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Absolutely continuous copulas with given sub-diagonal section

Enrique de Amo[a,*], Manuel Díaz Carrillo[b], Juan Fernández Sánchez[c]

[a] *Departamento de Álgebra y Análisis Matemático, Universidad de Almería, Carretera de Sacramento s/n, 04120 La Cañada de san Urbano, Almería, Spain*
[b] *Departamento de Análisis Matemático, Universidad de Granada, Campus de Fuentenueva s/n, 18071 Granada, Spain*
[c] *Grupo de Investigación de Análisis Matemático, Universidad de Almería, Carretera de Sacramento s/n, 04120 La Cañada de san Urbano, Almería, Spain*

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Abstract

Recently, Durante and Jaworski (2008) [6] have proved that the class of absolutely continuous copulas with a given diagonal section is non-empty in case that the diagonal function is such that the set of points where this coincides with the identity function has null-measure. In this paper, we show that if we consider sub-diagonals (or super-diagonals), then the framework changes. Concretely, for each sub-diagonal function there exists an absolutely continuous copula having this function as sub-diagonal section.

Keywords: Diagonal section; Rectangular patchwork; Ordinal sum

1. Introduction

The importance of copulas in statistical analysis and modelling is described by Sklar's Theorem, which states that the joint distribution function H of the random pair (X, Y) with respective marginals F and G can be expressed by $H(x, y) = C(F(x), G(y))$ for all (x, y) in the unit square, where C is a copula that is uniquely determined on $\text{Ran}(F) \times \text{Ran}(G)$.

By applying this result, to determine function H we can proceed in two steps: first, the marginals are chosen; then the dependence is modelled by means of a suitable copula C.

As B. Schweizer says, "since 1958 to 1976, virtually all the results concerning copulas were obtained in connection with the study and development of the theory of probabilistic metric spaces". But the ability to find those properties that are invariant under monotone transformations, joined to other properties either as the capability to express tail dependence or association measures as functions of itself, and to be stable by small input errors, have made that copulas be a tool of wide applications in many other fields. See for instance [3,17,26], and the references therein.

Nowadays there is great interest in the study and construction of copulas with given values in certain sections (see [6–9,13,16,20,24]). On the other hand, it is well known that under the assumption of the absolute continuity of the copula some estimation procedures and goodness-of-fit tests (see [18]) take place; therefore, it is important to know under what conditions we can find an absolute continuous copula for given values on a section. This problem has been

* Corresponding author. Tel.: +34 950 015278; fax: +34 950 015480.
E-mail addresses: edeamo@ual.es, enrideamo@gmail.com (E. de Amo), madiaz@ugr.es (M. Díaz Carrillo), juanfernandez@ual.es (J. Fernández Sánchez).

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considered in [6]. In this spirit, following the work by [25], we are concerned with the study of copulas with a given sub-diagonal. Before discussing our results we summarize a few basic notions and properties that will be useful later on (Section 2). In Section 3, by the combination of ordinal sums and rectangular patchwork techniques, we present an alternative approach to determine under which conditions absolutely continuous copulas exist with a described diagonal section [6, Theorem 3.1].

In Section 4, by exploiting the above cited techniques and the generalized shuffles, we present the main result of the paper, namely, for a given sub-diagonal, it is possible to find an absolutely continuous copula having it as sub-diagonal section. Finally, we conclude with Section 5 where several related open problems are proposed.

2. Preliminaries

A *bivariate copula* (briefly, a *copula*) is a function $C: \mathbb{I}^2 \to \mathbb{I}$ (with $\mathbb{I} = [0,1]$) satisfying the following conditions:

(C1) $C(x,0) = C(0,x) = 0$ for all $x \in \mathbb{I}$,
(C2) $C(x,1) = C(1,x) = x$ for all $x \in \mathbb{I}$,
(C3) for all x_1, x_2, y_1, y_2 in $\mathbb{I}$ with $x_1 \le x_2$ and $y_1 \le y_2$,

$$V_C([x_1,x_2]\times[y_1,y_2]) = C(x_2,y_2) - C(x_1,y_2) - C(x_2,y_1) + C(x_1,y_1) \ge 0.$$

Conditions (C1) and (C2) express the boundary properties of C, (C3) is the 2-increasing property of C, and $V_C([x_1,x_2]\times[y_1,y_2])$ is called the *C-volume* of the rectangle $[x_1,x_2]\times[y_1,y_2]$. For details on copulas and their applications we refer to [12,22,27].

A copula C induces a probability measure μ_C on $\mathbb{I}^2$ defined for each rectangle $R \subseteq \mathbb{I}^2$ by $\mu_C(R) = V_C(R)$, and is extendible in a standard way to the σ-algebra of all Borel sets in $\mathbb{I}^2$. As usual, λ denotes the Lebesgue measure on the σ-algebra of Borel sets, without confusion both in 1 or 2 dimensions.

A copula C is said to be singular (resp. absolutely continuous), if the associated measure μ_C is singular (resp. absolutely continuous) with respect to the Lebesgue measure λ. The associated measure μ_C has no atoms because the marginals are uniformly distributed, and the Lebesgue decomposition is $\mu_C = \mu_C^a + \mu_C^s$, where μ_C^a is its absolutely continuous component and μ_C^s its singular component.

A measure μ is said to be *concentrated* in A, if the complement of A is a set of μ-measure zero. The copula C *concentrates its mass in* A if $\mu_C(A) = 1$.

Classical examples of copulas are $M(x,y) = \min\{x,y\}$, $\Pi(x,y) = xy$, and $W(x,y) = \max\{x+y-1,0\}$ expressing, respectively, comonotonicity, independence and counter-monotonicity between two random variables. Both M and W are singular, and Π is absolutely continuous.

The *diagonal section* of a copula C is the function $\delta_C: \mathbb{I} \to \mathbb{I}$ defined by $\delta_C(t) = C(t,t)$. On the other hand, a *diagonal* is a function $\delta: \mathbb{I} \to \mathbb{I}$ which satisfies the following conditions:

(D1) $\delta(1) = 1$,
(D2) $\delta(t) \le t$ for all $t \in \mathbb{I}$,
(D3) δ is increasing,
(D4) $|\delta(v) - \delta(u)| \le 2|v-u|$ for all $u, v \in \mathbb{I}$.

Note that the diagonal section δ_C of any copula C is a diagonal function. We say that a diagonal δ satisfies the (s)-*property* if $\delta(t) < t$ in $]0,1[$.

We denote by $\mathcal{C}$ (resp., $\mathcal{C}^{ac}$) the class of all copulas (resp., absolutely continuous copulas), and for a fixed diagonal δ, we denote by $\mathcal{C}_\delta$ (resp., $\mathcal{C}_\delta^{ac}$) the class of all copulas (resp., absolutely continuous copulas) whose diagonal section is δ.

The diagonal section of a copula C has probabilistic interpretations: δ_C is the restriction to $\mathbb{I}$ of the distribution function of $\max(X,Y)$ whenever (X,Y) is a random pair distributed as C. Furthermore, δ_C provides some information about the tail dependence of the random pair (X,Y) whose associated copula is C (see [18,22]).

As we mentioned above, special attention has been devoted to the construction of copulas with a given diagonal section. Specifically, given a suitable $\delta: \mathbb{I} \to \mathbb{I}$, the class $\mathcal{C}_\delta$ of copulas having diagonal section equal to δ has been the object of several investigations (see [6–9,13,16,20,24]).

As it is known, for any diagonal function δ, there exists at least one copula C with diagonal section $\delta_C = \delta$; for example (see [22,14,23]), the diagonal copula K_δ defined by

$$K_\delta(x, y) = \min\left\{x, y, \frac{\delta(x) + \delta(y)}{2}\right\}.$$

Another example is the Bertino copula, that is, $B_\delta(x, y) = \min\{x, y\} - \min_{t\in[u\wedge v, u\vee v]} \widehat{\delta}(t)$, where $\widehat{\delta}(t) = t - \delta(t)$ for all $t \in \mathbb{I}$, where $u \wedge v = \min\{u, v\}$ and $u \vee v = \max\{u, v\}$ (see [2,15]).

Therefore, $\mathcal{C}_\delta$ is non-empty for any diagonal section δ. In particular, if $\delta = \delta_M$, then $\mathcal{C}_{\delta_M} = \{M\}$.

Because the copulas K_δ and B_δ are singular, and therefore, they cannot be absolutely continuous, one is interested in determining under which conditions on a diagonal function δ, the set $\mathcal{C}^{ac}_\delta$ is non-empty. Durante and Jaworski [6, Theorem 3.1] provide an answer to this question with the following result.

Theorem 2.1. *Given any diagonal function δ, the set $\mathcal{C}^{ac}_\delta$ is non-empty if and only if $\lambda(\{t \in \mathbb{I} : \delta(t) = t\}) = 0$.*

The study of diagonal sections has been recently generalized to sub-diagonal sections by Quesada-Molina et al. in [25]. We recall basic notions and properties:

(a) Given a copula C and $x_0 \in]0, 1[$, the *sub-diagonal section* $\delta_{x_0,C}$ of C at x_0 is the function $\delta_{x_0,C} : [0, 1 - x_0] \to [0, 1 - x_0]$ defined by $\delta_{x_0,C}(t) = C(x_0 + t, t)$.

(b) Given $x_0 \in]0, 1[$, a *sub-diagonal function* δ_{x_0} is a function $[0, 1 - x_0] \to [0, 1 - x_0]$ with the following properties:

  (i) $\delta_{x_0}(1 - x_0) = 1 - x_0$,
  (ii) $0 \le \delta_{x_0}(t) \le t$ for every $t \in [0, 1 - x_0]$,
  (iii) $0 \le \delta_{x_0}(t) - \delta_{x_0}(t') \le t - t'$ for every $t, t' \in [0, 1 - x_0]$ with $t' \le t$.

If (X, Y) is a random pair distributed according to the copula C, the function $\delta_{x_0}/(1 - x_0)$ is the restriction to $[0, 1 - x_0]$ of the conditional distribution function of $\max(X - x_0, Y)$ given $Y \le 1 - x_0$.

From Corollary 3 in [25], it is known that given a sub-diagonal function δ_{x_0} there exists an infinity of copulas C whose sub-diagonal section $\delta_{x_0,C}$ coincides with δ_{x_0}.

For a fixed sub-diagonal function δ_{x_0}, we denote by $\mathcal{C}^{ac}_{\delta_{x_0}}$ the class of all absolutely continuous copulas whose sub-diagonal section is δ_{x_0}.

A new problem is still open in this context: Given a sub-diagonal function δ_{x_0}, does an absolutely continuous copula C exist whose sub-diagonal section $\delta_{x_0,C}$ coincides with δ_{x_0}? As we pointed at Introduction, the answer to this question is in the affirmative. We now state it as the main theorem of this paper.

Theorem 2.2. *Let $x_0 \in]0, 1[$ and δ_{x_0} be a sub-diagonal function. Then the set $C^{ac}_{\delta_{x_0}}$ is non-empty.*

The case corresponding to super-diagonal functions is analogous to this. Note that if C and C' are copulas with $C'(x, y) = C(y, x)$, then the sub-diagonal sections of C are the super-diagonal sections of C', and vice versa. Following the ideas used in [5,25, Subsection 2.3], we can obtain results for functions that we can call opposite sub-diagonals (super-diagonals).

Moreover, the techniques we use to prove Theorem 2.2 allow us to give a new proof of Theorem 2.1 in the third section.

We give a proof of Theorem 2.1 based on two central tools; specifically: ordinal sums and rectangular patchworks.

The definition and properties of rectangular patchwork can be found in [10] (see also [4]). Let us recall that notation $\widetilde{C} = (\langle R_i, C_i\rangle)^C_{i\in\mathfrak{I}}$ is used for the rectangular patchwork of the copula C on rectangles R_i and copulas C_i, and that $\widetilde{C}$ is absolutely continuous when C and every C_i are absolutely continuous for all $i \in \mathfrak{I}$.

One of the oldest patchwork construction for copulas is the ordinal sum (see [22, p. 63]). This method is essentially based on a kind of "patchwork procedure", consisting of redefining the value that a copula assumes on a square in the form $[a, b]^2$, by plugging a suitable rescaling of another copula. It is obtained by considering C as the copula M and suitable squares $R_i = [a_i, b_i]^2$, with $0 \le a_i < b_i \le 1$. We recall the definition of an ordinal sum of copulas.

Definition 2.3. Let $\{[a_i, b_i]\}_{i\in\mathfrak{J}}$ be a countable family of intervals with pairwise intersections containing, at most, a common endpoint. Let $\{C_i\}_{i\in\mathfrak{J}}$ be a collection of copulas. Then, the ordinal sum C of $\{C_i\}_{i\in\mathfrak{J}}$ with respect to the family $\{[a_i, b_i]\}_{i\in\mathfrak{J}}$ is the copula defined for all $(x, y) \in \mathbb{I}^2$, by

$$C(x,y) = \begin{cases} a_i + (b_i - a_i)C_i\left(\dfrac{x-a_i}{b_i-a_i}, \dfrac{y-a_i}{b_i-a_i}\right) & (x,y) \in [a_i, b_i]^2, \\ M(x,y) & \text{otherwise.} \end{cases} \tag{1}$$

The W-ordinal sums are defined in a similar way (see [25,4,21]).

Theorem 3.2.1. in [22, p. 63] ensures that a copula C is an ordinal sum if and only if there exists $t \in]0,1[$ such that $C(t,t) = t$.

We need the generalized shuffles of a copula (introduced in [11]) to prove Theorem 2.2. Durante et al. [11, Section 4] generalize the notion of shuffles of Min in order to allow the use of the probability mass of any copula, not only of M, as the starting point for further rearrangements. Such a generalization is based on the use of special measure-preserving transformations. Let us recall that for a given measure space (Ω, F, μ), a measurable space (Ω_1, F_1), and a measurable function $f : \Omega \to \Omega_1$, the *push-forward* of μ under f is the set function $f * \mu$ defined by $(f * \mu)(A) = \mu(f^{-1}(A))$, for every $A \in F_1$.

Given any copula C, a copula D is a *shuffle* of C if there exists a measure-preserving transformation h such that $\mu_D = S_h * \mu_C$, where $S_h(x, y) := (h(x), y)$ for every $(x, y) \in \mathbb{I}^2$. We also denote the copula by $S_{h,C}$.

The next result, whose proof can be found in [1, Theorem 5.3.1], is an essential tool for the building of the copulas in Theorem 2.2.

Lemma 2.4 (*Disintegration theorem*). *Let μ be a probability measure on $\mathbb{I}^d$. Let π: $\mathbb{I}^d \to \mathbb{I}$ be a Borel function and let ν be the push-forward of μ under π. Then there exists a ν-a.e. uniquely determined Borel family of probability measures $\{\mu_x\}_{x\in\mathbb{I}}$ on $\mathbb{I}^d$ such that*

$$\mu_x(\mathbb{I}^d \setminus \pi^{-1}(x)) = 0 \quad \textit{for } \nu\text{-a.e. } x \in \mathbb{I}$$

and

$$\int_{\mathbb{I}^d} f(\mathbf{x})\, \mathrm{d}\mu(\mathbf{x}) = \int_{\mathbb{I}} \left(\int_{\pi^{-1}(x)} f(\mathbf{x})\, \mathrm{d}\mu_x(\mathbf{x}) \right) \mathrm{d}\nu(x),$$

for every Borel map $f : \mathbb{I}^d \to [0, +\infty]$.

To make ends meet, let us set $d = 2$ and $\pi(x, y) = x$. Now, if we apply the above lemma to the shuffle D of the copula C, obtained by means of the function h, then we have $\mu_{C,x} = \mu_{D,h(x)}$.

Finally, we state a result that will be helpful for proving Theorem 2.2.

Lemma 2.5. *Let A and B be subsets of the unit interval $\mathbb{I}$ with the same Lebesgue measure $\alpha > 0$. Then, there exist subsets $A^+ \subset A$ and $B^+ \subset B$ of measure α, and a strictly increasing bijection $f : A^+ \longrightarrow B^+$ that is measure-preserving.*

Proof. Let us set $\alpha = \lambda(A)$, and define an absolutely continuous function $f : \mathbb{I} \to [0, \alpha]$ given by $f(x) = \int_0^x \chi_A(t)\, \mathrm{d}t$. Therefore, the subset $A^z \subset A$ of points x satisfying that $f'(x) = 1$ has a measure α.

If $x, y \in A^z$, then $x < y$ implies $f(x) < f(y)$. Clearly, $f(x) \leq f(y)$. In the case where $f(x) = f(y)$, then f would be constant in $[x, y]$, and as a consequence $f'(x^+) = f'(y^-) = 0$, which contradicts that $x, y \in A^z$. Therefore, f is a measure-preserving increasing bijection from A^z on $f(A^z)$. (Lemmas 1.2.4 and 1.2.5 in [19] ensure that f is measure-preserving.)

We can realize the same arguments with B obtaining another measure-preserving increasing bijection g between subsets B^z and $g(B^z)$ of measure α.

The set $H = f(A^z) \cap g(B^z)$ has measure α. Let us define sets $A^+ = f^{-1}(H)$ and $B^+ = g^{-1}(H)$. The desired strictly increasing bijection from A^+ on B^+ which is measure-preserving is given by $g^{-1} \circ f$. (Here, f^{-1} and g^{-1} denote the respective inverses of the bijections between A^z and $f(A^z)$, and B^z and $g(B^z)$.) □

3. The proof of the Durante–Jaworski theorem

We hereby prove Theorem 2.1 by a combination of the above techniques. Let us denote $S_{x_0} = \{(x_0 + t, t) : t \in [0, 1 - x_0]\}$. In particular, S_0 is the diagonal joining (0, 0) with (1, 1).

Lemma 3.1. *If a diagonal section δ satisfies the (s)-property, then there exists an absolutely continuous copula in $\mathcal{C}_\delta^{ac}$.*

Proof. Given the diagonal δ, let us consider the diagonal copula K_δ. As it is known, K_δ is singular and $\mu_C(S_0) = 0$ (see [14]).

We shall now consider a suitable covering of $\mathbb{I}^2 \backslash S_0$. Let $\{R_j\}_{j \in \mathfrak{J}}$ be a family of rectangles satisfying $\bigcup_{j \in \mathfrak{J}} R_j \supseteq \mathbb{I}^2 \backslash S_0$, $\lambda(R_j \cap R_s) = 0$, whenever $j \neq s$, and $R_j \cap S_0$ is, at most, a singleton. This family is denumerable, because we can find a rational number in the interior of each rectangle. Then, the copula $(\langle R_j, \Pi \rangle)_{j \in \mathfrak{J}}^C$ is an element of the family $\mathcal{C}_\delta^{ac}$. □

Lemma 3.2. *If the copula C is an ordinal sum of absolutely continuous copulas C_i with respect to the family of intervals $\{[a_i, b_i]\}_{i \in \mathfrak{J}}$, and $\sum_{i \in \mathfrak{J}}(b_i - a_i) = 1 - \gamma$, then its singular component is concentrated on the diagonal with a mass equal to γ.*

Proof. Because C is an ordinal sum for the intervals $\{[a_i, b_i]\}_{i \in \mathfrak{J}}$, each rectangle contained in the complement of the set $\bigcup_i [a_i, b_i]^2 \cup S_0$ has a null C-volume.

Moreover, the copulas C_i are absolutely continuous, which implies that μ_C^s is concentrated on the diagonal of $\mathbb{I}^2$.

The absolutely continuous component μ_C^a is concentrated on $\bigcup_i [a_i, b_i]^2$. Therefore,

$$\mu_C^a(\mathbb{I}^2) = \mu_C^a\left(\bigcup_i [a_i, b_i]^2\right) = \sum_{i \in \mathfrak{J}} (b_i - a_i) = 1 - \gamma$$

and, as a consequence, $\mu_C^s(\mathbb{I}^2) = \gamma$.

Proof of Theorem 2.1. Given any diagonal δ, since δ is a continuous function, we can find intervals $J_i = [a_i, b_i]$, with $i \in \mathfrak{J}$ (where $\mathfrak{J}$ is a finite or countable index set), such that $\delta(a_i) = a_i$, $\delta(b_i) = b_i$ and $\delta(t) < t$ on $]a_i, b_i[$, and $\delta(t) = t$ otherwise. Let us observe that a copula C with a diagonal δ can be expressed in a unique way as a ordinal sum of copulas C^i with intervals J_i.

To prove the sufficient condition, let us consider the above family $\{J_i\}_{i \in \mathfrak{J}}$ of non-empty intervals on $\mathbb{I}$. Then, for any $i \in \mathfrak{J}$, let us define the function $\delta_i : \mathbb{I} \to \mathbb{I}$, by

$$\delta_i(x) = \frac{\delta\left(\dfrac{x - a_i}{b_i - a_i}\right) - a_i}{b_i - a_i}.$$

Note that δ_i is a diagonal function. But, by applying Lemma 3.1, δ_i is the diagonal of an absolutely continuous copula C^i, and the ordinal sum of the copulas C^i in the intervals $[a_i, b_i]$ is a copula whose absolutely continuous component has a mass equal to $\sum_{i \in \mathfrak{J}}(b_i - a_i)$. But the set of points t with $\delta(t) = t$ is a null-set and, therefore, $\sum_{i \in \mathfrak{J}}(b_i - a_i) = 1$.

The necessary condition follows from Lemma 3.2. □

4. Sub-diagonal sections

The goal of this section is to prove Theorem 2.2. Its proof is based upon results on copulas with a given sub-diagonal (obtained by Quesada-Molina et al. in [25]), and in the use of rectangular patchworks and the generalized shuffles, respectively introduced by Durante et al. in [10,11].

Proof of Theorem 2.2. We divide the proof into several steps.

First step: *Selecting a copula D with sub-diagonal section δ_{x_0}.*

Set

$$D(x,y)=\begin{cases}\dfrac{(x_0+v-1)u}{x_0} & (u,v)\in[0,x_0]\times[1-x_0,1],\\ (1-x_0)K_{\overline{\delta}_{x_0}}\left(\dfrac{u-x_0}{1-x_0},\dfrac{v}{1-x_0}\right) & (u,v)\in[x_0,1]\times[0,1-x_0],\\ W(u,v) & \text{otherwise,}\end{cases}$$

where $\overline{\delta}_{x_0}$ is the diagonal function given by $\overline{\delta}_{x_0}(t)=\delta_{x_0}((1-x_0)t)/(1-x_0)$. Then, [25, Proposition 2] ensures that D is a copula with sub-diagonal equal to δ_{x_0}, which is a W-ordinal sum copula. This copula has the following properties:

(a) Its absolutely continuous component has a density equal to $1/x_0$ in $[0,x_0]\times[1-x_0,1]$, and equal to zero otherwise.
(b) Its singular component is concentrated in a subset $A'\subset[x_0,1]\times[0,1-x_0]$.

Second step: *There exists a copula D_1 with sub-diagonal section δ_{x_0}, such that its singular component is concentrated on a set*

$$G=\{(a,x_0+a):a\in A\subset[x_0,1]\}.$$

The function $D^*(x,y)=D(x_0+(1-x_0)x,(1-x_0)y)/(1-x_0)$ is a copula, because D is a W-ordinal sum of copulas. If D^* satisfies the (s)-property, then Lemma 3.1 ensures that there is an absolutely continuous copula with the same diagonal that D^*. If D^* does not satisfy the (s)-property, then it is an ordinal sum on the intervals $[a_i,b_i]$ of copulas with the (s)-property. If we proceed on each square $[a_i,b_i]^2$ as in the proof of Lemma 3.1, we obtain that there exists a copula D_1^* which is absolutely continuous or its singular component is concentrated in $\{(t,t):\delta_{D^*}(t)=t\}$ such that $\delta_{D^*}=\delta_{D_1^*}$. Now, the copula D_1 is the W-ordinal sum given by

$$D_1(x,y)=\begin{cases}\dfrac{(x_0+v-1)u}{x_0} & (u,v)\in[0,x_0]\times[1-x_0,1],\\ (1-x_0)D_1^*\left(\dfrac{u-x_0}{1-x_0},\dfrac{v}{1-x_0}\right) & (u,v)\in[x_0,1]\times[0,1-x_0],\\ W(u,v) & \text{otherwise,}\end{cases}$$

and $A=\{(1-x_0)t+x_0:\delta_{D_1^*}(t)=t\}$.

Third step: *There exists an absolutely continuous copula D_2 with sub-diagonal δ_{x_0}.*

Let A be as above, and $\alpha=\lambda(A)$. Then, we can consider two possibilities:

(a) $\lambda(A)\le x_0$.

By Lemma 2.5, there exists a measure-preserving increasing bijection h_1 from a subset $A^+\subset A$ of measure α on a subset of $[0,\alpha]$. Let us define $h^1:\mathbb{I}\to\mathbb{I}$ by

$$h^1(x)=\begin{cases}h_1(x) & \text{if } x\in A^+,\\ h_1^{-1}(x) & \text{if } x\in h_1(A^+),\\ x & \text{otherwise.}\end{cases}$$

Let us note that $h^1\circ h^1(x)=x$.

This function is also measure-preserving, and it provides the copula S_{h^1,D_1} as a generalized shuffle. If we set $\pi(x,y)=x$, because the Lebesgue measure λ is the one-dimensional marginal of a copula, as a consequence of the Disintegration Theorem, we have that

$$\begin{aligned}D_1(t+x_0,t)&=\int_0^1 1_{[0,t+x_0]}\left(\int_0^1 1_{[0,t]}\,\mathrm{d}\mu_{D_1,x}(y)\right)\mathrm{d}x=\int_{[0,t+x_0]}\mu_{D_1,x}(t)\,\mathrm{d}x\\ &=\int_{[x_0,t+x_0]\cap A^+}\mu_{S_{h^1,D_1},h^1(x)}(t)\,\mathrm{d}x+\int_{[x_0,t+x_0]\cap\overline{A^+}}\mu_{S_{h^1,D_1},x}(t)\,\mathrm{d}x\end{aligned}$$

$$= \int_{h^1([x_0,t+x_0]\cap A^+)} \mu_{S_{h^1,D_1},x}(t)\,dx + \int_{[x_0,t+x_0]\cap\overline{A^+}} \mu_{S_{h^1,D_1},x}(t)\,dx$$

$$= \int_{[0,x_0]} \mu_{S_{h^1,D_1},x}(t)\,dx + \int_{[x_0,t+x_0]} \mu_{S_{h^1,D_1},x}(t)\,dx = S_{h^1,D_1}(t+x_0,t),$$

where $\overline{A^+} = [0,1]\backslash A^+$. The last but two equality follows from the fact that h^1 is a measure-preserving bijection, and the last but one equality follows from the fact that the measures $\mu_{S_{h^1,D_1},x}(t)$ vanish out of $h^1([x_0,t+x_0]\cap A^+)$ and $[x_0,t+x_0]\cap\overline{A^+}$. Therefore, S_{h^1,D_1} has a sub-diagonal equal to δ_{x_0}. Moreover, its singular component is concentrated in the rectangle $R_1 = [0,x_0]\times[0,1-x_0]$.

Finally, if we consider the rectangular patchwork $(\langle R_1, \Pi\rangle)^{S_{h^1,D_1}}$, by using similar arguments to those in [11, Proposition 12], we then obtain the desired absolutely continuous copula D_2.

(b) $\lambda(A) \geq x_0$.

Let $m \in \mathbb{N}$ such that $\alpha/m \leq x_0$. Now, we define m sets

$$A_i = \left\{ x \in a : \frac{i-1}{m}\alpha \leq \lambda([1-x_0,x]\cap A) < \frac{i}{m}\alpha \right\},$$

with $i \in \{1,\ldots,m\}$.

Therefore, $\lambda(A_1) = \alpha/m \leq x_0$, and by repeating the procedure shown above in (a), we obtain a copula $D_{1,1}$ with singular components is concentrated in $\{(a, x_0+a) : a \in \bigcup_{i=2}^m A_i\}$.

Next, by using Lemma 2.5, we can find a measure-preserving increasing bijection h_{12} from subsets A_1^+ and A_2^+ of A_1 and A_2, respectively, with $\lambda(A_1^+) = \lambda(A_2^+) = (1-x_0)/m$. Now, we extend h_{12} to $h^{12} : \mathbb{I} \to \mathbb{I}$ by

$$h^{12}(x) = \begin{cases} h_{12}(x) & \text{if } x \in A_1^+, \\ h_{12}^{-1}(x) & \text{if } x \in h_{12}(A_1^+), \\ x & \text{otherwise.} \end{cases}$$

This h^{12} is a measure-preserving bijection in $\mathbb{I}$. Therefore, we can define the generalized shuffle as $D'_{12} = S_{h^{12},D_{11}}$.

The singular component of this copula D_{12} is concentrated in

$$G^1 = \left\{ (a, a-x_0) : a \in \bigcup_{i=3}^m A_i \right\} \cup A_1 \times A'_2,$$

where $A'_2 = \{x : x + x_0 \in A_2\}$. Note that $A_1 \times A'_2$ is included in a rectangle R_{12} of the sides that are parallel to the axes, whose interior has an empty intersection with S_{x_0}. Once again, because the Disintegration Theorem and h^1 be a measure-preserving bijection, we have that the rectangular patchwork $(\langle R_{12}, \Pi\rangle)^{D'_{12}}$ is a copula with sub-diagonal δ_{x_0} and with singular component concentrated in the adherence of the set

$$G^1 = \left\{ (a, a-x_0) : a \in \bigcup_{i=3}^m A_i \right\}.$$

If we repeat this procedure with the sets A_2 and A_3, then we obtain a copula D_{22} with sub-diagonal δ_{x_0}, whose singular component is concentrated in the adherence of the set $G^2 = \{(a, a-x_0) : a \in \bigcup_{i=4}^m A_i\}$. By iteration, we find the copula $D_{m-1,2}$. This is the desired copula D_2. □

5. Conclusions

Several constructions of copulas with given section have been studied for the last few years. In this paper, the authors continue the studies initiated by Quesada-Molina et al. in [25], showing that for every sub-diagonal function there exists an absolutely continuous copula whose diagonal subsection is the given sub-diagonal. The situation differs from that obtained by Durante and Jaworski in [6] for diagonal functions.

For a given copula C and diagonal section δ_{x_0}, we can ask if C and the copula D_2 obtained in the proof of Theorem 2.2 are close with respect to the distance d_∞ (the supremum distance). Because D_2 is based on the use of W-ordinal sums and diagonal copulas, they will not be generally close, although in case C it is absolutely continuous. For instance, if $x_0 = 1/2$ and C is the ordinal sum of Π in the intervals $[0, 1/2]$ and $[1/2, 1]$, following the proof of Theorem 2.2, for the copula D_2 we obtain $d_\infty(\Pi, D_2) = 1/2$, which is the maximum value that can be reached. One question to be considered in future investigations is to determine if for a given singular copula, it is possible to find an absolutely continuous copula with the same section δ_{x_0}, and close to C (w.r.t. d_∞). If the answer is yes, then we will have to look for an algorithm to find the new copula.

Another question is to study the existence of a symmetric copula satisfying that the sub-diagonal function δ_{x_0} is its sub-diagonal section. Similar ideas to those used in the proof of Theorem 2.2 allow us to give an affirmative answer in the case of $x_0 \in]1/2, 1[$. However, it is an open problem if $x_0 \in]0, 1/2]$.

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