know from P13 that there is a graded coalgebra map $a : E_{-D}(P) \to C$ induced by the C-comodule structure map of P. The forgotten map $U(a) : e_{-D}(U(P)) \to U(C)$ is an isomorphism of coalgebras by [T,Th. 2.5] (again, we have used that $U(E_D(P)) \cong e_{-D}(U(P))$ because $U(P)$ is quasi-finite). Thus, a is a graded coalgebra isomorphism. ∎

Remark: Analogously, this theorem has a simmetric version for g.

Lemma 4.7 *Let (C, D, P, Q, f, g) be a strict graded Morita-Takeuchi context. Then, C and D are graded equivalent.*

Proof: Since the forgotten context is a strict Morita-Takeuchi context the functors

$$-\square_C U(P) : \mathbf{M}^C \to \mathbf{M}^D \qquad h_{-D}(U(P), -) : \mathbf{M}^D \to \mathbf{M}^C$$

are inverse equivalences. We are going to show that the functors associated to the above ones

$$-\square_C^{gr} P : \mathbf{gr}^C \to \mathbf{gr}^D \qquad H_{-D}(P, -) : \mathbf{gr}^D \to \mathbf{gr}^C,$$

are graded equivalences. Let $Z \in \mathbf{gr}^C$ and we consider the following composition map:

$$H_{-D}(P, Z\square_C^{gr} P) \xrightarrow{\delta} Z\square_C^{gr} H_{-D}(P, P) \xrightarrow{1\square_C^{gr} a} Z\square_C^{gr} C \xrightarrow{1 \otimes \varepsilon} Z$$

where $a : H_{-D}(P, P) \to C$ is the graded coalgebra map induced by the C-comodule structure of ${}_C P_D$. From the above theorem, we know that a is an isomorphism of graded coalgebras and that $U(P)_D$ is quasi-finite injective. By P11 δ is a graded isomorphism and thus the above composition is a graded isomorphism. Given $Z \in \mathbf{gr}^D$, we have:

$$H_{-D}(P, Z) \cong H_{-D}(P, Z\square_D^{gr} D) \xrightarrow{\delta} Z\square_D^{gr} H_{-D}(P, D) \cong Z\square_D^{gr} Q$$

The last isomorphism is given by the preceding theorem and also δ is a graded isomorphism because $U(P)$ is quasi-finite injective. Thus,

$$H_{-D}(P, Z)\square_C^{gr} P \cong Z\square_D^{gr} Q\square_C^{gr} P \xrightarrow{1\square^{gr} g} Z\square_D^{gr} D \cong Z$$

Since the context is strict, g is a graded isomorphism. Hence, the functors $-\square_C^{gr} P$ and $H_{-D}(P, -)$ are inverse equivalences and as both functors are graded, they are graded equivalences. ∎

Let $P \in \mathbf{gr}^D$ such that P is gr-quasi-finite and we set $C = E_{-D}(P)$, P becomes a graded (C, D)-bicomodule. We write $Q = H_{-D}(P, D)$ and

$$g = \theta : D \to Q\square_C^{gr} P$$

$$f : C = H_{-D}(P, P) = H_{-D}(P, P\square_D^{gr} D) \xrightarrow{\delta} P\square_D^{gr} H_{-D}(P, D) = P\square_D^{gr} Q$$

Proposition 4.8 *(C, D, P, Q, f, g) is a graded Morita-Takeuchi context.*

Proof: f and g are graded bicolinear maps by P11 and P8, $f\square^{gr}1 = 1\square^{gr}g$ by definition of δ and $1\square^{gr}f = g\square^{gr}1$ by P12. ∎.

The Morita-Takeuchi context constructed above from a gr-quasifinite comodule P, is called the Morita-Takeuchi context associated to P.

Proposition 4.9 *Let $P \in gr^D$ be a gr-quasifinite comodule, and (C, D, P, Q, f, g) the Morita-Takeuchi context associated to P. Then the context is strict if and only if $U(P)$ is an injective cogenerator.*

Proof: Since the forgotten context is strict then by [T, Prop. 3.2, 3.3], $U(P)$ is an injective cogenerator. Conversely, if $U(P)$ is an injective cogenerator, then by [T, Prop. 3.2, 3.3], $U(f)$ and $U(g)$ are isomorphism, so that f and g are graded isomorphisms. ∎

Now, we are able to characterize when two graded coalgebras are graded equivalent.

Theorem 4.10 *Let C and D be graded coalgebras by a group G. The following assertions are equivalent:*

i) *C and D are graded equivalent.*

ii) *There exists a graded equivalence $L : \mathbf{M}^C \to \mathbf{M}^D$.*

iii) *There is a graded comodule $P \in \mathbf{gr}^D$ such that $U(P)$ is a quasi-finite injective cogenerator and $E_{-D}(P) \cong C$ as graded coalgebras.*

iv) *There is a strict graded Morita-Takeuchi context (C, D, P, Q, f, g).*

Proof:

i) $\Rightarrow$ *iii)* Suppose that $V : \mathbf{gr}^C \to \mathbf{gr}^D$ and $V' : \mathbf{gr}^D \to \mathbf{gr}^C$ are inverse graded equivalences. From Proposition 4.4, we know that $V(-) \cong -\square_C^{gr}V(C)$ and $V'(-) \cong -\square_D^{gr}V'(D)$ and since they are inverse equivalences we have that V' is a left adjoint functor of V and thus $V(C)$ is injective. Moreover, as $-\square_C^{gr}V(C)$ has a left adjoint, from Proposition 3.10 we conclude that $V(C)$ is gr-quasi-finite. We set $P = V(C)$. P is an injective gr-quasi-finite graded D-comodule, from Example 3, $U(P)$ is a quasi-finite injective D-comodule.

Since $\oplus_{g\in G}C(g)$ is a cogenerator in $\mathbf{gr}^C$, then $V(\oplus_{g\in G}C(g)) = \oplus_{g\in G}V(C)(g)$ is a cogenerator in $\mathbf{gr}^D$. Hence $D \hookrightarrow \oplus_{g\in G}P(g)^{(J)}$ and so $U(D) \hookrightarrow U(P)^{(I)}$, that is, $U(P)$ is a cogenerator in $\mathbf{M}^D$.

Next, we show that $E_{-D}(P) \cong C$ as graded coalgebras. For $W \in \mathbf{gr}_k$ and $X, Y \in \mathbf{gr}^C$, with X gr-quasi-finite we have a diagram:

$$\begin{tikzcd}
Com_{gr-C}(Y, W\otimes^{gr}X) \arrow[d, shift right, "V"'] \arrow[r, phantom, "\cong"] & Hom_{gr}(H_{-C}(X,Y), W) \quad (1) \\
Com_{gr-D}(V(Y), W\otimes^{gr}V(X)) \arrow[u, shift right, "V^{-1}"'] \arrow[r, phantom, "\cong"] & Hom_{gr}(H_{-D}(V(X), V(Y)), W) \quad (2)
\end{tikzcd}$$

Remember that since V is graded, then $V(W\otimes^{gr}X) \cong W\otimes^{gr}V(X)$ and $V(C)$ is a left graded C-comodule with structure map $V(\Delta) : V(C) \to C \otimes V(C)$.

We denote by $\theta_C : C \to H_{-C}(C,C)\otimes^{gr}C$ the image of $1_{H_{-C}(C,C)}$ in (1) and by $\theta_{V(C)} : V(C) \to H_{-D}(V(C), V(C))\otimes^{gr}V(C)$ the image of $1_{H_{-D}(V(C),V(C))}$ in (2). By the universal property of the adjoints, there are graded linear maps, $a : H_{-D}(V(C), V(C)) \to H_{-C}(C,C)$ and $b : H_{-C}(C,C) \to H_{-D}(V(C), V(C))$ such that $V(\theta_C) = (a \otimes 1)\theta_{V(C)}$ and $V'(\theta_{V(C)}) = (b \otimes 1)\theta_C$. It is easy to check that a and b are inverse to each other and hence they are isomorphism.

The maps $\Delta : C \to C\otimes^{gr}C$ and $V(\Delta) : V(C) \to C\otimes^{gr}V(C)$ induce graded coalgebra maps $d : H_{-C}(C,C) \to C$ (this is an isomorphism) and $e : H_{-D}(V(C), V(C)) \to C$ such that $\Delta = (d \otimes 1)\theta_C$ and $V(\Delta) = (e \otimes 1)\theta_{V(C)}$. By applying V to $\Delta = (d \otimes 1)\theta_C$, we have that:

$$\begin{aligned}
V(\Delta) &= V(d \otimes 1)V(\theta_C) = (d \otimes 1)V(\theta_C) \\
&= (d \otimes 1)(a \otimes 1)\theta_{V(C)} = (da \otimes 1)\theta_{V(C)}.
\end{aligned}$$

The uniqueness of e yields $e = da$ and so e is an isomorphism because d and a are so. Hence $E_{-D}(P) \cong C$ as graded coalgebras.

iii) $\Rightarrow$ *iv)* It follows from Proposition 4.9.

iv) $\Rightarrow$ *i)* This is Lemma 4.7

iii) $\Rightarrow$ *ii)* Since $E_{-D}(P) \cong^{gr} C$ as graded coalgebras and $U(P)$ is quasi-finite then $e_{-D}(U(P)) \cong C$. Thus, $U(P)$ is a quasi-finite injective cogenerator and $e_{-D}(U(P)) \cong C$. By [T, Th. 3.5] the functor $-\square_C U(P) : \mathbf{M}^C \to \mathbf{M}^D$ is an equivalence and this equivalence is graded because it has the associated graded functor $-\Box_C^{gr}P$.

ii) $\Rightarrow$ *iv)* Let $L : \mathbf{M}^C \to \mathbf{M}^D$ be a graded equivalence with associated graded functor $V : \mathbf{gr}^C \to \mathbf{gr}^D$. From [T, Prop. 2.1] and Proposition 4.4 we obtain that $L(-) \cong -\Box_C L(C)$ and $V(-) \cong \Box_C^{gr} V(C)$. Since $LU(C)$ is a quasi-finite injective cogenerator, then $UV(C) = LU(C)$ is a quasi-finite injective cogenerator. Now we only need to apply Proposition 4.9. ∎

Remark. Theorem 4.10 establishes that two graded coalgebras C and D are graded equivalent if and only if there is a graded equivalence between $\mathbf{M}^C$ and $\mathbf{M}^D$. It is natural to ask which is the relationship between C and D graded equivalent and C and D equivalent. Examples of graded coalgebras C, D such that C and D are equivalent but not graded equivalent can be constructed following the same ideas of the examples given in [GG] and [RI], since these are of finite dimension.

We remember that $M \in \mathbf{M}^C$ is said to be gradable if there is $M' \in \mathbf{gr}^C$ such that $U(M') = M$. We denote by $\mathcal{G}$ the full subcategory of gradable C-comodules. We have the following corollaries:

Corollary 4.11 *Let C and D be graded coalgebras and $V : \mathbf{M}^C \to \mathbf{M}^D$ a graded equivalence, then $V(\mathcal{G})$ is the full subcategory of gradable D-comodules.*

Corollary 4.12 *Suppose that every quasi-finite injective C-comodule is gradable and C is Morita-Takeuchi equivalent to a coalgebra D. Then, there is a graded structure on D and C is graded equivalent to D.*

5 Picard Groups for Graded Coalgebras

Definition 5.1 *Let C be a cocommutative coalgebra.*

i) A graded C-coalgebra is a graded coalgebra $D = \oplus_{g\in G} D_g$ which is in addition a C-coalgebra, i.e, there is a cocentral coalgebra map $\epsilon : D \to C$.

ii) Let D, E be graded C-coalgebras, a graded (D, E)-bicomodule over C is a (D, E)-bicomodule ${}_D M_E$ over C which is graded, i.e, $M = \oplus_{g\in G} M_g$ and the following square is commutative:

$$\begin{array}{ccccc} M & \xrightarrow{\rho_E} & M \otimes E & \xrightarrow{1\otimes\epsilon} & M \otimes C \\ \Big\downarrow {\scriptstyle D\rho} & & & & \Big\downarrow {\scriptstyle =} \\ D \otimes M & \xrightarrow{\epsilon\otimes 1} & C \otimes M & \xrightarrow{\tau} & M \otimes C \end{array}$$

where τ is the twist map.

iii) *A graded (D, E)-bicomodule M over C is called invertible if the functor $-\square_D^{gr} M : \mathbf{gr}^D \to \mathbf{gr}^E$ is a graded equivalence. From Theorem 4.10, equivalently, there exists a graded (E, D)-bicomodule over C, ${}_E N_D$, and graded bicomodule maps $f : D \to M\square_E^{gr} N$ and $g : E \to N\square_D^{gr} M$ such that $(D, E, {}_D M_E, {}_E N_D, f, g)$ is a strict graded Morita-Takeuchi context. Then, ${}_E N_D \cong^{gr} H_{-E}(M, E)$.*

Definition 5.2 *Let D be a graded C-coalgebra and E a graded coalgebra.*

i) *We define $grPic_C(D)$ as the set of all graded bicomodule isomorphism classes $[M]$ of invertible graded bicomodules ${}_D M_D$ over C. This set becomes a group with the multiplication induced by the graded cotensor product,i.e, $[M][N] = [M\square_D^{gr} N]$. The identity element is $[D]$ and for $[M] \in grPic_C(D)$ the inverse $[M]^{-1} = H_{-D}(M, D)$.*

ii) *$grPicent(D)$ is defined to be $grPic_{Z(D)}(D)$.*

iii) *We define $Pic^{gr}(E) = Pic_k(E \rtimes kG)$. (We know that $\mathbf{gr}^E \cong \mathbf{M}^{E\rtimes kG}$).*

Proposition 5.3 *Let D be a graded coalgebra, $grPic_k(D)$ is a subgroup of $Pic^{gr}(D)$.*

Proof: We can identify $grPic_k(D)$ with the isomorphism classes of graded self-equivalences of $\mathbf{gr}^D$ and $Pic^{gr}(D)$ with the isomorphism classes of self-equivalences of $\mathbf{gr}^D$. Hence $grPic(D)$ is a subgroup of $Pic^{gr}(D)$. ∎

Remark: The theory of Picard group for graded rings presents some additional dificulties when the group is infinite. Rings with local units and unital modules over them appear in this setting and they are a fundamental tool for the computation of some graded Picard groups (see [Be], [BeR1], [BeR2], [HR] for more information on this topic). The coalgebra case is easier that the algebra case because, when the group G is infinite, $C \rtimes kG$ is a perfectly well behaved coalgebra with counit. Thus the Picard group theory developed in [TZ] may be applied to $C \rtimes kG$.

Proposition 5.4 *Let D, E be graded C-coalgebras and Γ, Λ graded coalgebras.*

i) There exist group morphisms $\phi : grPic_C(D) \to Pic_C(D)$ and $\psi : grPicent(D) \to Picent(D)$.

ii) Suppose that D, E are graded equivalent over C, then $grPic_C(D) \cong grPic_C(E)$, $grPicent(D) \cong grPicent(E)$.

iii) If Γ, Λ are graded equivalent, then $Pic^{gr}(\Gamma) \cong Pic^{gr}(\Lambda)$.

iv) If Γ is strongly graded, then $Pic^{gr}(\Gamma) \cong Pic(\Gamma_e)$.

Proof: *i)* If ${}_DM_D$ induces a graded equivalence from $\mathbf{gr}^D$ to $\mathbf{gr}^D$, then by Theorem 4.9. $U(M)$ induces an equivalence from $\mathbf{M}^D$ to $\mathbf{M}^D$. Thus, the map from $grPic_C(D)$ to $Pic_C(D)$, $[M] \mapsto [U(M)]$ is a group morphism. Analogously, for $grPicent(D)$.

ii) It follows by the same ideas of [TZ, Th. 2.5].

iii) If Γ, Λ are graded equivalent then we have that $\mathbf{M}^{\Gamma \rtimes kG} \cong \mathbf{gr}^\Gamma \simeq \mathbf{gr}^\Lambda \cong \mathbf{M}^{\Lambda \rtimes kG}$. Thus, $\Gamma \rtimes kG$ and $\Lambda \rtimes kG$ are Morita-Takeuchi equivalent and from [TZ, Th. 2.5], $Pic^{gr}(\Gamma) \cong Pic^{gr}(\Lambda)$.

iv) If Γ is strongly graded, from [NT, Th. 5.4] there is an equivalence of categories $\mathbf{gr}^\Gamma \simeq \mathbf{M}^{\Gamma_e}$. Then, we have that $\mathbf{M}^{\Gamma \rtimes kG} \cong \mathbf{gr}^\Gamma \simeq \mathbf{M}^{\Gamma_e}$ and thus $\Gamma \rtimes kG$ and Γ_e are Morita-Takeuchi equivalent coalgebras. From [TZ, Th. 2.5], $Pic^{gr}(\Gamma) \cong Pic(\Gamma_e)$. ∎

Let $grAut_C(D)$ denote the group of graded C-automorphisms of the graded C-coalgebra D. An C-automorphism $f \in grAut_C(D)$ is said to be gr-inner if there is an unity in D_e^* such that $f(d) = (u \otimes 1 \otimes u^{-1})(\Delta \otimes 1)\Delta(d)$ for all $d \in D$. We denote by $grInn_C(D)$ the group of inner C-automorphisms of D. It is easy to check that $grInn_C(D)$ is a normal subgroup of $grAut_C(D)$. The factor group $grAut_C(D)/grInn_C(D)$ is called the group of outer C-automorphisms and it is denoted by $grOut_C(D)$.

Let ${}_DM_D$ be a graded bicomodule over C and $f, g \in grAut_C(D)$, we denote by ${}_fM_g$ the bicomodule constructed in the following way: as vector space ${}_fM_g = M$ and the structure maps are defined by:

$$ {}_D\rho(m) = \sum_{(m)} f(m_{(-1)}) \otimes m_{(0)} \qquad \rho_D(m) = \sum_{(m)} m_{(0)} \otimes g(m_{(1)}) $$

for all $m \in M$. We have the following fact, ${}_{f'}({}_fM_g)_{g'} = {}_{f'f} M_{gg'}$. Analogously to the non graded case, cf. [TZ, Lem. 2.6], we can prove the following lemma:

Lemma 5.5 *Let D be a graded C-coalgebra, $f, g, h \in grAut_C(D)$ and we consider the (D, D)-bicomodule ${}_fD_g$. Then we have the following isomorphisms as graded (D, D)-bicomodules:*

i) ${}_fD_g \cong^{gr} {}_{fh}D_{gh}$

ii) ${}_fD_1 \square_D^{gr} {}_gD_1 \cong^{gr} {}_{fg}D_1$

iii) ${}_fD_1 \cong^{gr} {}_1D_1 \Leftrightarrow f \in grInn_C(D)$.

Theorem 5.6 *Let D be a graded C-coalgebra and E a graded coalgebra. We consider the smash coproduct associated to E, $E \rtimes kG$ and we set $Inn^{gr}(E) = Inn(E \rtimes kG)$, $Aut^{gr}(E) = Aut(E \rtimes kG)$ and $Out^{gr}(E) = Aut^{gr}(E)/Inn^{gr}(E)$, then we have exact sequences:*

$$ 1 \longrightarrow grInn_C(D) \longrightarrow grAut_C(D) \xrightarrow{\alpha} grPic_C(D) $$

$$ 1 \longrightarrow Inn^{gr}(E) \longrightarrow Aut^{gr}(E) \xrightarrow{\beta} Pic^{gr}(E) $$

where $\alpha(f) = [\ {}_fD_1]$ for all $f \in grAut_C(D)$ and $\beta(g) = [\ {}_gE \rtimes kG_1]$ for all $g \in Aut^{gr}(E)$. $grOut_C(D)$ and $Out^{gr}(E)$ are subgroups of $grPic_C(D)$ and $Pic^{gr}(E)$, respectively.

Proof: i),ii) of the above lemma give that α is a well-defined group morphism and iii) gives the exactness of the first sequence. The second sequence is just [TZ, Th. 2.7] applied to $E \rtimes kG$. ∎

Corollary 5.7 *Let $[M], [N] \in grPic_C(D)$ and $[P], [Q] \in Pic^{gr}(E)$.*

i) $M_D \cong^{gr} N_D$ *if and only if* $[N] \in [M]Im(\alpha)$, *that is,* $N \cong^{gr} {}_fM_1$ *as graded bicomodules for some* $f \in grAut(D)$.

ii) $P_{E\rtimes kG} \cong Q_{E\rtimes kG}$ *if and only if there is* $g \in Aut^{gr}(D)$ *such that* $Q \cong {}_gP_1$ *as* $E \rtimes kG$*-bicomodules.*

Proof: The first item is analogous to [TZ, Cor. 2.8] and the second is [TZ, Cor. 2.8] applied to $E \rtimes kG$. ∎

Next, we are able to interpret $grPic_C(D)$ in terms of graded C-automorphisms of D and $grPicent_D(D)$ when D is cocommutative and the group G is abelian.

Let D be a cocommutative C-coalgebra graded by an abelian group G, if M is a right graded D-comodule with structure map ρ_D and $\tau : M\otimes D \to D\otimes M$ is the twist map, M becomes a graded left D-comodule with the structure map ${}_D\rho = \tau\rho_D$ (Note that if G is not abelian, then ${}_D\rho$ is not graded). Also, every graded left D-comodule may be viewed as a graded right D-comodule in the same form. Similar facts happen for the map of graded right and left D-comodules.

Let ${}_DM_D$ be an invertible graded (D, D)-bicomodule over C. The structure maps $\rho_D : M \to M\otimes D$ and ${}_D\rho : M \to D\otimes M$ induce isomorphisms of graded C-coalgebras a and b such that the following triangles are commutative:

$$\begin{tikzcd} M \arrow[r, "{}_D\rho"] \arrow[d, "\theta"'] & D\otimes M \\ E_{-D}(M)\otimes M \arrow[ur, "a\otimes 1"'] & \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\rho_D"] \arrow[d, "\theta'"'] & M\otimes D \\ M\otimes E_{D-}(M) \arrow[ur, "1\otimes b"'] & \end{tikzcd}$$

We can identify $(E_{-D}(M), \theta) \cong (D, {}_D\rho)$ and $(E_{D-}(M), \theta') \cong (D, \rho_D)$. We consider the map $\tau\rho_D : M \to D\otimes M$ which is a left graded D-colinear map, then there is a unique graded C-coalgebra map $u : D \to D$ such that $\tau\rho_D = (u\otimes 1)\,{}_D\rho$. Analogously, for $\tau_D\rho : M \to M\otimes D$ there exists a unique graded C-coalgebra map $v : D \to D$ such that $\tau_D\rho = (1\otimes v)\rho_D$. We have:

$$\rho_D = (1\otimes u)\tau_D\rho = (1\otimes u)(1\otimes v)\rho_D = (1\otimes uv)\rho_D$$
$${}_D\rho = (v\otimes 1)\tau\rho_D = (v\otimes 1)(u\otimes 1)_D\rho = (vu\otimes 1)_D\rho.$$

Hence $uv = 1_D$ and $vu = 1_D$ and thus u is a graded C-automorphism of D. So, given $[M] \in grPic_C(D)$ there is a unique $u_M \in grAut_C(D)$ such that

$\tau\rho_D = (u_M \otimes 1)_D\rho$. Thus we can define a map $\phi : grPic_C(D) \to grAut_C(D)$ by $\phi([M]) = u_M$.

Theorem 5.8 *Let D be a cocommutative C-coalgebra graded by an abelian group G. Then, there is a splitting exact sequence of groups:*

$$1 \longrightarrow grPicent(D) \longrightarrow grPic_C(D) \xrightarrow{\phi} grAut_C(D) \longrightarrow 1$$

Hence $grPic_C(D) \cong grPicent(D) \ltimes grAut_C(D)$, where $\ltimes$ denotes the semidirect product of groups.

Proof: First, we see that ϕ is a group morphism. u_M is the unique map which verifies $\sum_{(m)} m_{(0)} \otimes m_{(1)} = \sum_{(m)} m_{(0)} \otimes u_M(m_{(-1)})$ for all $m \in M$. Let $[M], [N] \in grPic_C(D)$ and we consider $M\square_D^{gr}N$ and $z = x \otimes y \in M\square_D^{gr}N$. Then, we have:

$$\begin{aligned}\textstyle\sum_{(z)} z_{(0)} \otimes u_{M\square_D^{gr}N}(z_{(-1)}) &= \textstyle\sum_{(z)} z_{(0)} \otimes z_{(1)} = \sum_{(y)} x \otimes y_{(0)} \otimes y_{(1)} \\ &= \textstyle\sum_{(y)} x \otimes y_{(0)} \otimes u_N(y_{(-1)}) = \sum_{(x)} x_{(0)} \otimes y \otimes u_N(x_{(1)}) \\ &= \textstyle\sum_{(x)} x_{(0)} \otimes y \otimes u_N u_M(x_{(-1)}) = \sum_{(z)} z_{(0)} \otimes u_N u_M(z_{(-1)})\end{aligned}$$

where in the fourth equality we have used that $z \in M\square_D^{gr}N$. Therefore $u_{M\square_D^{gr}N} = u_N u_M$.

Now, we show that $Ker(\phi) = grPicent(D)$. If $[M] \in grPicent(D)$, then the structure maps of M verifies $\rho_D = \tau_D\rho$. Thus, $\tau\rho_D = {}_D\rho = (1_D \otimes 1)_D\rho$ and so $u_M = 1_D$, that is, $[M] \in Ker(\phi)$. Conversely, if $u_M = 1_D$, then $\tau\rho_D = {}_D\rho$ and $[M] \in grPicent(D)$.

Let $\alpha : grAut_C(D) \to Pic_C(D)$ be as in Theorem 5.6 and let $\sigma \in grAut_C(D)$. We set $M = {}_{\sigma^{-1}}D_1$. u_M is the unique map which verifies $\tau\rho_D = (u_M \otimes 1)_D\rho$ with $\rho_D = \Delta$ and ${}_D\rho = (\sigma^{-1} \otimes 1)\Delta$. Therefore, $u_M = \sigma$. ∎

Next, we apply our results in the study of graded Picard groups of gr-irreducible coalgebras. First, we need some definitions:

Definition 5.9 *Let D be a graded coalgebra and $X \in \mathbf{gr}^D$.*

i) A graded subcoalgebra $E \subseteq D$ is called gr-simple if it has no proper graded subcoalgebras.

ii) D is said to be gr-irreducible if it has a unique graded gr-simple subcoalgebra.

iii) X is called gr-free if $X \cong^{gr} \oplus_{g\in G} C(g)^{(I_g)}$ for some indexed sets I_g for all $g \in G$.

Lemma 5.10 *Let D be a coconmutative gr-irreducible graded coalgebra. Then every gr-injective D-comodule is gr-free.*

Proof: Let $X \in \mathbf{gr}^D$ gr-injective. We know that $X = \oplus_{i\in I} E^{gr}(S_i)$ with $S_i \in \mathbf{gr}^D$ gr-simple for all $i \in I$. Since D is gr-irreducible then it has a unique gr-simple subcoalgebra S and the only gr-simple comodules are of the form $S(g)$ with $g \in G$. Thus, we can set $X = \oplus_{g\in G} E^{gr}(S(g))^{(I_g)}$. But $E^{gr}(S(g)) = D(g)$ for all $g \in G$. Hence $X = \oplus_{g\in G} D(g)^{(I_g)}$ and so X is gr-free. ∎

Corollary 5.11 *Let D be a cocommutative gr-irreducible coalgebra graded by an abelian group G. Let S be the unique gr-simple subcoalgebra in D. Suppose that $S(g)$ is not isomorphic to S as graded comodules for any $g \in G$. Then, $grPic(D) \cong G \ltimes grAut(D)$*

Proof: We are going to prove that $grPicent(D) \cong G$ and we apply Theorem 5.8. For every $g \in G$ the graded bicomodule (the left structure is induced by the right structure) $D(g)$ is invertible with inverse $D(g^{-1})$ and if $D(g) \cong^{gr} D$ then $S(g) \cong soc(D(g)) \cong soc(D) \cong S$ which is a contradiction with the hypothesis. Hence $g = e$ and $G \hookrightarrow grPicent(D)$. Also, if M is a graded invertible comodule, then M is gr-injective and by the above lemma M is gr-free. Set $M = \oplus_{g\in G} D(g)^{(I_g)}$. Since M is invertible there is $N \in \mathbf{gr}^D$ such that $M \square_D^{gr} N = \oplus_{g\in G} N(g)^{(I_g)} \cong^{gr} D$ and hence

$$soc^{gr}(\oplus_{g\in G} N(g)^{(I_g)}) \cong^{gr} \oplus_{g\in G} soc^{gr}(N(g))^{(I_g)} \cong^{gr} soc^{gr}(D) \cong^{gr} S$$

So, we conclude that $M \cong^{gr} D(g)$ for some $g \in G$. ∎

Theorem 5.12 *Let C be a cocommutative gr-irreducible graded coalgebra. Then $Pic^{gr}(C) \cong Out^{gr}(C)$.*

Proof: Denote by $A : \mathbf{gr}^C \to \mathbf{M}^{C \rtimes kG}$ and $B : \mathbf{M}^{C \rtimes kG} \to \mathbf{gr}^C$ the isomorphisms between both categories. Suppose that $[M] \in Pic^{gr}(C)$, then M is a quasi-finite injective cogenerator $C \rtimes kG$-bicomodule. Hence $B(M)$

is a quasi-finite injective cogenerator as graded right C-comodule. Since C is gr-irreducible, by the above lemma $B(M)$ is gr-free, that is, $B(M)\cong^{gr} \oplus_{g\in G} C(g)^{(I_g)}$. From [T, Prop. 4.5], I_g is a finite index set and $|I_g| \geq 1$ for all $g \in G$ because $B(M)$ is a quasi-finite injective cogenerator. We can write $B(M)\cong^{gr} \oplus_{g\in G} C(g) \oplus V$ with $V \in \mathbf{gr}^C$, then $M \cong AB(M) \cong A(\oplus_{g\in G}C(g)) \oplus A(V)$. But $A(\oplus_{g\in G}C(g)) \cong C \rtimes kG$ as right $C \rtimes kG$-comodules. Thus $M \cong C \rtimes kG \oplus W$ with $W = A(V)$ as right $C \rtimes kG$-comodules.

On the other hand, since M is invertible, there is an invertible $C \rtimes kG$-comodule N such that $M\square_{C\rtimes kG}N \cong C \rtimes kG$ as right $C \rtimes kG$-comodules. Then, $(C \rtimes kG\square_{C\rtimes kG}N)\oplus(W\square_{C\rtimes kG}N) \cong C \rtimes kG$. Setting $Z = W\square_{C\rtimes kG} N$ we have that $N \oplus Z \cong C \rtimes kG$, and thus $soc(N) \oplus soc(Z) \cong soc(C \rtimes kG)$. The argument of the above paragraph applied to N gives that $N \cong C \rtimes kG \oplus W'$ and then $soc(N) \cong soc(C \rtimes kG) \oplus soc(W')$. Combining this with the fact that N is a quasi-finite injective cogenerator and $C \rtimes kG$ contains all simples of the category, from [T, Prop. 4.5], we deduce that $soc(Z) = \{0\}$ and therefore $Z = \{0\}$. Using that $-\square_{C\rtimes kG}N$ is an equivalence, $Z = W\square_{C\rtimes kG}N = \{0\}$ implies $W = \{0\}$. So that, $M \cong C \rtimes kG$ as right $C \rtimes kG$-comodules. Finally, from Corollary 5.7 ii), it follows that $[M] \in Im(\beta)$ and thus $Out^{gr}(C) \cong Pic^{gr}(C)$. ∎

EXAMPLES:

1.- We consider the coalgebra generated by two elements c_0, c_1 with comultiplication and counit given by:

$$\Delta(c_0) = c_0 \otimes c_0,\ \Delta(c_1) = c_0 \otimes c_1 + c_1 \otimes c_0 \quad \varepsilon(c_0) = 1,\ \varepsilon(c_1) = 0$$

We see C graded by $\mathbb{Z}_2$ with $C_0 = kc_0$ and $C_1 = kc_1$. Consider the associated smash coproduct $C \rtimes k\mathbb{Z}_2$. It is easy to check that the only automorphisms of $C \rtimes k\mathbb{Z}_2$ are given by:

$$\begin{array}{ll}
c_0 \rtimes 0 \mapsto c_0 \rtimes 0 & \quad c_0 \rtimes 0 \mapsto c_0 \rtimes 1 \\
c_0 \rtimes 1 \mapsto c_0 \rtimes 1 & \quad c_0 \rtimes 1 \mapsto c_0 \rtimes 0 \\
c_1 \rtimes 0 \mapsto ac_1 \rtimes 0 & \quad c_1 \rtimes 0 \mapsto bc_1 \rtimes 1 \\
c_1 \rtimes 1 \mapsto ac_1 \rtimes 1 & \quad c_1 \rtimes 1 \mapsto bc_1 \rtimes 0
\end{array}$$

where $a, b \in k^*$. Then, $Aut^{gr}(C) \cong k^* \times \mathbb{Z}_2$. The group of units of $(C \rtimes k\mathbb{Z}_2)^* \cong C^*\#(k\mathbb{Z}_2)^*$ (see [DNRV, Remark 1.7]) is k^* and $Inn^{gr}(C) = \{1\}$.

So that $Out^{gr}(C) = Aut^{gr}(C)/Inn^{gr}(C) \cong k^* \times \mathbb{Z}_2$. From Theorem 5.12, $Pic^{gr}(C) \cong Out^{gr}(C) \cong k^* \times \mathbb{Z}_2$. On the other hand, $grAut(C) \cong k^*$ and from Theorem 5.11, $grPic(C) \cong \mathbb{Z}_2 \times k^*$. By [TZ, Th. 2.7] $Pic(C) \cong Aut(C) \cong k^*$.

2.- Let C be the trigonometric coalgebra, that is, the coalgebra generated by two elements c, s with comultiplication and counit given by:

$$\Delta(c) = c \otimes c - s \otimes s,\ \Delta(s) = c \otimes s + s \otimes c \quad \varepsilon(c) = 1,\ \varepsilon(s) = 0$$

C becomes a $\mathbb{Z}_2$-graded coalgebra by setting $C_0 = kc$ and $C_1 = ks$. In fact, C is strongly graded. By Proposition 5.4 vi), $Pic^{gr}(C) \cong Pic(C_0) \cong Pic(k)$ since C_0 is isomorphic to k as coalgebra.

Before we give the last example, we note the following fact: if $C = \oplus_{g\in G} C_g$ is a graded coalgebra and $C \rtimes kG$ is its associated smash coproduct, then $grAut(C) \times G \subseteq Aut^{gr}(C)$. Given $(f, h) \in grAut(C) \times G$ we define $f_h(c \rtimes g) = f(c) \rtimes gh$ for every homogeneous $c \in C_k$ and $g \in G$. f_h is an automorphism of $C \rtimes kG$:

$$\begin{aligned}
\Delta f_h(c \rtimes g) &= \Delta(f(c) \rtimes gh) \\
&= \textstyle\sum_{f(c)} f(c)_{(1)} \rtimes deg(f(c)_{(2)}) gh \otimes f(c)_{(2)} \rtimes gh \\
&= \textstyle\sum_{(c)} f(c_{(1)}) \rtimes deg(c_{(2)}) gh \otimes f(c_{(2)}) \rtimes gh \\
&= (f_h \otimes f_h)\Delta(c \rtimes g)
\end{aligned}$$

$$\varepsilon(f_h(c \rtimes g)) = \varepsilon(f(c) \rtimes g) = \varepsilon(f(c)) = \varepsilon(c)$$

3.- Let C be the power divided coalgebra $C = \{c_0, c_1, c_2, ...\}$ with $\Delta(c_i) = \sum_{j=1}^{i} c_j \otimes c_{i-j}$ and $\varepsilon(c_i) = \delta_{0,i}$ for all $i \geq 0$. C is graded by $\mathbb{Z}$ if we set $C_i = kc_i$ for all $i \geq 0$ and $C_i = \{0\}$ for all $i < 0$.

It is easy to check that every gr-automorphism of C is of the form $c_i \mapsto a^i c_i$ with $a \in k^*$. Hence $grAut(C) \cong k^*$. From Corollary 5.11, $grPic(C) \cong \mathbb{Z} \times k^*$.

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