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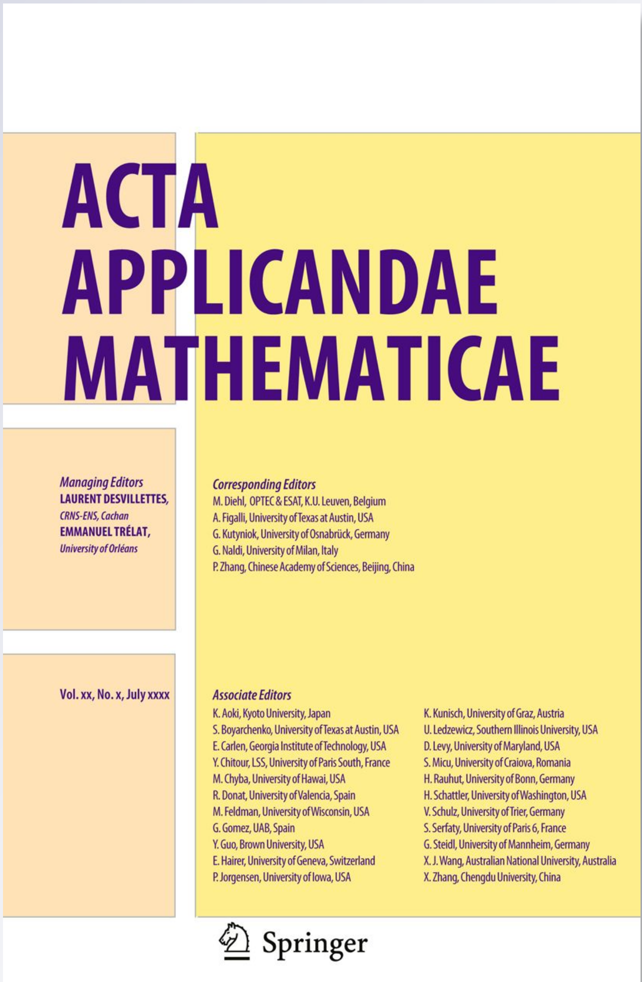

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Singular Functions with Applications to Fractal Dimensions and Generalized Takagi Functions

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Abstract We study a class of singular functions via a generalized dyadic system and Hausdorff dimensions are calculated for several sets related with these functions. Furthermore, we introduce a class of monotonic type on no-interval and almost everywhere differentiable functions that includes—as an exceptional case—the continuous nowhere differentiable Takagi function (multiplied by 2) among them.

Keywords Singular function · Generalized dyadic system · Schauder basis · (Simply) normal number · Hausdorff (fractal) dimension · MTNI function

1 Introduction

The first example of a continuous nowhere differentiable function was published by du Bois-Reymond in 1875. This example, given by the formula

$$W_{a,b}(x) := \sum_{k=0}^{+\infty} a^k \cos\bigl(b^k \pi x\bigr), \qquad 0 < a < 1, ab > 1 + \frac{3}{2}\pi, b + 1 \in 2\mathbb{Z},$$

was due to Weierstrass. Afterwards, Hardy proved that it is still a continuous nowhere differentiable function if $0 < a < 1$, $ab > 1$.

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Later, in 1903, Takagi gave another example of a continuous nowhere differentiable function as follows:

$$T(x) := \sum_{k=0}^{+\infty} \frac{d(2^k x)}{2^k}, \quad \forall x \in \mathbb{R}, \tag{1}$$

where $d(x)$ denotes the distance of x from the nearest integer.

Several properties of this function have been studied in depth, for example one-side derivatives, maxima, level sets, etc. as can be seen in [5, 6, 13, 24].

The first example of a singular function (i.e. a monotone increasing and continuous function whose derivatives vanish a.e.) was independently published by Cantor and Scheefer twelve years after the functions $W_{a,b}$ were introduced. In 1904, Minkowski gave an example that allowed him to enumerate the quadratic irrational numbers. Moreover, he established a bijection between rationals and numbers in the unit interval $\mathbb{I} := [0, 1]$ whose dyadic representation is finite via Farey's sequence. Denjoy showed the relation between Minkowski's representation system for real numbers and the representation by simple continuous fractions. More recently, Viader et al. in [30], showed this function as the asymptotic distribution function of an enumeration of the rationals in $\mathbb{I}$.

The family of functions $\{S_a\}$ we are going to study were introduced, simultaneously, by Césaro in 1906 and Hellinger in 1907. They have been studied from a wide variety of viewpoints (for example, geometric, arithmetic, probabilistic, or as functional equations), as can be seen in [3, 7, 10, 20, 27–29, 31]. One application for plastic deformation can be found in [8]. Other related references with respect to these functions can be found in [21].

In 1984, Hata and Yamaguti [19] showed that functions S_a and Takagi function T are related through the formula $\frac{\partial S_a}{\partial a}(x) = 2T(x)$.

In Fig. 1, from left to right, and from top to bottom, we show the graphs of $W_{.3,5}$, T, C and $S_{.15}$:

Singular functions come up in a wide variety of contexts. We mention here three topics where these functions appear, and seem to be far from each other.

We find these functions to relate representation number systems, in a similar way to the Minkowski function in [9, 25].

Under certain conditions, the Riesz products in the Theory of Trigonometric Series are singular functions (see [32, p. 208]).

On the other hand, the natural need to analyze transient data has led researchers to look for new tools that could provide some information about the change of scale when they observed decomposable events in nature, such as fingerprints or self-similar behavior in iterated processes. This leads to a special class of functions, called wavelets (that can be traced back to Haar in the early 20th century as particular examples), which provides a satisfactory answer to the scale problem (see [14]).

Much of the enthusiasm for wavelets comes from their potential applications. Among other fields, wavelets have found use in image processing, in restoration of recordings, and in seismology. In the context of wavelets in fractals, singular functions appear for example as conjugating homeomorphisms (see [11, Example 4.6]) or Perron-Frobenius measures (see [15]).

An overview of singular functions can be found, for example, in [21].

Our main purpose is to study the class of singular functions $\{S_a : a \neq 1/2\}$ with the aid of a generalized dyadic system, analyzing, among other properties, the Hausdorff dimensions of a certain set related with them. We also work on the path traced by Hata and Yamaguti, and we show how generalized Takagi functions can be obtained.

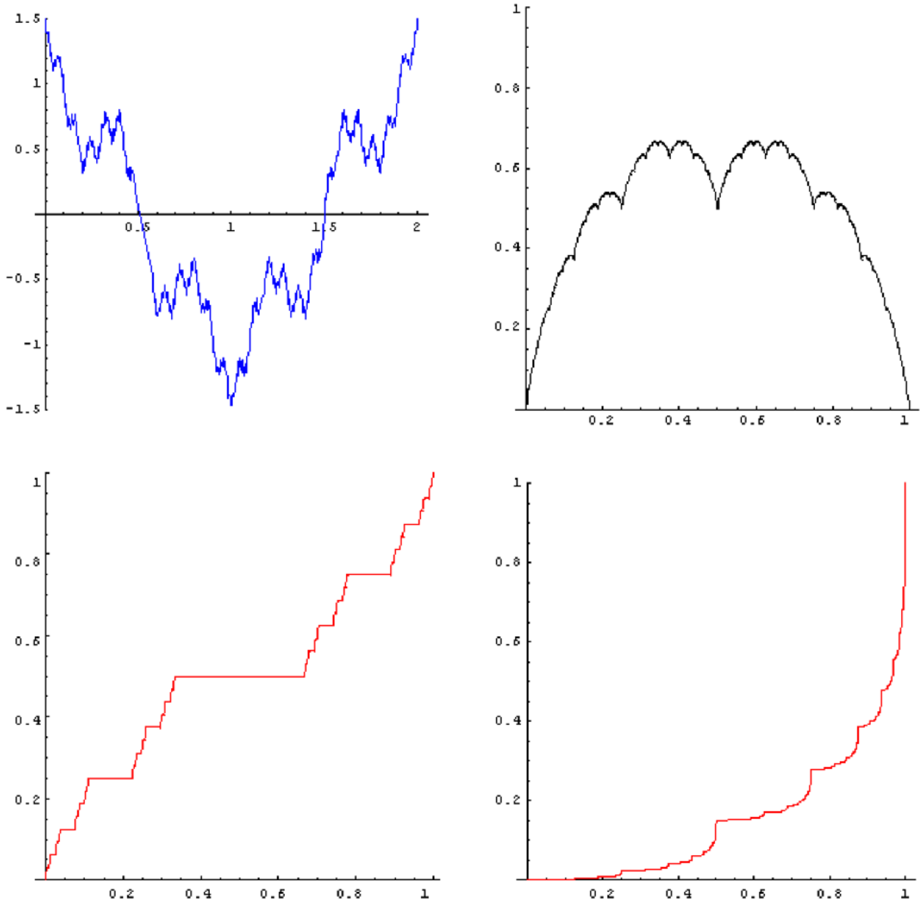


Fig. 1 $W_{.3,5}$, Takagi, Cantor and $S_{.15}$'s graphs

The organization of the paper is as follows. In Sect. 2, we describe the family of singular functions S_a (parameterized by $a \in]0,1[$), and we also calculate the coefficients of S_a for the Schauder's basis S. We prove that if $a \neq 1/2$, then S_a does not admit a non-zero derivative at any $x \in \mathbb{I}$.

In Sect. 3, the Hausdorff dimensions for some subsets in $\mathbb{I}$ are calculated, using two number representation systems to express S_a.

In Sect. 4, we introduce a number representation system and a system of functional equations. Afterwards, we study the quasi-inverse function of the unique bounded solution of that system of functional equations, that is a Cantor-type function.

The last section concerns the study of the function T_a (a generalized Takagi function) given by the derivative of S_a with respect to the variable a, and we establish that if $a \neq 1/2$, the function T_a is of monotonic type on no interval and $T_a'(x) = 0$ a.e.

In a series of papers several notions are considered that measure some degrees of pathology in the class of continuous nowhere monotone functions (see, for example, [12] and the references therein). We will study two of these notions for the functions T_a and other functions related with them.

For our further consideration, we now introduce some preliminary notation. As usual, $\mathcal{C}(\mathbb{I})$ denotes the collection of functions $f : \mathbb{I} \to \mathbb{R}$ that are continuous. We say that f is *of monotonic type on an interval* $J \subset \mathbb{I}$ if the function $f_m(x) = f(x) + mx$ is monotone on J for some $m \in \mathbb{R}$. Let us denote by MNI those functions in $\mathcal{C}(\mathbb{I})$ that are monotone on no interval (or nowhere monotone) and by MTNI those functions in $\mathcal{C}(\mathbb{I})$ that are of monotonic type on no interval. In [12], the relationships between these notions are given. In particular, MTNI implies MNI.

We use standard terms of Measure and Fractal Geometry whose definitions may be found in [16].

2 The Family $\{S_a : a \in]0, 1[\}$ of Singular Functions

The functions S_a we will study in this section are, together with the Cantor function, the best known examples of singular functions. In [20, 28], a class of functions h_a is studied as the limit of a sequence of functions with polygonal graphs. For their definition, the authors use an operator π_a on segments $\overline{AB}$ of coordinates $A = (x, y)$ and $B = (x + \triangle x, y + \triangle y)$, being $\pi_a(\overline{AB})$ the polygonal $\overline{ACB}$, the union of the segments $\overline{AC}$ and $\overline{CB}$, with $C = (x + \triangle x/2, y + a\triangle y)$. The graph of f_0 is the segment joining $(0, 0)$ with $(1, 1)$; the graph of f_1 is obtained when π_a is applied to the previous segment. Afterwards, the graph of f_2 follows from the action of π_a on the two segments obtained in the graph of f_1; and applying π_a over the 2^2 segments of the graph of f_2, we obtain f_3. Thus, the function h_a is defined by

$$h_a(x) := \lim_{n\to\infty} f_n(x).$$

We will study h_a from a different point of view with respect to the references cited above. The study follows as a relationship between the dyadic and the generalized dyadic systems introduced in [4, Sect. 3]. This is a representation system for numbers in $]0, 1]$, generalizing the dyadic (or binary) one.

As it is known, the dyadic system permits the expression of any real number in $]0, 1]$ through a series in the form $x = \sum_{n=1}^{+\infty} \frac{1}{2^{m_n}}$, where (m_n) is a strictly increasing sequence of positive integers. In order to generalize this representation, we introduce two numbers k and $1 - k$ (with $k \in]0, 1[$), and we obtain expansions in the form $x = \sum_{n=1}^{+\infty}(1 - k)^n k^{r_n}$, with $r_n \in \mathbb{Z}^+$ and $r_n \le r_{n+1}$. The coefficients r_n depending on x, sometimes will be written as $r_n(x)$. Properties of this system of representation of numbers in $\mathbb{I}$ are similar to those of the dyadic representation system. The representation is unique except for a denumerable set of numbers x for which there are exactly two, one of which is finite. We call the new representation the *generalized dyadic representation number system.*

Note that in the case $k = 1 - k = 1/2$, both representations are the same for numbers in $]0, 1]$. We recall the following result.

Proposition 1 ([4, Prop. 3]) *Let* $k \in]0, 1[$. *If* $x \in]0, 1]$, *then there exists a unique increasing sequence of positive integers* $1 \le r_0 \le r_1 \le \cdots \le r_n \le \cdots$, *such that*

$$x = \sum_{n=0}^{+\infty} (1 - k)^n \, k^{r_n}.$$

The expansion $x = \sum_{n=0}^{+\infty}(1-k)^n k^{r_n}$ is unique, but in the stationary case (i.e. $r_j > r_{j-1}$ and $r_n = r_j$, if $n \geq j$), we have the equality

$$x = \sum_{n=0}^{+\infty}(1-k)^n\, k^{r_n} = \sum_{n=0}^{j-1}(1-k)^n\, k^{r_n} + (1-k)^j\, k^{r_j - 1}.$$

This finite expression can be considered as a second expansion in this system for the number x.

An outstanding property for the dyadic representation is the following result that generalizes the Borel Theorem of Normal Numbers (see [18, p. 125]). As usual, λ denotes the Lebesgue measure on the reals.

Theorem 1 ([4, Th. 13]) *The set N_a of points satisfying*

$$\lim_{n\to\infty}\frac{r_n(x)}{n} = \frac{k}{1-k},$$

is a set of λ-measure 1.

Definition 1 The elements of N_a are called normal numbers in the generalized dyadic system. If $a = 1/2$, we denote N_a by N, and its elements are called the normal numbers in the base-2.

Definition 2 Let $x \in \mathbb{I}$, and $\sum_{n=0}^{+\infty}\frac{1}{2^{m_n}}$ be its dyadic expansion. For each $a \in]0,1[$, we set

$$S_a(x) := \sum_{n=0}^{+\infty} a^{m_n - n}(1-a)^n .$$

Let us observe that $0 < a, 1-a < 1$ and $m_n \geq n$, ensure the convergence for the series. This definition is implicit in some papers (see for instance [27–29]).

We can prove with a small amount of calculus that this family of functions satisfies the following properties.

Proposition 2 *For S_a, with $a \in]0,1[$, we have*:

i. *S_a is well defined.*
ii. *S_a is an increasing function.*
iii. *S_a is continuous.*
iv. *If $a \neq 1/2$, then S_a is a singular function.*
v. *S_a is the unique bounded solution for the system of functional equations*

$$\begin{cases} f\left(\frac{x}{2}\right) = af(x) \\ f\left(\frac{1+x}{2}\right) = a + (1-a)\,f(x) \end{cases} \tag{2}$$

We remark that S_a and h_a satisfy the system (2); therefore, they coincide.

A sequence (y_n) in a normed space $(Y, \|\circ\|)$ is a *basis of Schauder* if, for every y in Y, there is a unique sequence of scalars (α_n) such that $y = \sum_{j=0}^{+\infty}\alpha_j y_j$. That is,

$$\lim_{n\to\infty}\left\| y - \sum_{j=0}^{n}\alpha_j y_j \right\| = 0.$$

We use the basis of Schauder in the linear space $\mathcal{C}(\mathbb{I})$ endowed with the sup-norm, which has the following description: $B_0(x) := x$, $B_1(x) := 1 - x$, and $B_{n,k}$ are functions that vanish out of $[\frac{n}{2^k}, \frac{n+1}{2^k}]$, and whose graph, when x runs on it, is given by the equal sides of the isosceles triangle determined by $(\frac{n}{2^k}, 0)$, $(\frac{n+1}{2^k}, 0)$, and $(\frac{2n+1}{2^{k+1}}, 1)$. They can be expressed as:

$$B_{n,k}(x) := 2^k \left(\left| x - \frac{n}{2^k} \right| + \left| x - \frac{n+1}{2^k} \right| - \left| 2x - \frac{2n+1}{2^k} \right| \right),$$

with $0 \leq n \leq 2^k - 1$, $k \geq 1$.

This basis will be denoted by S. We have that $B_0 = y_0$, $B_1 = y_1$ and $B_{n,k} = y_{2^k+n+1}$, in S. For the sake of clarity, we use the notation α_0, α_1 and $\alpha_{n,k}$, for the coefficients in the series expansion.

If $f \in \mathcal{C}(\mathbb{I})$, its coefficients can be calculated as:

$$\alpha_0 = f(1), \qquad \alpha_1 = f(0), \qquad \alpha_{n,k} = f\left(\frac{2n+1}{2^{k+1}}\right) - \frac{1}{2}\left(f\left(\frac{n}{2^k}\right) + f\left(\frac{n+1}{2^k}\right)\right) \tag{3}$$

In particular, the expression for the Takagi function given in (1) shows that the corresponding coefficients in the basis S are

$$\alpha_0 = \alpha_1 = 0 \quad \text{and} \quad \alpha_{n,k} = \frac{1}{2^k}, \tag{4}$$

not depending on n.

The reader can find either the proof that S is a basis, or the validity of the relations (3) in [26, Chp. 6].

Proposition 3 *Schauder's basis for S_a functions yields coefficients in the form of*

$$\alpha_0 = 1, \qquad \alpha_1 = 0 \quad \textit{and} \quad \alpha_{k,j} = a^m (1-a)^n \left(a - \frac{1}{2} \right),$$

where n is the number of terms in the binary expansion of j, and $m = k - n$.

Proof Let us consider n even:

$$\frac{n}{2^k} = \sum_{j=0}^{t} \frac{1}{2^{m_j}}; \qquad \frac{n+1}{2^k} = \sum_{j=0}^{t} \frac{1}{2^{m_j}} + \frac{1}{2^k}; \qquad \frac{2n+1}{2^{k+1}} = \sum_{j=0}^{t} \frac{1}{2^{m_j}} + \frac{1}{2^{k+1}}$$

and then,

$$S_a\left(\frac{n}{2^k}\right) = \sum_{j=0}^{t} a^{m_j - j}(1-a)^j;$$

$$S_a\left(\frac{n+1}{2^k}\right) = \sum_{j=0}^{t} a^{m_j - j}(1-a)^j + a^{k-t-1}(1-a)^{t+1};$$

$$S_a\left(\frac{2n+1}{2^{k+1}}\right) = \sum_{j=0}^{t} a^{m_j - j}(1-a)^j + a^{k-t}(1-a)^{t+1}.$$

Then, the coefficients are:

$$\begin{aligned}\alpha_{k,n} &= f\left(\frac{2n+1}{2^{k+1}}\right) - \frac{1}{2}\left(f\left(\frac{n}{2^k}\right) + f\left(\frac{n+1}{2^k}\right)\right)\\ &= a^{k-t}(1-a)^{t+1} - \frac{1}{2}a^{k-t-1}(1-a)^{t+1}\\ &= \left(a - \frac{1}{2}\right)a^{k-t-1}(1-a)^{t+1}.\end{aligned}$$

If $\frac{n}{2^k} = \sum_{j=0}^{t} \frac{1}{2^{m_j}}$, then $n = \sum_{j=0}^{t} 2^{k-m_j}$; that is, the binary expansion for n needs $t+1$ terms.

We proceed in a similar way if n is odd, and the proof is complete. □

Theorem 2 *If $a \in]0,1[\setminus\{1/2\}$, then S_a does not admit a non-zero derivative.*

Proof The proof rests on a geometric argument. To be specific, the idea is that the iterative application of the operator π_a has $S_a(\frac{1}{2})$, $S_a(\frac{1}{2^2})$ and $S_a(\frac{1}{2}+\frac{1}{2^2})$; and $S_a(\frac{1}{2^3})$, $S_a(\frac{1}{2^2}+\frac{1}{2^2})$, $S_a(\frac{1}{2}+\frac{1}{2^3})$ and $S_a(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3})$ as fixed points. And so on, with respective applications.

For each iteration, we find divisions on the X and Y axes. We denote by u_n and v_n the extremes for the interval including x in the n-th application of π_a. The corresponding Y-interval is $[S_a(u_n), S_a(v_n)]$, which includes the point $S_a(x)$. If $S_a'(x) = \alpha \neq 0$ exists, then, for n large enough:

$$\frac{S_a(v_n) - S_a(u_n)}{v_n - u_n} = \frac{a^k(1-a)^{n-k}}{\frac{1}{2^n}} \simeq \alpha;$$

and wherever the $n+1$-th interval is:

$$\frac{S_a(v_{n+1}) - S_a(u_{n+1})}{v_{n+1} - u_{n+1}} = \begin{cases} \frac{a^{k+1}(1-a)^{n-k}}{\frac{1}{2^{n+1}}} \simeq 2a\alpha \\ \frac{a^k(1-a)^{n+1-k}}{\frac{1}{2^{n+1}}} \simeq 2(1-a)\alpha \end{cases}$$

Therefore, the limit can only exists if $a = 1/2$. □

3 Application to Fractal Dimension

This section is devoted to describing several sets related to the function S_a and to compute their Hausdorff dimensions. If $f : \mathbb{I} \to \mathbb{I}$ is a continuous bijection, then it is known that the following statements are equivalent (see for instance [20, pp. 288–290]):

a. f is a singular function.
b. f maps a set of λ-measure one onto a set of λ-measure zero.
c. f maps a set of λ-measure zero onto a set of λ-measure one.

For S_a, we will find examples of sets satisfying **b**. and **c**. above. Let us recall that N denotes the set of normal numbers in the base-2.

Theorem 3 *If $a \neq 1/2$, then $S_a(N)$ is a set of λ-measure zero.*

Proof If $x \in N$, then $m_n = 2n + o(n)$. Applying S_a:

$$S_a(x) = \sum_{n=0}^{+\infty} a^{m_n - n} (1-a)^n .$$

For $S_a(x)$, we have $r_n = m_n - n$, and thus $\frac{r_n}{n} \to 1$. But, for the generalized dyadic system, this limit must be equal to $\frac{a}{1-a}$ at points in a set of λ-measure 1.

Consequently, if $a \neq \frac{1}{2}$, the set N (which is Lebesgue-measurable of λ-measure 1) is mapped by S_a on a set of λ-measure zero; to be specific, the set of points where $\frac{r_n}{n} \to 1$. □

Lemma 1 *If $x \in N_a$, then its image under S_a^{-1} satisfies that, in its binary expansion, among the first k digits, the number of 0s is approximately ak, and the number of 1s is approximately $(1-a)k$.*

Proof If $S_a^{-1}(x) = \sum_{n=0}^{+\infty} \frac{1}{2^{m_n}}$, we have

$$\frac{m_n}{n} = \frac{r_n + n}{n} \to \frac{a}{1-a} + 1 = \frac{1}{1-a},$$

because there are exactly $n+1$ digits equal to 1 among the first m_n digits of $S_a^{-1}(x)$. Now, for k, with $m_n \leq k \leq m_{n+1}$, it follows that

$$\frac{n}{m_{n+1}} \leq \frac{n}{k} \leq \frac{n}{m_n},$$

and the Sandwich rule (or *the squeeze theorem*) ensures the result. □

Corollary 1 *If $a \neq 1/2$, then S_a maps*

$$A_a = \left\{ x = \sum_{n=0}^{+\infty} \frac{1}{2^{m_n}} : \lim_{n\to\infty} \frac{n}{m_n} = 1 - a \right\}$$

of λ-measure zero onto a set of λ-measure one.

Proof If $a \neq 1/2$, then $\lambda(A_a) = 0$, because they are not normal numbers, and $S_a(A)$ is a set of λ-measure 1, because $S_a(A_a) = N_a$. □

The following result concerning Hausdorff dimensions is a consequence of a theorem by Besicovitch (see [16, Prop. 10.1]).

Theorem 4 *S_a maps a set of λ-measure zero onto a set of λ-measure one. The Hausdorff dimension of the first set is $\log_2[\frac{1}{a^a}\frac{1}{(1-a)^{1-a}}]$.*

Proof The Hausdorff dimension of A_a is $\log_2[\frac{1}{a^a}\frac{1}{(1-a)^{1-a}}]$ (see [16, Prop. 10.1]). □

As Fig. 2 shows, the Hausdorff dimension of A_a tends to 0 on 0 and 1, and is 1 if $a = 1/2$, as can be expected, because the function $S_{1/2}$ is the identity.

We require the following useful Frostman-type lemma (see [16, pp. 60–61]). Here $B_r(x)$ denotes the closed ball with centre x and radius r.

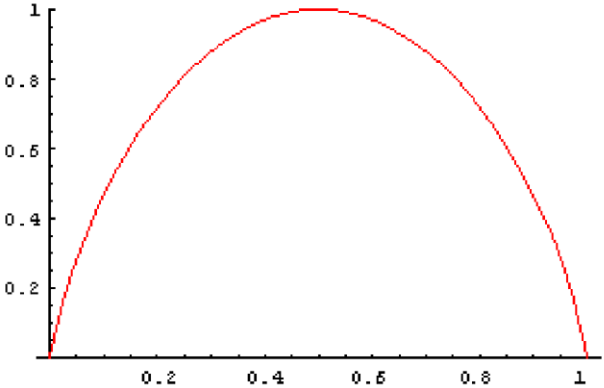


Fig. 2 Graph of the Hausdorff dimensions

Lemma 2 *Let F be a Borel set in $\mathbb{R}$, and let μ be a probability measure on $\mathbb{R}$ such that $\mu(F) = 1$. Under these assumptions,*

i. *If the upper limit* $\limsup_{r\to 0} \frac{\mu(B_r(x))}{r^s}$ *is bounded on F, then* $\dim_{\mathrm{H}}(F) \le s$.
ii. *If there exists a positive real c such that* $\liminf_{r\to 0} \frac{\mu(B_r(x))}{r^s} > c > 0$ *on F, then* $\dim_{\mathrm{H}}(F) \ge s$.

We use this lemma but with cylinders of the generalized dyadic representation number system in [4] instead of balls.

Theorem 5 *S_a maps N onto a set of λ-measure zero and with Hausdorff dimension* $\frac{-\ln 4}{\ln a(1-a)}$.

Proof We here use the function S_a^{-1}. If

$$y = \sum_{n=0}^{+\infty} a^{r_n} (1-a)^n ,$$

we then consider the interval

$$\left[\sum_{k=0}^{n} a^{r_k} (1-a)^k , \sum_{k=0}^{n} a^{r_k} (1-a)^k + a^{r_n - 1} (1-a)^{n+1}\right],$$

and the corresponding image under S_a^{-1} is another interval, specifically:

$$\left[\sum_{k=0}^{n} \frac{1}{2^{r_k+k}}, \sum_{k=0}^{n} \frac{1}{2^{r_k+k}} + \frac{1}{2^{r_n+n}}\right].$$

Note that the respective lengths of both intervals are given by the numbers $a^{r_n-1}(1-a)^{n+1}$ and $\frac{1}{2^{r_n+n}}$.

Therefore, the Hausdorff dimension is given by the number

$$\sup\left\{\beta > 0 : \lim_{n\to\infty} \frac{\frac{1}{2^{r_n+n}}}{[a^{r_n-1}(1-a)^{n+1}]^{\beta}} < +\infty\right\}.$$

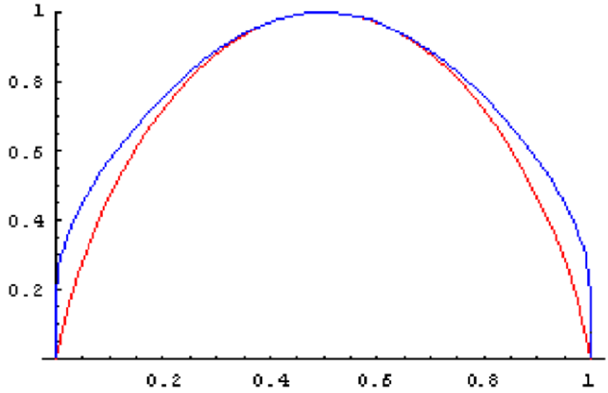


Fig. 3 Comparison of the Hausdorff dimensions

Taking logarithms:

$$\lim_{n\to\infty} \frac{-(r_n+n)\ln 2-\beta(r_n-1)\ln a-\beta(n+1)\ln(1-a)}{n} n<+\infty,$$

and since $\frac{r_n}{n}\to 1$, this gives $\beta=\frac{-\ln 4}{\ln a(1-a)}$, which is the Hausdorff dimension. $\square$

Figure 3 shows Hausdorff dimensions of sets we have considered in Theorems 4 and 5.

The graph of the Hausdorff dimension of functions $S_a(N)$, as shown by Fig. 3, is very similar to that one given in Fig. 2: this is not smaller than that and both coincide if and only if $a=1/2$.

This function is a particular case in a wider class of functions that we find in the Harmonic Analysis of Fractals (see for example [23]). It is clear that the unit interval $\mathbb{I}$ is a post critically finite set for the pair of functions given by $F_1(x)=x/2$ and $F_2(x)=x/2+1/2$. Following the ideas in [23, examp. 3.1.4], the associated harmonic functions related with the harmonic framework in this example are, precisely, the functions S_a.

In the case where the pair of similarities is $F_1(x)=ax$ and $F_2(x)=(1-a)x+a$, $a\in\,]0,1[$, the harmonic functions are a generalization of the previous ones. They are in the form $S_{a,b}:=S_b\circ S_a^{-1}$ (see [7]), and apply $x=\sum_{n=0}^{+\infty}a^{r_n}(1-a)^n$ on $S_{a,b}(x)=\sum_{n=0}^{+\infty}b^{r_n}(1-b)^n$. The use of these representations in two generalized dyadic systems provides the following result.

Theorem 6 *If $a\neq b$, then $S_{a,b}$ maps a set of λ-measure zero onto a set of λ-measure one. The Hausdorff dimension of the first set is*

$$\frac{b\ln b+(1-b)\ln(1-b)}{b\ln a+(1-b)\ln(1-a)}.$$

Proof The set of λ-measure 0 is that of the points with limit $\frac{b}{1-b}$ for their corresponding sequences of ratios $\frac{r_n}{n}$. To obtain its Hausdorff dimension, if $x=\sum_{n=0}^{+\infty}a^{r_n}(1-a)^n$, then we consider the interval

$$\left[\sum_{k=0}^{n}a^{r_k}(1-a)^k,\sum_{k=0}^{n}a^{r_k}(1-a)^k+a^{r_n-1}(1-a)^{n+1}\right].$$

The image of this interval under $S_{a,b}$ is the interval

$$\left[\sum_{k=0}^{n} b^{r_k}\,(1-b)^k\,,\sum_{k=0}^{n} b^{r_k}\,(1-b)^k + b^{r_n-1}\,(1-b)^{n+1}\right].$$

The respective lengths of these intervals are

$$a^{r_n-1}\,(1-a)^{n+1} \quad \text{and} \quad (1-b)^{r_n-1}\,(1-b)^{n+1}\,.$$

Therefore, its Hausdorff dimension is given by

$$\sup\left\{\beta>0:\ \lim_{n\to\infty}\frac{b^{r_n-1}(1-b)^{n+1}}{[a^{r_n-1}(1-a)^{n+1}]^{\beta}}<+\infty\right\}.$$

If we proceed as before, then the limit

$$\lim_{n\to\infty}\frac{(r_n-1)\ln b+(n+1)\ln(1-b)-\beta\left[(r_n-1)\ln a+(n+1)\ln(1-a)\right]}{n}n$$

is finite. Finally, $\frac{r_n}{n}\to\frac{b}{1-b}$ gives

$$\dim_{\mathrm{H}}(A)=\frac{b\ln b+(1-b)\ln(1-b)}{b\ln a+(1-b)\ln(1-a)}.$$

□

4 Relationship Between Cantor and S_a Functions

The study of the system (2) in Proposition 2 is generalized in [7] to a system in the form

$$\begin{cases} f\left(\frac{x}{2}\right)=af(x) \\ f\left(\frac{1+x}{2}\right)=b+cf(x) \end{cases} \tag{5}$$

with positive real parameters a, b, c satisfying relations $0<a<b<1$, and $b+c=1$.

In this section, we study the more general system

$$\begin{cases} f(\alpha x)=af(x) \\ f(\alpha+(1-\alpha)x)=b+cf(x) \end{cases} \tag{6}$$

with $\alpha\in\,]0,1[$. We introduce a number representation system which generalizes another system of representation **B** used by the authors in [2]. The new system is also denoted by **B**, and it is defined as follows:

Every number $x\in\mathbb{I}$ can be written in the form:

$$\beta_0+\delta_1\beta_1+\delta_2\beta_2+\cdots+\delta_i\beta_i+\cdots,$$

where $\beta_i\in\{0,a,b\}$ and $\delta_0=1$,

$$\delta_i=\begin{cases} a\delta_{i-1}, & \text{if } \beta_{i-1}=0,\\ (b-a)\delta_{i-1}, & \text{if } \beta_{i-1}=a,\\ c\delta_{i-1}, & \text{if } \beta_{i-1}=b.\end{cases}$$

This representation is unique except for a denumerable set where there are two expressions. A property of this system is that the set of points in the unit interval satisfying the existence of the limits

$$\lim_{n\to\infty} \frac{\text{Card}\{i \le n : \beta_i = 0\}}{n} = a$$
$$\lim_{n\to\infty} \frac{\text{Card}\{i \le n : \beta_i = a\}}{n} = b - a$$
$$\lim_{n\to\infty} \frac{\text{Card}\{i \le n : \beta_i = b\}}{n} = c$$

is of λ-measure 1.

Proposition 4 *The system of functional equations* (6) *has one and only one bounded solution in*]0, 1]. *We denote it by* $f_{a,c;\alpha}$. *Furthermore, if*

$$x = \sum_{n=0}^{+\infty} a^{r_n} (1-a)^n ,$$

then

$$f_{a,c;\alpha}(x) = \beta_0 + \delta_1\beta_1 + \delta_2\beta_2 + \cdots + \delta_i\beta_i + \cdots$$

with

$$\beta_i = \begin{cases} b, & \text{if } i+1 \in \{r_n + n\}_{n\ge 0}, \\ 0, & \text{otherwise} \end{cases}$$

taking $m_{-1} = 0$ *and* $f_{a,c;\alpha}(0) = 0$.

Proof It is based upon the Contraction Mapping Theorem and a direct checking that the equations are satisfied. □

These functions map a set of λ-measure 1 onto another denoted by $Nu_{a,c;\alpha}$, of λ-measure zero. To study the Hausdorff dimension of $Nu_{a,c;\alpha}$ we introduce the quasi-inverse function of a monotone function; that is: if $g : \mathbb{I} \to \mathbb{I}$ is a monotone function, its quasi-inverse is defined by $g^{(-1)}(x) = \sup\{t : g(t) \le x\}$.

Let us note that $f^{(-1)}_{a,c;\alpha}$ is a generalized Cantor-type function. As an example, Fig. 4 shows the graph of the function $f^{(-1)}_{.2,.5;.4}$.

In the particular case of the Cantor function, c satisfies $c(x) = f^{(-1)}_{1/3,1/3;.5}(x)$, for all $x \in \mathbb{I}$.

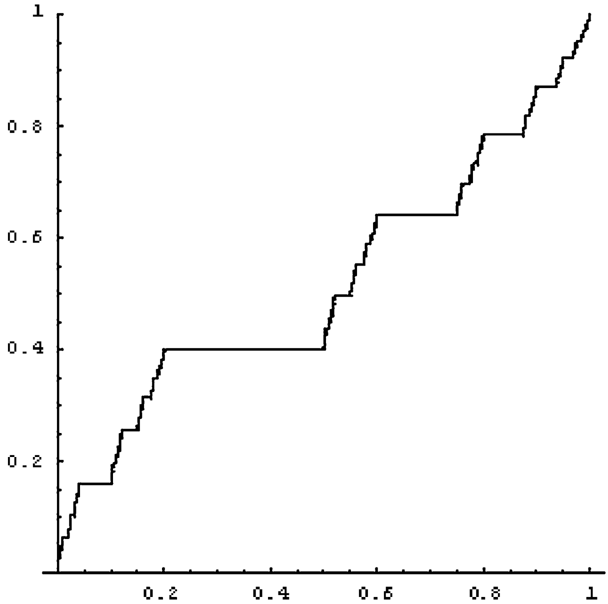
The moments of functions $f^{(-1)}_{t,t;\tau}$, are studied in [17, (2.11)]. These functions are called the Cantor-Riesz-Nágy functions.

Theorem 7 $f^{(-1)}_{a,c;\alpha}$ *is the unique bounded solution for the system of functional equations*

$$f(x) = \begin{cases} \alpha f(\frac{x}{a}), & \text{if } 0 \le x \le a, \\ \alpha, & \text{if } a < x < b, \\ \alpha + (1-\alpha) f(\frac{x-b}{c}) & \text{if } b \le x \le 1. \end{cases} \tag{7}$$

For $x = \beta_0 + \delta_1\beta_1 + \delta_2\beta_2 + \cdots + \delta_i\beta_i + \cdots$ *with* $\beta_i \in \{0, b\}$, *the corresponding series expansion for* $f^{(-1)}_{a,c;\alpha}(x)$ *is* $\sum_{n=0}^{+\infty} \alpha^{r_n}(1-\alpha)^n$, *with* $r_n = j - n - 1$ *where* j *satisfies that*

Fig. 4 Graph of $f^{(-1)}_{.2,.5;.4}$



$n = \mathrm{Card}\{i \le j : \beta_i = b\}$. If x has a coefficient $\beta_i = a$, then $f^{(-1)}_{a,c;\alpha}(x) = f^{(-1)}_{a,c;\alpha}(s_x)$, where s_x denotes the greatest point with all its coefficients in $\{0, b\}$ and $s_x < x$.

Moreover, $f^{(-1)}_{a,c;\alpha}$ maps a set of λ-measure zero with a Hausdorff dimension

$$\frac{\alpha \ln \alpha + (1-\alpha)\ln(1-\alpha)}{\alpha \ln a + (1-\alpha)\ln c}$$

onto a set of λ-measure one.

Proof Direct calculations show that $f^{(-1)}_{a,c;\alpha}$ satisfies (7). Uniqueness follows from the definition of a functional that is a contraction.

We are considering the subset whose elements in the **B** system satisfy that $\delta_i \in \{0, b\}$ and the limits

$$\lim_{n\to\infty} \frac{\mathrm{Card}\{i \le n : \beta_i = 0\}}{n} = \alpha,$$
$$\lim_{n\to\infty} \frac{\mathrm{Card}\{i \le n : \beta_i = b\}}{n} = 1-\alpha,$$

exist with these values.

The calculation of the Hausdorff dimension follows, such as in the above theorems, applying the Frostman lemma to cylinders in the representation system **B**. In this case, they are in the form

$$\left[\beta_0 + \delta_1\beta_1 + \delta_2\beta_2 + \cdots + \delta_i\beta_i,\ \beta_0 + \delta_1\beta_1 + \delta_2\beta_2 + \cdots + \delta_i\beta_i + \delta_i\gamma_i\right]$$

with

$$\gamma_i = \begin{cases} a, & \text{if } \beta_i = 0, \\ c, & \text{if } \beta_i = b. \end{cases}$$

□

Corollary 2 *$Nu_{a,c;.5}$ has Hausdorff dimension $\frac{-2\ln 2}{\ln a+\ln(c)}$, and the Hausdorff dimension of $Nu_{t,t;\tau}$ is $\frac{\tau\ln\tau+(1-\tau)\ln(1-\tau)}{\ln t}$.*

We recall that the Cantor ternary set C is the subset of elements $\sum_{k=0}^{+\infty}\frac{u_k}{3^k}$ in the unit interval $\mathbb{I}$ with $u_k \in \{0, 2\}$ for all k. Therefore, as a consequence of the above theorem, we have the following result.

Corollary 3 *Let us consider the elements in the Cantor set satisfying the following relations*:

$$\lim_{n\to\infty}\frac{\operatorname{Card}\{k\le n : u_k = 0\}}{n} = \alpha,$$
$$\lim_{n\to\infty}\frac{\operatorname{Card}\{k\le n : u_k = 2\}}{n} = 1-\alpha.$$

Then, this subset has Hausdorff dimension

$$-\frac{\alpha\ln\alpha + (1-\alpha)\ln(1-\alpha)}{\ln 3}.$$

Using similar techniques it is possible to obtain a result for a more general class of Cantor-type sets.

Theorem 8 *If m is an integer greater than 2, and C^A denotes the set of points in $\mathbb{I}$ such that their base-m expansion $\sum_{k=0}^{+\infty}\frac{u_k}{m^k}$ has all its digits in the set $A = \{a_1, \ldots, a_n\}$, then the Hausdorff dimension of the set of points in C^A such that $\lim_{n\to\infty}\frac{\operatorname{Card}\{k\le n:u_k=0\}}{n} = \alpha_i$ is equal to*

$$-\frac{\sum_{i=1}^{n}\alpha_i\ln\alpha_i}{\ln m}.$$

5 A Generalized Takagi Function

Up to this point, we have considered the functions S_a as depending on x:

$$x = \sum_{n=0}^{+\infty}\frac{1}{2^{m_n}} \longrightarrow S_a(x) := \sum_{n=0}^{+\infty} a^{m_n-n}(1-a)^n$$

They are functions in the two variables x and a. Interchanging their roles, we now consider these functions depending on the parameter a, and we study their derivatives with respect to this variable. Several properties of these functions, e.g. their maxima and their relations with binary digital sums have been studied in [1, 22].

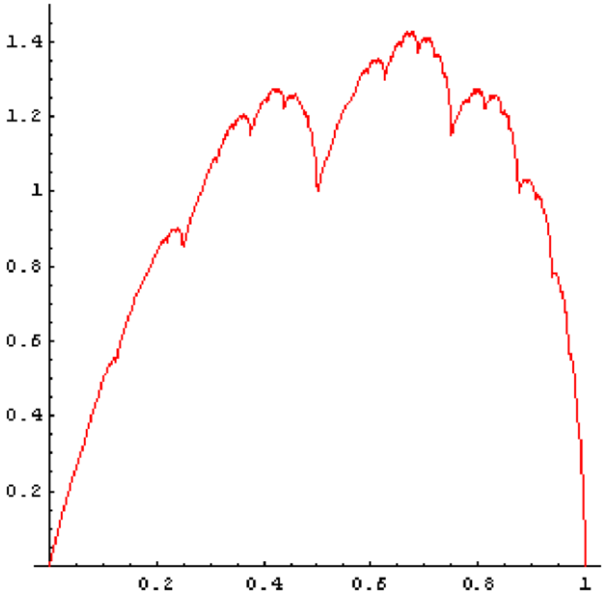
Definition 3 Let $T_a : \mathbb{I} \longrightarrow \mathbb{R}$ be given as

$$T_a(x) := \frac{\partial S_a}{\partial a}(x) = \sum_{n=0}^{+\infty}\left[(m_n-n)\,a^{m_n-n-1}(1-a)^n - n a^{m_n-n}(1-a)^{n-1}\right].$$

Through calculations it is possible to check that this formula for $T_a(x)$ is also valid for x with double representation. Figure 5 shows the graph of $T_{.35}$.

The next result generalizes the equalities given in (4).

Fig. 5 Graph of $T_{.35}$



Proposition 5 (Identities for T_a) *Let $k \in \mathbb{Z}^+$, $0 \le n \le 2^k - 1$, and set $\frac{n}{2^k} = \frac{b_k}{2^k} + \frac{b_{k-1}}{2^{k-1}} + \cdots + \frac{b_1}{2^1}$, $a_j \in \{0, 1\}$. If d is the number of 0s among the values b_j, and r denotes the corresponding number of 1s, then*

$$T_a\left(\frac{2n+1}{2^{k+1}}\right) - \left((1-a)T_a\left(\frac{n}{2^k}\right) + aT_a\left(\frac{n+1}{2^k}\right)\right) = a^r(1-a)^d.$$

Proof We distinguish four cases:

Case I: n is even. Let us consider $n = 2^c\alpha$ with odd α, $1 \le c$, and $0 < n < 2^k - 1$. Its expression in the base-2 is $\frac{n}{2^k} = \sum_{j=0}^{d-1} \frac{1}{2^{m_j}}$, with $m_{d-1} = k - c$. Hence, $\frac{n+1}{2^k} = \sum_{j=0}^{d-1} \frac{1}{2^{m_j}} + \frac{1}{2^{m_{d-1}+c}}$; and $\frac{2n+1}{2^k} = \sum_{j=0}^{d-1} \frac{1}{2^{m_j}} + \frac{1}{2^{m_{d-1}+c+1}}$.

Taking the two expressions above into account, and applying the definition of T_a, we have

$$T_a\left(\frac{2n+1}{2^{k+1}}\right) - \left((1-a)T_a\left(\frac{n}{2^k}\right) + aT_a\left(\frac{n+1}{2^k}\right)\right)$$
$$= a^{m_{d-1}+c-d}(1-a)^d = a^{k-d}(1-a)^d = a^r(1-a)^d.$$

Similar calculations give the result in the other three cases.

Case II: n is odd with $n < 2^k - 1$.

Case III: $n = 0$.

Case IV: $n = 2^k - 1$. □

This result together with (4), allow us to relate the Takagi function with T_a. To be specific, the equality $2T = T_{\frac{1}{2}}$ is valid.

Theorem 9 *The family given by*

$$g_1(x) := S_a(x)$$
$$g_2(x) := 1 - S_a(x)$$
$$g_{k,j}(x) := \begin{cases} S_a\left(2^{k+1}\left(x - j/2^k\right)\right), & x \in \left[\frac{j}{2^k}, \frac{2j+1}{2^{k+1}}\right] \\ 1 - S_a\left(2^{k+1}\left(x - \frac{2j+1}{2^{k+1}}\right)\right), & x \in \left[\frac{2j+1}{2^k+1}, \frac{j+1}{2^k}\right] \\ 0, & \text{otherwise} \end{cases}$$

$$\text{for } k = 0, 1, 2, \ldots;\ j = 0, 1, 2, \ldots, 2^k - 1$$

provides a Schauder's basis for the space $\mathcal{C}(\mathbb{I})$ of continuous functions in $\mathbb{I}$. We denote this basis by ${}^aS$.

The proof follows in a similar way to that given for S in [26, Ch. 6].

Theorem 10 *T_a is the unique bounded function satisfying*

$$\begin{cases} G\,(x/2) = aG\,(x) + S_a\,(x) \\ G\left(\frac{1+x}{2}\right) = (1-a)\,G\,(x) + 1 - S_a\,(x)\,. \end{cases}$$

Proof It is a direct consequence of the Contraction Mapping Theorem or, on the other hand, as an application of the system of (2) to S_a. □

These functional equations give the following result by induction.

Theorem 11 *The coefficients in the series expansion for T_a in the basis ${}^aS$ are given by*

$$\alpha_0 = 0, \quad \alpha_1 = 0, \quad \alpha_{k,j} = a^{k-n}\,(1-a)^n\,,$$

where n denotes the number of 1*s in the binary expansion for j.*

Theorem 12 *If $a \neq \frac{1}{2}$, then there exists the derivative $T_a'(x)$ and it is zero a.e. x in $\mathbb{I}$.*

Proof Given $x \in N$, with representation $x = \sum_{i=1}^{n} \frac{1}{2^{m_i}} + \sum_{j=n+1}^{\infty} \frac{1}{2^{m_j}}$, we consider $x_n = \sum_{i=1}^{n} \frac{1}{2^{m_i}} + \sum_{j=n+1}^{\infty} \frac{1}{2^{m_j'}}$, with $m_{n+1} \neq m_{n+1}'$. At these points, we have:

$$\begin{aligned} |T_a(x) - T_a(x_n)| \le a^{m_n - n - 1}\,(1-a)^n & \\ \times \Big(\big|(m_{n+1} - (n+1))\,a^{m_{n+1} - m_n - 1}\,(1-a) - (n+1)\,a^{m_{n+1} - m_n} + \cdots\big| & \\ + \left|\left(m_{n+1}' - (n+1)\right)a^{m_{n+1}' - m_n - 1}\,(1-a) - (n+1)\,a^{m_{n+1}' - m_n} + \cdots\right|\Big)\,. & \end{aligned}$$

We want to find an upper bound for the ending terms:

$$\begin{aligned} &\big|(m_{n+1} - (n+1))\,a^{m_{n+1} - m_n - 1}\,(1-a) - (n+1)\,a^{m_{n+1} - m_n} \\ &\quad + (m_{n+2} - (n+2))\,a^{m_{n+2} - m_n - 2}\,(1-a)^2 - (n+2)\,a^{m_{n+2} - m_n - 1} + \cdots\big| \\ &= O\left(\sum_{j=1}^{\infty} m_{n+j} a^{m_{n+j} - m_n - j}\,(1-a)^j + \sum_{j=1}^{\infty} (n+j) a^{m_{n+j} - m_n - j}\,(1-a)^j\right) \end{aligned}$$

$$= O\left(\sum_{j=1}^{\infty} \left(m_{n+j} - m_n - j\right) a^{m_{n+j}-m_n-j} (1-a)^j \right.$$

$$\left. + \sum_{j=1}^{\infty} (m_n + n + 2j) a^{m_{n+j}-m_n-j} (1-a)^j \right).$$

Because $m_{n+j} - m_n - j \geq m_{n+1} - m_n - 1$ and $m_n \geq n$, it follows that

$$\sum_{j=1}^{\infty} (m_n + n + 2j) a^{m_{n+j}-m_n-j} (1-a)^j = O\left(m_{n+1} a^{m_{n+1}-m_n}\right),$$

since xa^x is bounded and decreasing for suitable conditions, and thus

$$\sum_{j=1}^{\infty} \left(m_{n+j} - m_n - j + 1\right) a^{m_{n+j}-m_n-j+1} (1-a)^j = O\left(m_{n+1} a^{m_{n+1}-m_n}\right).$$

Therefore,

$$\begin{aligned} |T_a(x) - T_a(x_n)| &= O\left(a^{m_n-n} (1-a)^n \left(m_{n+1} a^{m_{n+1}-m_n}\right)\right) \\ &= O\left(a^{m_{n+1}-n} (1-a)^n m_{n+1}\right) = O\left(a^{m_n-n} (1-a)^n m_n\right), \end{aligned}$$

because $x \in N$. The constant involved in O depends on a. Again, taking into account that $x \in N$, then

$$\begin{aligned} |T_a(x) - T_a(x_n)| &= O\left(a^{m_n-n} (1-a)^n m_n\right) \\ &= O\left((a(1-a))^{\frac{m_n}{2}} a^{\frac{m_n}{2}-n} (1-a)^{n-\frac{m_n}{2}} m_n\right) \\ &= O\left((a(1-a))^{\frac{m_n}{2}} \left(\frac{a}{1-a}\right)^{\frac{m_n}{2}-n} m_n\right). \end{aligned}$$

Finally, we have:

$$\begin{aligned} \left|\frac{T_a(x) - T_a(x_n)}{x - x_n}\right| &= O\left(\frac{[a(1-a)]^{\frac{m_n}{2}} \left(\frac{a}{1-a}\right)^{\frac{m_n}{2}-n} m_n}{\frac{1}{2^{m_n+o(n)}}}\right) \\ &= O\left(2^{m_n+o(n)} [a(1-a)]^{\frac{m_n}{2}} \left(\frac{a}{1-a}\right)^{o(n)} m_n\right) \\ &= O\left(\left([4a(1-a)]^{1+o(1)} \left(\frac{a}{1-a}\right)^{o(1)} 2^{o(1)}\right)^n (2n + o(n))\right). \end{aligned}$$

Since $4a(1-a) < 1$ if $a \neq 1/2$, for n large enough, it follows that

$$[4a(1-a)]^{1+o(1)} \left(\frac{a}{1-a}\right)^{o(1)} 2^{o(1)} < \alpha < 1.$$

Hence, if $x_n \to x$, the quotient converges to zero. Therefore, $T_a'(x)$ exists and is zero. $\square$

Proposition 6 *If $a \neq \frac{1}{2}$, then T_a is an MTNI function.*

Proof It is sufficient to consider open intervals. Let us consider an arbitrary open interval $J \subset \mathbb{I}$, $\alpha = \sum_{j=0}^{n} \frac{1}{2^{m_j}} \in J$ and k such that $\beta = \sum_{j=0}^{n} \frac{1}{2^{m_j}} + \frac{1}{2^{m_n+k}} \in J$. Then,

$$\begin{aligned} T_a(\beta) - T_a(\alpha) + p(\beta-\alpha) &= (m_n + k - n - 1)a^{m_n+k-n-2}(1-a)^{n+1} \\ &\quad - (n+1)\, a^{m_n+k-n-1}(1-a)^n + p\frac{1}{2^{m_n+k}} \\ &= (1-a)^n\, a^{m_n+k-n-2}[(m_n+k-n-1)(1-a) \\ &\quad - (n+1)\, a] + p\frac{1}{2^{m_n+k}}. \end{aligned}$$

For each p we can find a k such that $T_a(\beta) - T_a(\alpha) + p(\beta-\alpha) > 0$.

We can also choose α such that $\gamma = \alpha + \sum_{j=1}^{k} \frac{1}{2^{m_n+j}}$ and $\delta = \gamma + \frac{1}{2^{m_n+k+1}}$, belong to J for all $k \in \mathbb{Z}^+$. Under these conditions, we have

$$\begin{aligned} T_a(\delta) - T_a(\gamma) + p(\gamma-\delta) = a^{m_n-n-1}(1-a)^{n+k} \cdot [(m_n-n)(1-a) \\ -\ (n+k+1)\, a] + p\frac{1}{2^{m_n+k+1}}. \end{aligned}$$

For each p, it is possible to find k such that the last term is negative. Therefore, T_a is an MTNI function. $\square$

Corollary 4 *If $a \neq \frac{1}{2}$, then T_a is an MNI function.*

Definition 4 For all positive integer k, let be ${}_kT_a(x) := \frac{\partial^k S_a}{\partial a^k}(x)$ for $a \in \,]0,1[\setminus 1/2$, and $x \in \,]0,1[$.

In a similar way to that in Theorem 12, we obtain the following result.

Theorem 13 ${}_kT_a'(x) = 0$ *a.e.* $x \in \mathbb{I}$.

Theorem 14 *If k is odd, then ${}_kT_a$ is an MTNI function.*

Finally, with the aid of the functions $S_{a,b}$, we give a last generalization obtaining a bi-parametric family of nowhere differentiable functions.

Definition 5 For parameters $a, b \in \,]0,1[$, set $T_{a,b} := \frac{\partial S_{a,b}}{\partial b}$.

Theorem 15 *$T_{a,b}$ is the unique bounded function satisfying*

$$\begin{cases} G(ax) = bG(x) + S_{a,b}(x), \\ G(a + (1-a)x) = (1-b)\, G(x) + 1 - S_{a,b}(x). \end{cases}$$

Theorem 16 *$T_{a,a}$ is a nowhere differentiable function.*

Proof On the one hand, if x has a finite expansion of n terms, we consider $y_k := x + a^k(1-a)^{n+1}$. Then,

$$\frac{T_{aa}(y_k) - T_{aa}(x)}{y_k - x} \longrightarrow \infty$$

if $k \to +\infty$.

On the other hand, if x has no a finite expansion, then we consider n such that $r_{n+1} > r_n$. Let us consider the truncated expansion for x

$$x_1 := a^{r_0} + \cdots + a^{r_n}(1-a)^n,$$

and

$$\overline{x}_1 := x_1 + a^{r_n}(1-a)^{n+1}.$$

Then $x_1 < x < \overline{x}_1$, and the derivative $T'_{aa}(x)$ does not exist, because

$$\lim_{n\to\infty} \frac{T_{aa}(\overline{x}_1) - T_{aa}(x_1)}{\overline{x}_1 - x_1} = \infty,$$

and the proof is complete for $x \in]0, 1[$.

Finally, for $x \in \{0, 1\}$, the one side derivatives complete the result. □

Remark 1 The above property is true for a wider class of functions:

$${}_kT_{a,a} := \left.\frac{\partial^k S_{a,b}}{\partial b^k}\right|_{b=a}.$$

Applying the same approaches as those of the previous results, we obtain:

Theorem 17 *If $a \neq b$, then $T_{a,b}$ is an MTNI function.*

Theorem 18 *If $a \neq b$, then there exists a set of λ-measure 1 such that $T_{a,b}$ has derivatives at any point x with $T'_{a,b}(x) = 0$.*

Function T_a is a strange one: it is a function without monotonicity on any subinterval whose derivative vanishes on a set of λ-measure 1.

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