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Original article

PCF self-similar sets and fractal interpolation

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Abstract

The aim of this paper is to show, using some of Barnsley's ideas, how it is possible to generalize a fractal interpolation problem to certain post critically finite (PCF) compact sets in $\mathbb{R}^n$. We use harmonic functions to solve this fractal interpolation problem.

MSC: 28A78; 58E20

Keywords: Post critically finite set; Self-similar structure; Iterated function system; Harmonic function on a fractal

1. Introduction

A long-standing question in mathematical analysis on fractals is the existence of a "Laplace operator" on a given self-similar set. In the last decades several approaches have been developed from both probabilistic and analytic viewpoints. The analytic approach goes back to Kigami (see [11,10]), where a general theory of analysis on post-critically finite (PCF) self-similar sets is developed when assuming the existence of the so-called self-similar harmonic structure or, equivalently, a self-similar Laplace operator.

On the other hand, Barnsley introduced in [1] a fractal interpolation method based on Hutchinson [9] and using iterated function system (IFS) theory. A function $f : \mathbb{I} \to \mathbb{R}$, where $\mathbb{I} := [0, 1]$ is a real closed interval, is named by Barnsley as a *fractal function* if its graph is a fractal set. Furthermore, fractals can be realized as the attractor of an IFS.

This method has been used in the unit interval $\mathbb{I}$ to generalize Hermite functions by fractal interpolation, and to study spline fractal interpolation functions (see [13,14]).

In other direction, considering the polynomials of degree 1 as classical harmonic functions on $\mathbb{I}$ and replacing them on the Sierpinski gasket (SG) (respectively, on the Koch Curve (KC)) by harmonic functions of fractal analysis, an analog to the Barnsley fractal interpolation result for SG (respectively, KC) is obtained in [5] (respectively [15]).

We find an essential difference between the result by Barnsley and the others in [5,15]. For real intervals, the graph of the function is possible to be obtained as the attractor of an IFS, but in the cases of Sierpinski gasket and Koch Curve, it is impossible to ensure that the corresponding graph of the interpolating function is the attractor of an IFS. In this paper we state a generalized fractal interpolation problem in PCF that includes the preceding cases. We start

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to study the problem with a family of bounded functions. Afterwards, we describe sufficient conditions to obtain the solution of the interpolation problem via an IFS. The last result generalizes to PCF the ones obtained in [5,15]. The benefit of using the set of harmonic functions of a PCF in interpolation is that they ensure we can find exactly one of them under the stated assumptions, however for other kinds of families, for instance polynomials, there exist cases where neither the uniqueness nor the existence of such solution are satisfied.

The paper is organized as follows: in Section 2 we review the notations and preliminary facts of the objects we investigate. Section 3 contains the main results of this paper. We state the conditions to solve a generalized Barnsley interpolation problem on a compact $K \subset \mathbb{R}^n$, and we generalize the results that are already known for real closed intervals (Theorem 2.1). This study is applied to PCF self-similar structures (Theorem 3.6), and to harmonic structures (Theorem 3.12) in the sense of Kigami (see [12, Chapter 6]). The results we obtain can be applied to the Lagrangian interpolation problem (Corollary 3.8). We give some classic examples; other examples are new. They show how our study works.

2. Notions and definitions

We review in this section, for the reader's convenience, the main definitions and properties concerning the IFS theory, fractal interpolation, and self-similar and harmonic structures, that we use below. For a more detailed account, see [12, Section 1.3] and [2, 6].

2.1. *Self-similar set generated by a finite system of similitudes*

Within fractal geometry, the method of iterated function systems (see [3,9]) is a relatively easy way to generate fractal sets.

Let (X, d) be a complete metric space. A function $f: X \to X$ is called a *similitude* (or a *contracting similarity*) of ratio λ if there is $\lambda \in]0, 1[$, such that $d(f(x), f(y)) = \lambda d(x, y)$, for all $x, y \in X$. A similitude transforms subsets of X into geometrically similar sets. An *iterated function system*, or IFS for short, is a collection of a complete metric space (X, d) together with a finite set of similitudes $F_i : X \to X$, $i = 1, 2, \ldots, N$. It is often convenient to write an IFS as $\{X : F_1, F_2, \ldots, F_N\}$. For an introduction to representation techniques of many fractals by function systems, see [7,8], and for a study of the speed of convergence of the "approximate fractal" and the limit fractal set (in terms of preselected parameters) you can see [6].

The following result ensures the uniqueness and existence of self-similar sets (see, for example [9]): Let $\{X : F_1, F_2, \ldots, F_N\}$ be an IFS. Then there exists a unique non-empty compact subset K of X that satisfies

$$K = \bigcup_{i=1}^{N} F_i(K).$$

The compact K is called the *self-similar set* or the *attractor set with respect to* the system $\{X : F_1, F_2, \ldots, F_N\}$.

For a finite family $\{F_1, F_2, \ldots, F_N\}$ of similitudes acting on X, we write $|F_i(x) - F_i(y)| = \lambda_i |x - y|$ for numbers $\lambda_i \in]0, 1[$, $i = 1, \ldots, N$.

2.2. *Fractal interpolation functions*

Suppose that a set of data points

$$\Gamma := \{(x_i, u_i) \in \mathbb{R}^2 : i = 0, 1, 2, \ldots, N; N \geq 2\}$$

is given, where $x_0 < x_1 < \cdots < x_i < x_{i-1} < \cdots < x_N$. An *interpolation function* corresponding to this set of data is a continuous function $h : [x_0, x_N] \to \mathbb{R}$ such that $h(x_i) = u_i$ for every $i = 0, 1, 2, \ldots, N$. We say that function h interpolates the data, and that the graph of h (denoted by $G(h)$), passes through interpolation points (x_i, u_i). Barnsley [2, Chapter 6] explains how one can construct an IFS in $\mathbb{R}^2$ such that its attractor is $G(h)$, with h interpolating the data.

We note $\langle N \rangle := \{1, 2, 3, \ldots, N\}$. Let $F_i : [x_0, x_N] \to [x_{i-1}, x_i]$ be contractive homeomorphisms defined by $F_i(x) = m_i x + n_i$, where $m_i = (x_{i-1} - x_i)/(x_0 - x_N)$ and $n_i = (x_0 x_i - x_N x_{i-1})/(x_0 - x_N)$ for $i \in \langle N \rangle$. Let $M_i : [x_0, x_N] \to \mathbb{R}$

be a continuous function satisfying $M_i(x_0)=0$, $M_i(x_N)=1$ for $i\in\langle N\rangle$ and let us suppose that there exists $c>0$ such that

$$|M_i(x)-M_i(y)|\leq c|x-y|,\quad \text{forall}\, x,\, y.$$

Let ρ_i be numbers, where $|\rho_i|<1$. Next, we define functions $r_i(x)=m_i'M_i(F_i^{-1}(x))+n_i'$, in $[x_{i-1},\, x_i]$, where $m_i'=u_i-u_{i-1}+\rho_i(u_0-u_N)$ and $n_i'=u_{i-1}-\rho_i u_0$. Finally, for $i\in\langle N\rangle$, let $P_i:[x_0,x_N]\times\mathbb{R}\longrightarrow\mathbb{R}^2$ be a map defined by $P_i(x,y)=(F_i(x),\,\rho_i y+r_i(x))$.

Barnsley shows in [2, Chapter 6] a fractal interpolation theorem that we introduce using terminology we gave above.

Theorem 2.1. *([2, Chapter 6]) Let Γ be a set of data and let P_i be the functions defined above. Then there exists a unique continuous function $h:[x_0,x_N]\to\mathbb{R}$ which interpolates the set of data Γ, satisfying $h(F_i(x))=\rho_i h(x)+r_i(x)$ for $x\in[x_0, x_N]$, $i\in\langle N\rangle$ and such that its graph H is the attractor of an IFS determined by functions P_i, that is $H=\cup_{i\in\langle N\rangle}P_i(H)$.*

The function h whose graph G(h) is the attractor of an IFS as described in Theorem 2.1, is called a *fractal interpolation function*, or a FIF, for short.

Example 2.2. Takagi's nowhere differentiable function T (see, for example [18]) is a FIF with $\Gamma=\{(0,0),(1/2,1/2),(1,0)\}$. Another example is the function studied in [4,17]. Now, the set of data points is $\Gamma=\{(0,0),(1/2,a),(1,1)\}$. If $a\neq 1/2$, then this is a singular function, that is, a continuous increasing function whose derivative vanishes a.e.

Example 2.3. Although the above examples have an atypical performance with respect to derivation, it is not the usual. For example, function $f(x)=x(x-1)$ corresponds to $\Gamma=\{(0,0),(1/2,1/4),(1,0)\}$.

In order to make our framework clearer, we give several more examples. Let us note that the unit interval $\mathbb{I}$ can be seen as a particular case of self-similar set. Other examples of self-similar sets are the following.

Example 2.4. The Sierpinski gasket is the unique compact in $\mathbb{C}$ that is invariant under the contraction functions system $\{f_n:\mathbb{C}\to\mathbb{C}:n=0,1,2\}$, given by

$$\begin{cases} f_0(z)=\dfrac{z}{2};\\ f_1(z)=1+\dfrac{z-1}{2};\\ f_2(z)=\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}+\dfrac{z-1/2+i\sqrt{3}/2}{2};\end{cases}$$

Example 2.5. The family of von Koch-type curves.

Let $\rho\in\mathbb{C}$, $0<Im(\rho)\leq\sqrt{3}/6$ and $Re(\rho)=1/2$ ([16, p. 100]). Then

$$K_\rho(t):=\begin{cases}\rho\overline{K(2t)}, & \text{if}\, 0<t<1/2\\ (1-\rho)\overline{K(2t-1)}+\rho, & \text{if}\, 1/2<t<1\end{cases}$$

where $\overline{z}$ denotes de conjugate of z.

Example 2.6. Hata's tree-like set (see [12, Ex.1.2.9]).

Let $X=\mathbb{C}$, and define $f_1(z)=c\overline{z}$, $f_2(z)=(1-|c|^2)\overline{z}+|c|^2$, where $|c|, |c-1|\in]0,1[$. The self-similar set with respect to $\{f_1,f_2\}$ is called Hata's tree-like set.

These sets can be seen in Fig. 1.

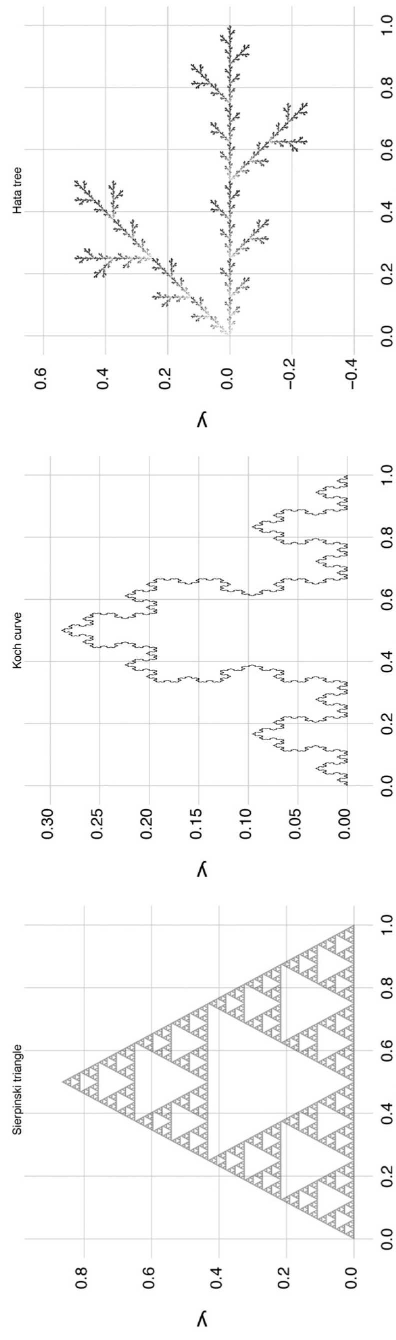

Fig. 1. Sierpinski gasket, von Koch Curve, Hata tree

2.3. PCF self-similar structures

The notion of self-similar structure has been introduced to give a topological description of self-similar sets (other related terms are "nested fractal" or "finitely ramified fractal"). The basic properties of PCF self-similar structures can be found in [11] and [12, Section 1.3].

The *one-sided shift space* Σ is defined by

$$\Sigma := \langle N\rangle^{\mathbb{Z}^+} = \left\{ w_1 w_2 w_3 \ldots : w_i \in \langle N\rangle \text{ for } i \in \mathbb{Z}^+ \right\},$$

that is, the collection of one-sided infinite sequences with symbols in the set $\{1, 2, 3, \ldots, N\}$. For $i \in \langle N\rangle$ let us define a map $\sigma_i : \Sigma \to \Sigma$ by $\sigma_i(w_1 w_2 w_3 \ldots) = i w_1 w_2 w_3 \ldots$, and also define the shift map $\sigma : \Sigma \to \Sigma$ by $\sigma(w_1 w_2 w_3 \ldots) = w_2 w_3 \ldots$.

If we choose an appropriate metric, it turns out that σ_i is a similitude and Σ is the self-similar set with respect to $\{\sigma_1, \sigma_2, \ldots, \sigma_N\}$.

Now, let K be a compact metric space and, for each $i \in \langle N\rangle$, $F_i : K \to K$ be a continuous injection. Then, $L := (K, \langle N\rangle, \{F_i\}_{i\in\langle N\rangle})$ is called a *self-similar structure* on K if there exists a continuous surjection $\pi : \Sigma \to K$ such that $F_i \circ \pi = \pi \circ \sigma_i$ for every $i \in \langle N\rangle$. Denote $W_m := \langle N\rangle^m$ the set of words of length $m \in \mathbb{Z}^+$, and $W_* = \bigcup_{m\geq 0} W_m$. Each word $w = w_1 \ldots w_m \in W_m$ defines a continuous injection $F_w : K \to K$ by $F_w := F_{w_1} \circ \cdots \circ F_{w_m}$ whose image $F_w(K)$ is denoted by K_w.

Let L be a self-similar structure on K. The critical set $C_L \subset \Sigma$ and the post critical set $P_L \subset \Sigma$ are respectively defined by

$$C_L = \pi^{-1}\left(\bigcup_{\substack{i \neq j \\ i, j \in \langle N\rangle}} (F_i(K) \cap F_j(K)) \right)$$

and $P_L = \cup_{n\geq 1} \sigma^n(C_L)$.

A self-similar structure L is called *post critically finite* (PCF for short) if P_L is a finite set. The boundary set of K is defined as $V_0 := \pi(P_L)$. The cardinal of V_0 is denoted by v, and its elements as p_i, with $i \in \langle v\rangle$. If we define $V_m = \bigcup_{i\in W_m} F_i(V_0)$ and $V_* = \bigcup_{m\geq 0} V_m$, then we have that $V_m \subset V_{m+1}$ and that K is the closure of V_*.

Example 2.7. Sierpinski gasket and Hata's tree-like set are PCF self-similar structures.

Another example is a close interval divided as N subintervals. Let $[x_0, x_N]$ be a closed interval, and let us consider points $x_0 < x_1 < \cdots < x_N$ and functions $F_i : [x_0, x_N] \to [x_{i-1}, x_i]$ defined as in Section 2.2. In this case, $P_L = \{1^\omega, N^\omega\}$.

2.4. Harmonic structures

A very detailed exposition on harmonic structures and harmonic functions defined on a PCF self-similar structure can be seen in [10]; and their respective proofs can be found in [12, Section 3.3]. The latter reference contains several examples of these structures, two among them are Hata's tree and Sierpinski gasket. We here restrict ourselves to cite the two results we use below.

Let $L = (K, \langle N\rangle, \{F_i\}_{i\in\langle N\rangle})$ be a PCF self-similar structure with a harmonic structure on K. (K is assumed to be connected.) Then, the following statements hold:

(a) For a given set $\{(p_i, u_i) \in V_0 \times \mathbb{R} : i = 1, 2, \ldots, v\}$, there exists one, and only one, harmonic function u^* defined on L satisfying that $u^*(p_i) = u_i$.
(b) If u^* is an harmonic function defined in L, then there exists one, and only one, continuous function $u : K \to \mathbb{R}$ satisfying $u(x) = u^*(x)$ for all $x \in V_*$.

The set of functions u satisfying (b) is denoted by $\mathcal{A}(L)$.

3. Results

3.1. A generalized fractal interpolation

Throughout this section $L=(K, \langle N\rangle, \{F_i\}_{i\in\langle N\rangle})$ is a PCF self-similar structure. Elements in V_0 will be denoted by $p_1, \ldots, p_v$ and those in V_1 by $t_1, \ldots, t_A$, under the assumption that $p_1 = t_1, \ldots, p_v = t_v$. We also use notation $t_{i,k}=F_i(p_k)$. In the case of interval $[x_0, x_N]$ introduced in Section 2.2, we have $t_1=p_1=x_0$, $t_2=p_2=x_N$, $t_i=x_{i-2}$ with $i=3, \ldots, N+1$.

Definition 3.1. Let $L=(K, \langle N\rangle, \{F_i\}_{i\in\langle N\rangle})$ be a PCF self-similar structure, $\mathbf{u} = (u_1, \ldots, u_A) \in \mathbb{R}^A$, let ρ_i be numbers in $]0, 1[$, $i=1, \ldots, N$, and S be a family of functions defined in K. We call a generalized Barnsley interpolation problem (a Barnsley problem, for short) to find N functions $r_i\in S$, and a function h defined in K satisfying that $h(t_i)=u_i$, with $i\in\langle A\rangle$ and

$$h(F_i(x)) = \rho_i h(x) + r_i(x),$$

for all $x\in K$.

Barnsley's problem generalizes classical Barnsley's fractal interpolation where we have $t_1=p_1=x_0$, $t_2=p_2=x_N$, $t_i=x_{i-2}$ with $i=3, \ldots, N+1$ and S consists of the set of affine functions. Possibly, the most interesting interpolation problems are those where the uniqueness of the solution is ensured. For this reason, we are interested in the study of conditions that imply a unique solution for Barnsley's problem, as in [5,15].

Proposition 3.2. *Let* $L:=(K, \langle N\rangle, \{F_i\}_{i\in\langle N\rangle})$ *be a PCF self-similar structure, and let* $\{r_i\}$ *be a family of N bounded functions defined in* $K\setminus V_*$ *Let* $\{\rho_i\}_{i\in\langle N\rangle}$ *be real numbers in* $]0, 1[$. *Then, there exists a unique bounded function h in* $K\setminus V_*$ *satisfying the functional relations*

$$h(F_i(x)) = \rho_i h(x) + r_i(x), \text{ with } x \in K\setminus V_*. \tag{1}$$

Proof. The proof is based on the Banach fixed point theorem. Let us note that, for a given function, to satisfy functional relations (1) is equivalent to satisfy:

$$h(x) = \rho_i h(F_i^{-1}(x)) + r_i(F_i^{-1}(x))$$

in case $x\in F_i(K)\setminus V_*$.

Let $\mathcal{B}(K\setminus V_*)$ be the class of all bounded functions in $K\setminus V_*$ equipped with the supremum norm $|\circ|$. Let us define the operator

$$H : \mathcal{B}(K\setminus V_*) \to \mathcal{B}(K\setminus V_*)$$

such that, for each f, the image $H(f)$ is given by

$$H(f)(x) = \rho_i f(F_i^{-1}(x)) + r_i(F_i^{-1}(x)), \quad \text{foreach } x \in F_i(K)\setminus V_*$$

Moreover, if $x\in F_i(K)\setminus V_*$, then

$$|H(f)(x) - H(g)(x)| = |\rho_i||f(F_i^{-1}(x)) - g(F_i^{-1}(x))| \le |\rho_i||f - g|.$$

As a consequence, $||H(f) - H(g)| \le |\rho|\,|f - g||$, and

$$\rho = \max\{|\rho_i| : i \in \langle N\rangle\} < 1,$$

that is, H is a contraction. Therefore, the existence and uniqueness of a bounded function h that satisfies the functional relations (1) in the statement is guaranteed. □

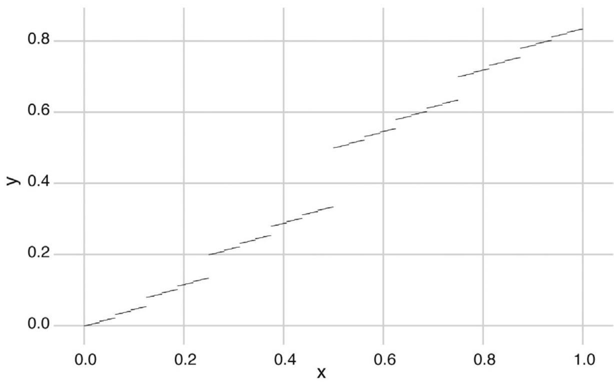

Fig. 2. Case in which $a = 0.4$, $b = 0.5$, $c = 0.4$

Proposition 3.2 guaranties that for given functions r_i, a unique function h is built on $K \setminus V_*$. But, it is possible that this function is not well defined on V_1, because if x belongs to V_1 can exist w and v such that $\pi(w) = \pi(v) = x$. Then, the function h defined in Proposition 3.2 could have different values, i.e. $h(\pi(w)) \neq h(\pi(v))$.

The following example illustrates this fact.

Example 3.3. In [4] is studied the system of functional equations

$$\begin{cases} h\left(\dfrac{t}{2}\right) = ah(t), & \text{if } 0 < t < 1 \\ h\left(\dfrac{t+1}{2}\right) = b + ch(t), & \text{if } 0 < t < 1 \end{cases}$$

In the case $0 \neq a$, $|a| < 1$ and $|c| < 1$, the system has a unique left-continuous solution in $0 < t \leq 1$. The graph of this function at points without finite dyadic expansion coincides with that obtained in Proposition 3.2, in case $K = \mathbb{I}$, $N = 2$, $F_1(x) = x/2$, $F_2(x) = (x+1)/2$, $r_1(x) = 0$, $r_2(x) = b$, $\rho_1 = a$, and $\rho_2 = c$. For points with finite dyadic expansion the result is obtained via left-continuity extension.

We can ask the following question: is it possible to define the function in Example 3.3 at points with finite dyadic expansion in such a way the new function in $\mathbb{I}$ be an extension of that in Proposition 3.2? To this end, it is necessary that $f(0) = 0$, which implies $0 < a = 1 - c$ or $b = 0$. In this case, we obtain an affirmative answer, but this is false for the general case.

Fig. 2 shows the case in which $a = 0.4$, $b = 0.5$, $c = 0.4$:

To extend Proposition 3.2 for all the elements in K it is necessary to avoid problems for points in V_*. The next statement is true:

Proposition 3.4. *Let $L := (K, \langle N \rangle, \{F_i\}_{i \in \langle N \rangle})$ be a PCF self-similar structure, and let r_i, $i = 1, 2, \ldots, N$ be a finite family of bounded functions in V_* Let $\{\rho_i\}_{i \in \langle N \rangle}$ be real numbers in $]0, 1[$. If there exists a bounded function h in V_* satisfying that*

$$h(F_i(x)) = \rho_i h(x) + r_i(x), \quad \text{with } x \in V_*, \tag{2}$$

then it is necessary that $r_i(x) = h(F_i(x)) - \rho_i h(x)$ in the case in which $x \in V_0$.

For the reverse, given N bounded functions r_i in V_*, and a function h^* in V_1 satisfying $r_i(x) = h^*(F_i(x)) - \rho_i h^*(x)$, when $x \in V_0$, then there exists a unique function h defined in V_* satisfying (2) that extends h^*.

Proof. The former statement is immediate. For the latter, if $x \in V_0$ then the assumptions ensure that Eq. (2) is true. If $x \in V_n \setminus V_1$, then there exists one and only one i satisfying that $x \in F_i(K)$ and we can define $h(x) := \rho_i h(F_i^{-1}(x)) + r_i(F_i^{-1}(x))$. Therefore, this is a sufficient condition. □

The above results can be now stated as:

Theorem 3.5. *Let $L := (K, \langle N\rangle, \{F_i\}_{i\in\langle N\rangle})$ be a PCF self-similar structure, and let a finite family of bounded functions $r_1, \ldots, r_N$ in K be given. Then, for given A real values $\{u_j\}_{j\in\langle A\rangle}$, and $V_1 := \{t_1, \ldots, t_A\}$, there exists a unique bounded function h in K satisfying that $h(t_j) = u_j$, and*

$$h(F_i(x)) = \rho_i h(x) + r_i(x), \text{ with } i = 1, 2, \ldots, N,$$

if and only if $t_j = t_{i,k}$ then $u_j = \rho_i u_k + r_i(p_k)$.

When functions r_i are continuous, then h is also continuous.

3.2. Fractal interpolation on PCF self-similar structures

The result obtained above allows the study of the interpolation problem on PCF compact sets $K \subset \mathbb{R}^n$. When, for given functions F_i, there are parameters $0 < s_i < 1$ such that $|F_i(x) - F_i(y)| < s_i\,|x - y|$ for $x, y \in K$, we say that $L = (K, \langle N\rangle, \{F_i\}_{i\in\langle N\rangle})$ is a *contractive PCF self-similar structure*.

To obtain Theorem 3.5 we were not able to use similar ideas to those of Barnsley for FIF because the only condition imposed on the functions was that they were bounded. In the result that follows we impose conditions that allow the use of ideas close to those of this author.

As usual, we denote by $\mathcal{C}(K)$ the family of continuous functions defined in the compact set K.

Theorem 3.6. *Let $L = (K, \langle N\rangle, \{F_i\}_{i\in\langle N\rangle})$ be a contractive PCF self-similar structure and S a subclass of $\mathcal{C}(K)$ such that for each $u_i \in \mathbb{R}$, $i \in \langle v\rangle$, there exists an only function f of S such that $f(p_i) = u_i$. If there exists a constant k satisfying that $|f(x) - f(y)| \le k\,|x - y|$, with $x, y \in K$, for each $f \in S$, then, for a given set of real numbers $\{\rho_i\}_{i\in\langle N\rangle}$, $0 < |\rho_i| < 1$, a unique $h \in \mathcal{C}(K)$ exists satisfying that $h(t_j) = u_j$ for $t_j \in V_1$ and $h(F_i(x)) = \rho_i h(x) + r_i(x)$, with $i \in \langle N\rangle$, such that r_i is the only function of S such that $r_i(p_k) = u_j - \rho_i u_k$, and index j corresponds to $t_j = t_{i,k}$.*

Moreover, the graph of h coincides with the attractor of the IFS in $\mathbb{R}^n \times \mathbb{R}$ determined by maps $P_i\ (x, y) = (F_i(x), \rho_i y + r_i(x))$, with $i \in \langle N\rangle$; that is

$$\mathrm{G}(h) = \cup_{i\in\langle N\rangle} P_i(\mathrm{G}(h)).$$

Proof. For each $i \in \langle N\rangle$, there is a constant k_i satisfying that

$$|r_i(x) - r_i(y)| \le k_i|x - y|.$$

Let β be a constant such that $s_i^2 + 4k_i^2/\beta^2 < 1$ and $2\rho_i/\beta < 1$ for all i .

Theorem 3.5 ensures the existence of a unique continuous function f satisfying that $f(t_j) = u_j/\beta$, with $t_j \in V_1$, and $f(F_i(x)) = \rho_i f(x) + r_i'(x)$, with $i \in \langle N\rangle$, where r_i' is the unique function of S satisfying that $r_i'(p_k) = u_j/\beta - \rho_i(u_k)/\beta$, and index j corresponds to $p_j = F_i(p_k)$. The graph of f is a fixed point for the IFS system defined by maps $P_i^*(x, y) = (F_i(x), (\rho_i y + r_i(x))/\beta)$, $i \in \langle N\rangle$.

Each one of these maps is a contraction. Therefore, $P^*(A) := \cup_{i\in\langle N\rangle} P_i^*(A)$ is a contraction in the space $\mathcal{K}(\mathbb{R}^n)$ of compact sets in $\mathbb{R}^n$ endowed with the Hausdorff metric. Now, if G(f) is the attractor of P^*, then the graph of $h = \beta f$ is an attractor of the function P defined by$P(A) := \cup_{i\in\langle N\rangle} P_i(A)$. □

Notation 3.7. We denote by *FIF*(L, $\{\rho_i\}$, S) the class of functions h introduced in Theorem 3.6.

We recall that polynomial interpolation involves finding a polynomial of degree N that passes through the $N+1$ data points. For a given vector subspace V of polynomials in n variables, if we consider the subset below

$$\Gamma = \{(x_i, u_i) \in \mathbb{R}^n : i = 0, 1, 2, \ldots, m; \quad m \ge 2\},$$

the Lagrangian interpolation problem consists to find elements $q \in V$ satisfying that $q(x_i) = u_i$. Let us denote by Π_n the set of polynomials in two variables of degree, at most, n .

Corollary 3.8. *Let $L = (K, \langle N\rangle, \{F_i\}_{i\in\langle N\rangle})$ be a contractive PCF self-similar structure on a compact $K \subset \mathbb{R}^2$. Let V_0 be a set of points where the interpolation Lagrangian problem has a unique solution in Π_n . Then, there exists a unique* *continuous function* $h \in FIF(L, \{\rho_i\}, \Pi_n)$ *such that* $h(t_i) = u_i$ *with* $t_i \in V_1$, *and* $h(F_i(x)) = \rho_i h(x) + r_i(x)$ *for all* $i \in \langle N \rangle$, *where* r_i *is the unique function in* Π_n *satisfying that* $r_i(p_k) = u_j - \rho_i u_k$, *where index j means* $t_j = t_{i,k}$.

Example 3.9. In case of Hata's tree, we have $V_0 = \{0, 1, c\}$, and the above result is true for Π_1. Now, this situation is similar to that in Theorem 2.1, where lines are substituted by planes.

Proposition 3.10. *Let* $L = (K, \langle N \rangle, \{F_i\}_{i \in \langle N \rangle})$ *be a contractive PCF self-similar structure. If S is a vector space, then* $FIF(L, \{\rho_i\}, S)$ *is a vector space, as well; and its dimension is equal to the cardinal of* V_1.

Proof. We respectively write $h_{\mathbf{u}}$ and $r_{i,\mathbf{u}}$ for the interpolating function and functions that are associated with the problem of interpolation of Barnsley with vector **u**. Let us consider a scalar λ. If $h_{\mathbf{u}} \in FIF(L, \{\rho_i\}, S)$ and $r_{i,\mathbf{u}}$ denote their associated functions, then $\lambda h_{\mathbf{u}} = h_{\lambda \mathbf{u}}$. Therefore, $\lambda r_{i,\mathbf{u}} = r_{i,\lambda\mathbf{u}}$. When $h_{\mathbf{u}}, h_{\mathbf{v}} \in FIF(L, \{\rho_i\}, S)$ and $r_{i,\mathbf{u}}, r_{i,\mathbf{v}}$ are their respective associated functions, then $h_{\mathbf{u}} + h_{\mathbf{v}} = h_{\mathbf{u+v}}$ and $r_{i,\mathbf{u}} + r_{i,\mathbf{v}} = r_{i,\mathbf{u+v}}$.

A basis for this vector space is given by $\{h_{\mathbf{u}_i}\}_{i \in \langle N \rangle}$, where u_i is a vector with 0s, but with 1 at the i-th coordinate. □

3.3. Harmonic interpolation on PCF structures

Note that Corollary 3.8 seems to be a natural extension of Theorem 2.1 using polynomials, but this requires a condition that cannot possibly be satisfied under the hypothesis that the Lagrangian interpolation problem has a unique solution V_0. Such an example is the well-known snowflake (see [12, p. 54]), which is given by contractions

$$F_i : \mathbb{C} \to \mathbb{C} \text{ given by } F_i(z) = \frac{z + 2p_i}{3}, \quad \text{where } i \in \langle 7 \rangle,$$

and $p_i = e^{(2\pi i\sqrt{-1})/6}$, for $i \in \langle 6 \rangle$ and $p_7 = 0$. In this case $V_0 = \{p_i\}_{i \in \langle 6 \rangle}$, and the existence and uniqueness of a solution of the Lagrangian interpolation problem does not have a general answer in Π_2.

Therefore, applications of Corollary 3.8 depend on the way the points are distributed in $\mathbb{R}^2$. There exist families of functions that ensure the existence and uniqueness of a function h satisfying $h(p_i) = u_i$. They are the harmonic functions in L.

On the other hand, harmonic functions have a disadvantage: it is impossible to ensure the existence of a parameter k satisfying $d(f(r), f(s)) \leq k d(r, s)$ for every $r, s \in K$. This is the reason why it is not possible to generalize Theorem 3.6 using these functions.

Example 3.11. It is immediate that the unit interval $\mathbb{I}$ is a self-similar set for the pair of functions given by $F_1(x) = x/2$ and $F_2(x) = x/2 + 1/2$. Therefore, we have a PCF self-similar structure. Following the ideas in [12, Example 3.1.4], the associated harmonic functions which are related to this example are, precisely, the functions in the Example 3.3 when $a = b = 1 - c$. For these functions, there is no k such that $|f(x) - f(y)| \leq k\,|x - y|$, for $x, y \in K$.

In this context, applying Theorem 3.5 to PCF self-similar structures, the following statement is true.

Theorem 3.12. *Let* $L = (K, \langle N \rangle, \{F_i\}_{i \in \langle N \rangle})$ *be a PCF self-similar structure with an harmonic structure, and* $\mathcal{A}(L)$ *be the family of functions satisfying (b) in Section 2.4. Then, there exists one and only one bounded function h in K such that* $h(t_i) = u_i$ *for* $t_i \in V_1$ *and*

$$h(F_i(x)) = \rho_i h(x) + r_i(x), \quad \text{with } i \in \langle N \rangle,$$

where $r_i : F_i(K) \to \mathbb{R}$ *is the unique function in* $\mathcal{A}(L)$ *satisfying* $r_i(p_k) = u_j - \rho_i u_k$, *where the index j corresponds to* $t_j = t_{i,k}$.

The class of functions h satisfying the statement in the above theorem is denoted by $\mathcal{H}(L, \{\rho_i\})$. The class $\mathcal{A}(L)$ is a vector space; therefore, as a consequence of Proposition 3.10, the following result is true.

Corollary 3.13. *Let* $L = (K, \langle N \rangle, \{F_i\}_{i \in \langle N \rangle})$ *be a PCF self-similar structure with an harmonic structure. Then,* $\mathcal{H}(L, \{\rho_i\})$ *is a vector space with a dimension equal to* $\dim(V_1)$.

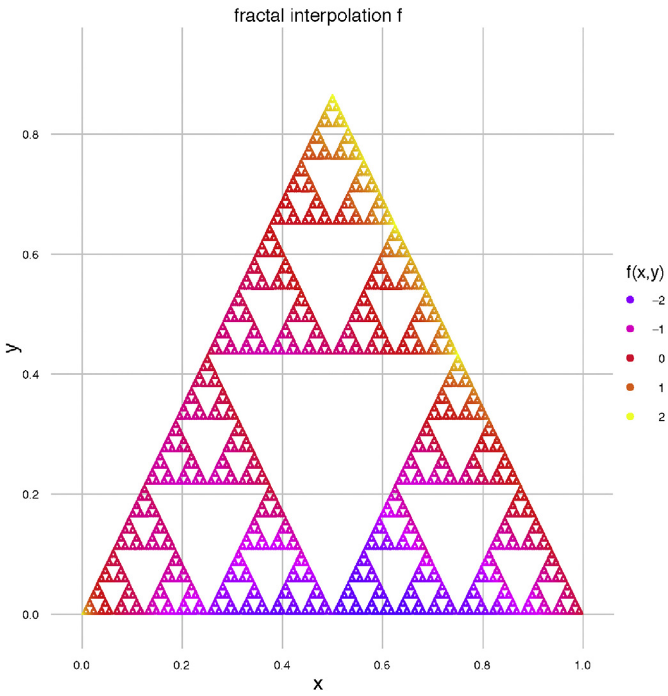

Fig. 3. Interpolation on the Sierpinski gasket

To end this section we present two simulations of the harmonic interpolation on PCF structures. Figs. 3 and 4 show the graph of the function that interpolates the values

$$\{2, 0, 2, 0, -2, 2\}$$

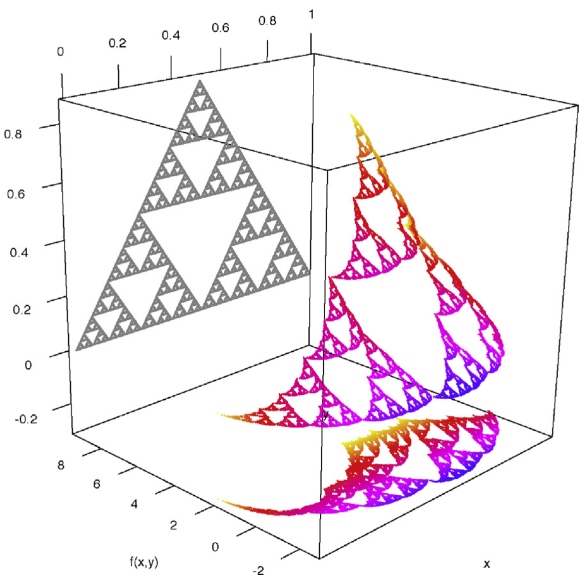

Fig. 4. Interpolation on the Sierpinski gasket 3d-plot

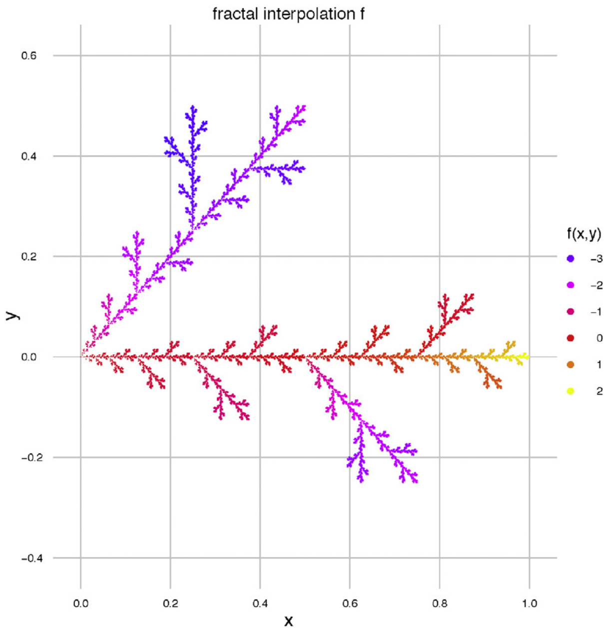

Fig. 5. Interpolation on the Hata tree

at the points

$$V_1 = \left\{(0,0), (1,0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right), \left(\frac{1}{2}, 0\right), \left(\frac{3}{4}, \frac{2+\sqrt{3}}{4}\right)\right\}$$

in the Sierpinski gasket. With the aim of a better appreciation of the curvature, we include a perpendicular projection to the *OX* axe in the graph. Fig. 3 shows the colored values of the function, and Fig. 4 the three-dimensional representation.

In Figs. 5 and 6 we do the same as before, now for Hata's tree with parameter $c = (1 + i)/2$. In this case

$$V_1 = \left\{(0,0), (1,0), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, 0\right), \left(\frac{3}{4}, \frac{-1}{4}\right)\right\},$$

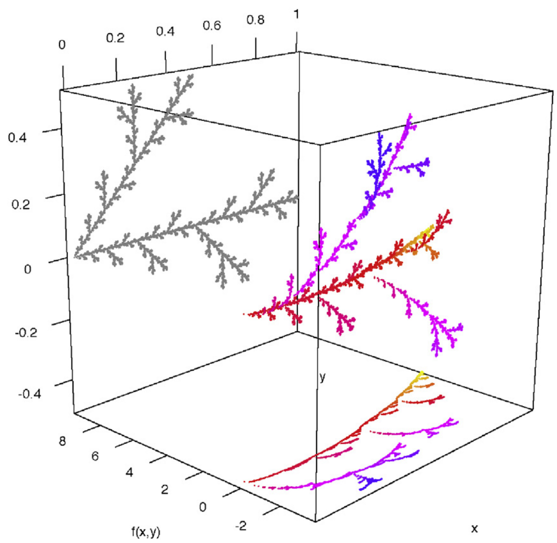

Fig. 6. Interpolation on the Hata tree 3d-plot

and the interpolated values are

$$\{0, 2, -2, 0, 2\},$$

and we use the regular harmonic structure where $h = \sqrt{3}/2$ (see [12, p. 71]).

4. Conclusions

This paper deals with the interpolation problem introduced by Barnsley for real functions on a real closed interval. We state a generalized fractal interpolation theory to post critical finite (PCF) compact sets in general Euclidean space, using the analytic approach on PCF self- similar harmonic structure developed by Kigami. Our results include Barnsley's fractal interpolation theory of Sierpinski gasket (by Celik et al.) and Koch Curve (by Paramanathan and Uthayakumar). The proposed theory is illustrated with classic and new examples.

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