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DISCONTINUITY SETS OF SOME INTERVAL BIJECTIONS

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Abstract: In this paper we prove that the discontinuity set of any bijective function from $[a, b[$ to $[c, d]$ is infinite. We then proceed to give a gallery of examples which shows that this is the best we can expect, and that it is possible that any intermediate case exists.

We conclude with an example of a (nowhere continuous) bijective function which exhibits a dense graph in the rectangle $[a, b[\times [c, d]$.

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1. Introduction

During our investigations [1] concerning on the application of Ergodic Theory of Numbers to generate singular functions (i.e., their derivatives vanish almost everywhere), we came across a bijection from a closed interval to another that it is not closed: $c : [0, 1] \longrightarrow [0, 1[$, given by (segments on some intervals):

$$c(x) := \frac{(n-1)!}{n} x - \frac{n! - 1}{n^2} + \frac{1}{n+1}$$

which maps $\left[1 - \frac{1}{n!}, 1 - \frac{1}{(n+1)!}\right[$ onto $\left[\frac{1}{n+1}, \frac{1}{n}\right[$ and $c(1) = 0$.

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Such a function provides very interesting properties as:

i. c preserves arithmetical properties for x;

ii. for composition $c^n := c \circ \overset{n}{\cdots} \circ c$ and for all x, the sequence $(c^n(n))_n$ has an associated density function (a.d.f.) which is a singular function; and

iii. this function c allows us to show an example of a non measurable Lebesgue set.

After this finding, we asked ourselves if there were other examples of bijections f of this explicit nature given by a formula and if their properties had been studied or not.

The literature we have read up to this moment is richer from the theoretical than the exemplary point of view (see [2] and [3]):

a. such a function f will never be continuous on all its domain of definition;

b. explicit examples of such bijections f are not exhibited

After this stage, questions concerning the nature of the set $D(f)$ of points where f is not continuous arise in a natural way.

Some interesting facts that are well known with respect to this discussion (see [2]) are the following:

(α) If f is monotonic, then $D(f)$ is a denumerable set (and, conversely, for each denumerable set A there exists a monotonic function f such that $A = D(f)$).

(β) There is no f such that $\mathbb{R}\backslash\mathbb{Q} = D(f)$ (it is a consequence of the Baire Category Theorem).

(γ) There exists f for which $D(f)$ is the Cantor Ternary Set.

(δ) For any f, $D(f)$ is a denumerable union of closed sets ($D(f) \in F_\sigma$ for short), and conversely, for all $E \in F_\sigma$, there is some function f such that $E = D(f)$. (Moreover, if the set E is closed, then the function f can be taken bounded.)

(ϵ) The set $D(f)$ is of first category (i.e. it is a denumerable union of sets M such that $\overset{\circ}{\overline{M}}$ is empty) if and only if f is continuous at a dense set.

A simple and basic fact grows up among all these ideas, if f is a bijection: clearly $D(f) \neq \emptyset$; but, what more can be said concerning its size?

We give a simple but, at the same time, complete solution in the sense that we prove the following:

i. For each bijection $f : [a, b[\longrightarrow [c, d]$, the set $D(f)$ is infinite.

ii. There are bijections $f, g : [a, b[\longrightarrow [c, d]$ such that on the one hand $D(f)$ is denumerable and on the other hand $D(g) = [a, b[$.

As a colophon, we introduce an example (the 8th) which flirts with the classical, but always up-to-date, idea of space-filling curves (see [4]): we carry out (of course, very far from the situation $D(f) = \emptyset$, now f will be nowhere continuous!) a bijection from $[0, 1[$ to $[0, 1]$, such that its graph is dense in the unit square $[0, 1]^2$.

Moreover we propose an Open Question to investigate whether a bijection $f : [a, b[\longrightarrow [c, d]$ exits or not, such that the set $D(f)$ is denumerable and has derivatives at a finite number of points.

2. The Result

Elementary topological techniques in Real Analysis show that one-to-one and onto continuous mappings between intervals on the real line $\mathbb{R}$ are homeomorphisms; i.e. f and f^{-1} are continuous bijections. (Such a function is forced to be strictly monotonic.) A first consequence of this fact is that if one of the intervals is open then the other one is open too. Thus, we have the following and well-known fact:

Fact. *There are no continuous bijections from $[a, b[$ to $[c, d]$; i.e. these intervals are not homeomorphic.*

Let us denote by $D(f)$ the set of points where f is not continuous (or discontinuous), i.e. $D(f)$ is the discontinuity set of f. We are interested in its size. We will consider, without loss of generality, bijections from $[0, 1[$ to $[0, 1]$. Let $\mathbb{N}$ denote the natural numbers, i. e. the set $\mathbb{Z}^+$ of all positive integers.

Firstly, we prove that for such bijections the set $D(f)$ is infinite.

Theorem. *No continuous bijection exists from $[0, 1[$ to $[0, 1]$ but a finite number of points, i.e.*

$$\{f : [0, 1[\longrightarrow [0, 1] \mid f \text{ bijection and } D(f) \text{ finite}\} = \emptyset.$$

Proof. By *Reductio ad absurdum*, we suppose there exists a bijection $f : [0,1[\longrightarrow [0,1]$, that is continuous but at points in a finite set. Let us denote the set $\{x_0, x_1, ..., x_p\} := D(f) \cup \{0\}$. We have no loss of generality if we consider the lexicographic order ($n < m \Rightarrow x_n < x_m$) on this finite set. (Note that $x_0 = 0$.) Let $x_{p+1} := 1$, and denote $I_k :=]x_{k-1}, x_k[$, for $k = 1, 2, ..., p+1$. Hence, there is a collection

$$\{f(I_k) : k = 1, 2, ..., p+1\}$$

of (nonempty and) pairwise disjoint open intervals, i.e.

$$[0,1] \setminus \{f(I_k) : k = 1, 2, ..., p+1\} = \{y_0, y_1, ..., y_p, y_{p+1}\}.$$

But this yields a contradiction: it is impossible that $\{f(x_0), f(x_1), ..., f(x_p)\}$ and $\{y_0, y_1, ..., y_p, y_{p+1}\}$ to be equipotent sets. $\square$

3. A Gallery of Examples

Among infinite sets, denumerable ones are the smallest. The next example presents a particular f that will be the smoothest we can expect to encounter.

Example 1. There exists a bijection f from $[0,1[$ to $[0,1]$ having (a constant) derivative at each point but a denumerable set in $[0,1[$.

Let $f : [0,1[\longrightarrow [0,1]$ be given by

$$f(x) := \begin{cases} 1, & x = 0 \\ x, & x \in]0,1[\setminus \left\{\frac{1}{n+1} : n \in \mathbb{N}\right\} \\ 1/n, & x \in \left\{\frac{1}{n+1} : n \in \mathbb{N}\right\} \end{cases}$$

This function is one-to-one and onto. Moreover, it is continuous on $]0,1[\setminus \left\{\frac{1}{n+1} : n \in \mathbb{N}\right\}$, because the local character of continuity. The same goes for derivability. In fact, $f'(x) = 1$ for all $x \in]0,1[\setminus \left\{\frac{1}{n+1} : n \in \mathbb{N}\right\}$. Furthermore, f is discontinuous on each $x \in \left\{\frac{1}{n+1} : n \in \mathbb{N}\right\} \cup \{0\}$. $\square$

The way to define the function f in Example 1 suggests a suitable method to find an explicit bijection from $[0,1]$ on $[0,1] \cap (\mathbb{R} \setminus \mathbb{Q})$.

The rest of the paper is devoted to exhibiting more examples showing that many intermediate cases are possible.

In the above example $f'\left([0,1[\setminus D(f)\right)$ splits to one point. This idea is very far from being the paradigm.

Example 2. There exists a bijection f from $[0,1[$ to $[0,1]$ such that for all $r>0$ there exist $x\in[0,1[$ such that $f'(x)=r$.

Let $f:[0,1[\longrightarrow[0,1]$ be given by

$$f(x):=\begin{cases}0, & x=0\\ \left[n(n+1)\right]^{n-1}\left(x-\frac{n-1}{n}\right)^{n}+\frac{1}{n+1}, & x\in\left]\frac{n-1}{n},\frac{n}{n+1}\right],n\in\mathbb{N}\end{cases}$$

Easy calculations show that, for each natural n:

$$f\left(\left]\frac{n-1}{n},\frac{n}{n+1}\right]\right)=\left]\frac{1}{n+1},\frac{1}{n}\right]$$

(in particular, one-to-one) and each one of the families $\left\{\left]\frac{n-1}{n},\frac{n}{n+1}\right]:n\in\mathbb{N}\right\}$ and $\left\{\left]\frac{1}{n+1},\frac{1}{n}\right]:n\in\mathbb{N}\right\}$ is a collection of mutually disjoint intervals and

$$]0,1[=\bigcup_{n=1}^{+\infty}\left]\frac{n-1}{n},\frac{n}{n+1}\right]\text{ and }]0,1]=\bigcup_{n=1}^{+\infty}\left]\frac{1}{n+1},\frac{1}{n}\right];$$

thus, f is one-to-one and onto from $[0,1[$ to $[0,1]$.

Moreover, this function has derivative at each point $x\in\left]\frac{n-1}{n},\frac{n}{n+1}\right[,n\in\mathbb{N}$, and satisfies

$$f'\left(\left]\frac{n-1}{n},\frac{n}{n+1}\right[\right)=]0,n[\,.$$

We omit small modifications in f, for the sake of simplicity we omit them, which would allow us to obtain a bijection such that $\{f'(x):f\text{ has derivative at }x\}=\mathbb{R}$. □

Example 3. There exists a bijection f from $[0,1[$ to $[0,1]$ such that it is continuous but a denumerable and dense set in $[0,1[$, and it has no derivative at any point in $[0,1[$.

Let us consider the sequence (a_n):

$$\frac{1}{2},\frac{1}{2^2},\frac{3}{2^2},\frac{1}{2^3},\frac{3}{2^3},\frac{5}{2^3},\frac{7}{2^3},\dots,\frac{1}{2^n},\frac{3}{2^n},\dots,\frac{2^n-1}{2^n},\dots;$$

that is,

$$a_n:=\frac{2n+1}{2^{E(\log_2 n)+1}}-1,\qquad\forall n\in\mathbb{N},$$

where E denotes the integral part of a real number:

$$E(x) := \max\{p \in \mathbb{Z} : p \leq x\}.$$

(The set $\{a_n : n \in \mathbb{N}\}$ is dense in $[0,1[$.)

Let $f : [0,1[\longrightarrow [0,1]$ be given by

$$f(x) := \begin{cases} x, & x \in [0,1[\setminus \{a_n : n \in \mathbb{N}\} \\ 1, & x = a_1 \\ a_n, & x \in \{a_{n+1} : n \in \mathbb{N}\} \end{cases}$$

The method of definition, already used before in Example 1, shows that this function is one-to-one and onto. Moreover, $D(f) = \{a_n : n \in \mathbb{N}\} \cup \{0\}$.

We will only prove that there are no derivatives of f at any point $a \in [0,1[\setminus \{a_n : n \in \mathbb{N}\}$.

On the one hand, with fixed but arbitrary a, if $x \in [0,1[\setminus \{a_n : n \in \mathbb{N}\}$, $x \neq a$, then

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = 1.$$

On the other hand, if we consider $x = 0.x_1x_2...x_n...$ with infinite binary expansion (i.e. the set $\{n \in \mathbb{N} : x_n = 1\}$ is infinite), then we will consider (b_n) the sequence on truncations of x given by

$$b_n := 0.x_1x_2...x_{n-1}x_n$$

with $x_n = 1$. (We can rearrange it if necessary.) Hence:

$$\frac{f(x) - f(b_n)}{x - b_n} = 1 + \frac{b_n - f(b_n)}{x - b_n} \geq 1 + \frac{1/2^{n-1}}{1/2^n} = 3.$$

Moreover, $f'(0)$ does not exist either, such as it is expressed by

$$\frac{f\left(\frac{5}{2^n}\right) - f(0)}{\frac{5}{2^n} - 0} = \frac{3}{2^n} \Big/ \frac{5}{2^n} = 3/5 \neq 1/3 = \frac{1}{2^n} \Big/ \frac{3}{2^n} = \frac{f\left(\frac{3}{2^n}\right) - f(0)}{\frac{3}{2^n} - 0}.$$

Similar arguments show that $f'(x)$ does not either exist when x is a rational with finite expansion.

If we consider $f(x) := 1 - x$, when $x \in [0,1[\setminus \{a_n : n \in \mathbb{N}\}$ (instead of the other definition), then f has no fixed points and $D(f) = [0,1[$ (see Example 7). $\square$

Open Question. Does a bijection f exist from $[0,1[$ to $[0,1]$ such that it is continuous but an infinite denumerable set in $[0,1[$ having derivatives at points of a finite set?

Example 4. There exists a bijection f from $[0,1[$ to $[0,1]$ such that it is continuous on a finite subset of $[0,1[$ having derivatives at each point.

Let us denote the following denumerable sets:

$$\begin{aligned}
&\{a_n : n \in \mathbb{N}\} := \{x \in \mathbb{Q} : 0 \leq x < 1/4\}\\
&\{b_n : n \in \mathbb{N}\} := \{x \in \mathbb{Q} : 3/4 < x < 1\}\\
&\{\alpha_n : n \in \mathbb{N}\} := \{y \in \mathbb{Q} : 0 \leq y < 5/16\}\\
&\{\beta_n : n \in \mathbb{N}\} := \{y \in \mathbb{Q} : [5/16, 13/16]\} \setminus \left\{x^2 + \tfrac{1}{4} : x \in \mathbb{Q}\right\}\\
&\{\gamma_n : n \in \mathbb{N}\} := \{y \in \mathbb{Q} : 13/16 < y \leq 1\},
\end{aligned}$$

and we redefine

$$\begin{aligned}
&\{x_n : n \in \mathbb{N}\} := \{a_n, b_n : n \in \mathbb{N}\}\\
&\{y_n : n \in \mathbb{N}\} := \{\alpha_n, \beta_n, \gamma_n : n \in \mathbb{N}\}.
\end{aligned}$$

With these previously fixed sequences, let us define

$$f(x) := \begin{cases} x, & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1[\\ x^2 + \frac{1}{4}, & x \in \mathbb{Q} \cap [1/4, 1/3] \\ y_n, & x = x_n, n \in \mathbb{N} \end{cases}$$

If we rearrange the initial sequences in a suitable order (permutations on $\mathbb{N}$), then f is continuous at $1/2$ with derivative $f'(1/2) = 1$ and $D(f) = [0,1[\setminus \{1/2\}$. □

Example 5. There exists a bijection f from $[0,1[$ to $[0,1]$ such that it is continuous on a finite subset of $[0,1[$ having no derivatives at any point.

The function $f : [0,1[\longrightarrow [0,1]$ given by

$$f(x) := \begin{cases} 1, & x = x_1 > 1/2 \\ x_n, & x = x_{n+1} = \frac{n+1}{n+2}, n \in \mathbb{N} \\ x, & x \in (\mathbb{Q} \setminus \{x_n : n \in \mathbb{N}\}) \cap [0,1[\\ 1 - x, & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1[\end{cases}$$

is an elementary sample. Note that $f\left(\frac{1}{2}\right) = \frac{1}{2} = \lim_{x \to 1/2} f(x)$ and the derivative at $1/2$ depends on going to by rationals or by irrationals:

$$\lim_{\substack{x \to 1/2 \\ x \in \mathbb{Q}}} f(x) = 1 \neq -1 = \lim_{\substack{x \to 1/2 \\ x \in \mathbb{R}\setminus\mathbb{Q}}} f(x).$$

Example 6. There exists a bijection f from $[0,1[$ to $[0,1]$ which is discontinuous everywhere (with fixed points).

Let us consider a strictly increasing sequence of rationals (x_n) such that

$$\frac{1}{2} =: x_1 < x_{n+1} < x_{n+2} \longrightarrow 1, n \in \mathbb{N}.$$

Let $f : [0,1[\longrightarrow [0,1]$ be given by

$$f(x) := \begin{cases} 1, & x = x_1 = 1/2 \\ x_n, & x = x_{n+1}, n \in \mathbb{N} \\ x, & x \in (\mathbb{Q} \setminus \{x_n : n \in \mathbb{N}\}) \cap [0,1[\\ 1-x, & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1[\end{cases}$$

Showing the desired properties for f is a funny Calculus exercise. $\square$

Example 7. There exists a bijection f from $[0,1[$ to $[0,1]$ which is discontinuous everywhere (without fixed points).

We return to the scheme in Example 3. If we modify the given function in the following way: $f(x) := 1-x$, when $x \in [0,1[\setminus \{a_n : n \in \mathbb{N}\}$, $f(a_1) := 1/4$, $f(a_2) := 1$, $f(a_3) := 1/2$, and it remains equally defined otherwise in $\{a_n : n \in \mathbb{N}, n \geq 4\}$, then f has no fixed points and $D(f) = [0,1[$.

(Note that we are forced to this surprising definition because if we do not change f on the sequence, then bad luck would play against us f being continuous at $a_3 = 3/4$. Check it yourself!) $\square$

Example 8. There exists a bijection f from $[0,1[$ to $[0,1]$ whose graph is dense in $[0,1[\times [0,1]$.

Let us consider the sequence (r_n) given by $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2}+\frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2}+\frac{1}{2^3}, \frac{1}{2^2}+\frac{1}{2^3}, \ldots$, that is

$$r_n := \frac{1}{2^{\alpha_1(n)+1}} + \frac{1}{2^{\alpha_2(n)+1}} + \frac{1}{2^{\alpha_3(n)+1}} + \ldots + \frac{1}{2^{\alpha_k(n)+1}},$$

where $n = 2^{\alpha_1(n)} + 2^{\alpha_2(n)} + 2^{\alpha_3(n)} + \ldots + 2^{\alpha_k(n)}$ (think of the binary expansion for natural numbers).

This sequence is dense and uniformly distributed in $[0,1[$.

On the other hand, we consider the sequence (a_n) in Example 3 once more. Let $f : [0,1[\longrightarrow [0,1]$ be given by

$$f(x) := \begin{cases} 1-x, & x \in [0,1[\setminus \{a_n : n \in \mathbb{N}\} \\ 0, & x = a_1 = 1/2 \\ r_n, & x \in \{a_{n+1} : n \in \mathbb{N}\} \end{cases}$$

(hence, f has no fixed points.) In this context, we can prove that for each rectangle

$$[a,b]\times[c,d]\subset[0,1[\times[0,1]$$

there exists $n\in\mathbb{N}$ such that $(a_{n+1},r_n)\in[a,b]\times[c,d]$; i.e.,

$$\{n\in\mathbb{N}:a_{n+1}\in[a,b],r_n\in[c,d]\}\neq\emptyset.$$

For the given interval $[c,d]$, there exist naturals $\alpha_1,\alpha_2,...,\alpha_k$ such that:

$$\left[\sum_{j=1}^{k}\frac{1}{2^{\alpha_j+1}},\sum_{j=1}^{k}\frac{1}{2^{\alpha_j+1}}+\frac{1}{2^{\alpha_k+1}}\right]\subset[c,d],$$

and for elements of (r_n) belonging to this interval, we have:

$$r_n=\sum_{j=1}^{k+n'}\frac{1}{2^{\alpha_j+1}}=\sum_{j=1}^{k}\frac{1}{2^{\alpha_j+1}}+\sum_{j=k+1}^{k+n'}\frac{1}{2^{\alpha_j+1}}$$

(where $0<\sum_{j=k+1}^{k+n'}\frac{1}{2^{\alpha_j+1}}<\frac{1}{2^{\alpha_k+1}}$) with

$$n=2^{\alpha_1}+2^{\alpha_2}+...+2^{\alpha_k}+2^{\alpha_{k+1}}+2^{\alpha_{k+2}}+...+2^{\alpha_{k+n'}}.$$

(Here, in fact, $\alpha_j=\alpha_j(n)$, for all j.)

But we can consider subsequences $\{a_{\sigma(n)}\}\subset[a,b]$ such that

$$\sigma(n)=2^{\alpha_1}+2^{\alpha_2}+...+2^{\alpha_k}+2^{\alpha_{k+1}}+2^{\alpha_{k+2}}+...+2^{\alpha_{k+\sigma(n)'}}$$

and hence,

$$f\left(a_{\sigma(n)+1}\right)=\sum_{j=1}^{k+\sigma(n)'}\frac{1}{2^{\alpha_j+1}}=\sum_{j=1}^{k}\frac{1}{2^{\alpha_j+1}}+\sum_{j=k+1}^{k+\sigma(n)'}\frac{1}{2^{\alpha_j+1}}=r_{\sigma(n)}.$$

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