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Copulas and associated fractal sets

E. de Amo[a,*], M. Díaz Carrillo[b], J. Fernández-Sánchez[a]

[a] University of Almería, Spain
[b] University of Granada, Spain

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ABSTRACT

The problem of constructing copulas whose supports are fractals has been studied by Fredricks, Nelsen and Rodríguez-Lallena [G.A. Fredricks, R.B. Nelsen, J.A. Rodríguez-Lallena, Copulas with fractal supports, Insurance Math. Econom. 37 (1) (2005) 42–48]. In this paper we continue on the path traced by these authors. We provide different types of families of self-similar copulas using techniques from Probability and Ergodic Theory to give properties on subsets of their fractal supports. In particular, we give new examples for those copulas and we analyze related topics with mutual singularity of the associated measures, Hausdorff dimension, and the connectedness of their supports.

1. Introduction

Copulas are mathematical objects which only have began to be studied in depth a few years ago. Since Sklar proved his celebrated theorem in 1959, the study of copulas and their applications has been revealed as a tool of great interest in several branches of mathematics. For an introduction to copulas see [8,17].

In the literature we have examined, all the examples of singular copulas we have found, are supported by sets with Hausdorff dimension 1. However, it is implicit in some papers (e.g. [23]) that the well-known examples of Peano and Hilbert curves, produce copulas with a fractal support, since the Hausdorff dimension of their graphs is 3/2 (see [15,24]).

Recently, Fredricks et al. [11], using an iterated function system, constructed families of copulas whose supports are fractals. In particular, they give sufficient conditions for the support of a self-similar copula to be a fractal with a Hausdorff dimension between 1 and 2. These copulas were studied by the authors in [1]. They expressed them in terms of measure-preserving functions.

Formulas for computing Hausdorff dimensions of this type of sets require that the functions involved in the iterated function system be similarities (see for example [9,10]).

In the way traced by Fredricks et al. [11], we present in this paper copulas whose supports are fractal sets, and we prove additional properties making use of results from Probability and Ergodic Theory.

From the literature we have read, the use of measure-preserving functions and techniques related to the Ergodic Theory in the study of copulas started with the proof of the existence of a correspondence between copulas and measure-preserving transformations in the unit interval $\mathbb{I}$ [18,25]; although they were already involved in the definition of the shuffle of Min (see [16]). Later on, other papers using measure-preserving functions have appeared (see [1,6,7,13]). In this paper we continue on the path traced by these other authors directly connecting the Theory of Copulas with techniques from Ergodic Theory; so that, we will frequently use concepts such as dynamical system, entropy, and so on.

* Corresponding author.
E-mail addresses: edeamo@ual.es (E. de Amo), madiaz@ugr.es (M. Díaz Carrillo), juanfernandez@ual.es (J. Fernández-Sánchez).

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We recall that each copula C induces a probability measure μ_C on the σ-algebra $\mathcal{B}(\mathbb{I}^2)$ of all the Borel sets in the unit square. Since in Ergodic Theory measure-preserving transformations are studied, we are interested in defining functions which have the property of being ergodic (see [26], among others).

In Section 3 we extend the study of those self-similar copulas initiated in [11]. Specifically, we provide a two-parameter family of copulas. Moreover, we give the first example in the literature we have examined, of mutually singular copulas with the same support (Proposition 3.24). As a consequence, we obtain a copula whose associated measure is concentrated in a set with a Hausdorff dimension less than that for the corresponding support (Proposition 4.8).

In Section 4 we study the case of copulas of full support whose associated measure has its mass concentrated in a set with a fractal dimension less than 2. We remark that all the examples we introduce in the previous sections are of copulas with connected supports.

Finally, with the techniques we have used in earlier sections, we obtain in Section 5 examples of copulas with disconnected fractal supports, and copulas with totally disconnected fractal supports (Propositions 5.7 and 5.8).

2. Notions and definitions

This section contains the background information we use throughout the paper.

Let us denote by $\mathbb{I}$ the closed unit interval $[0,1]$ and let $\mathbb{I}^2 = \mathbb{I} \times \mathbb{I}$ be the unit square. A *two-dimensional copula* (a *copula* for brevity) is a bivariate distribution function $C : \mathbb{I}^2 \to \mathbb{I}$ whose univariate marginals are uniformly distributed on $\mathbb{I}$. Each copula C induces a probability measure μ_C on $\mathbb{I}^2$ via the formula

$$\mu_C\big([a,b] \times [c,d]\big) = C(b,d) - C(b,c) - C(a,d) + C(a,c),$$

for all a,b,c,d in $\mathbb{I}$ with $a \leqslant b$ and $c \leqslant d$. The number $\mu_C([a,b] \times [c,d])$ is called the *C-volume* of the rectangle $[a,b] \times [c,d]$. Through standard measure-theoretic techniques, μ_C can be extended from the semi-ring of rectangles in $\mathbb{I}^2$ to the σ-algebra $\mathcal{B}(\mathbb{I}^2)$ of the Borel sets. The measure μ_C is doubly stochastic, and represents an infinite-dimensional generalization of doubly stochastic matrices. It originates from an idea by Birkhoff (see [4, problem 111] and [5,14,19]). As usual, λ denotes the restriction of the standard Lebesgue measure on the Euclidean plane to the σ-algebra $\mathcal{B}(\mathbb{I}^2)$.

Taking into account the correspondence between copulas and doubly stochastic measures, we can translate some measure-theoretic concepts and results into the language of copulas. In particular, the Lebesgue Decomposition Theorem produces (see [17,22]):

For any copula C, let $\mu_C = \mu_C^c + \mu_C^s$ where $\mu_C^c = \int_D \frac{\partial^2 C}{\partial u \partial v}\, d\lambda$, and $\mu_C^s = \mu_C - \mu_C^c$ for all $D \in \mathcal{B}(\mathbb{I}^2)$. Then, $\mu_C^c \ll \lambda$ (μ_C^c is an *absolutely continuous measure* with respect to λ), and $\mu_C^s \perp \lambda$ (they are *mutually singular measures*). Because the margins of C are uniform distributions, we deduce that μ_C^s has no atoms.

Just as in the case of the support of a joint distribution function, the *support* of a copula is the complement of the union of all open subsets of $\mathbb{I}^2$ with zero μ_C-measure. If $\mu_C \equiv \mu_C^s$, or equivalently,

$$\frac{\partial^2 C(u,v)}{\partial u \partial v} = 0$$

almost everywhere in $\mathbb{I}^2$, then μ_C is called *singular*, and the support of C has Lebesgue measure zero (see, for example, [17, §2.4]).

Definition 2.1. A *transformation matrix* is a matrix T with nonnegative entries, for which the sum of these is 1 and neither the row nor column sums are zero.

Following the paper by Fredricks et al. [11], we recall that each transformation matrix T determines a subdivision of $\mathbb{I}^2$ into subrectangles $R_{ij} = [p_{i-1}, p_i] \times [q_{j-1}, q_j]$, where p_i (respectively, q_j) denotes the sum of the entries in the first i columns (respectively, j rows) of T. For a transformation matrix T and a copula C, $T(C)$ denotes the copula that, for each (i,j), spreads its mass on R_{ij} in the same way as C spreads its mass on $\mathbb{I}^2$. In [11] the authors give the expression of $T(C)$ showing that is a contraction. Since the set of copulas endowed with the sup metric is a complete metric space, the Contraction-Mapping Theorem ensures that for each transformation matrix $T \neq [1]$, there is a unique copula C_T for which $T(C_T) = C_T$. Moreover $C_T = \lim_{n\to\infty} T^n(C)$, for any copula C.

Condition 2.2. Let T be a transformation matrix. We now consider the following conditions for T:

i) T has, at least, one zero entry.
ii) For each non-zero entry of T, the row and column sums through that entry are equal.
iii) There is, at least, one row or column of T with two non-zero entries.

Theorem 3 in [11], shows that if T is a transformation matrix satisfying Condition 2.2i), then C_T is singular (that is, $\mu_{C_T} \equiv \mu_{C_T}^s$).

We say that a copula C is *invariant* if $C = C_T$ for some transformation matrix T. An invariant copula C_T is said to be *self-similar* if T satisfies Condition 2.2ii).

Theorem 6 in [11] shows that the support of a self-similar copula C_T, with T satisfying Condition 2.2i) and iii), is a fractal with Hausdorff dimension between 1 and 2.

Given a measure space (X, Ω, μ), a measurable function $F : X \to X$ is said to be μ *measure-preserving* (or F *preserves* μ for short) if $\mu(F^{-1}(A)) = \mu(A)$, for all $A \in \Omega$.

If the σ-algebra Ω is generated by a family P which is closed under finite intersections, then a sufficient condition for F to be measurable and measure-preserving is that $F^{-1}(A) \in \Omega$ and $\mu(F^{-1}(A)) = \mu(A)$, for all $A \in P$ (see [2, p. 311]). The system (X, Ω, μ, F) will be called a *dynamical system*, and the main theorem in this context is the one known as Ergodic Theorem (see e.g. [12,20]).

The class of measure-preserving transformations contains special ones among them. We recall that F is said to be *ergodic* if each invariant set A (i.e. $F^{-1}(A) = A$) is trivial in the sense of having either measure 0 or 1. In this case, we say that (X, Ω, μ, F) is an *ergodic system*. If F is ergodic, then the Ergodic Theorem implies that, for μ-almost all $x \in X$, the orbit $\{F^n x\colon\ n \in N\}$ of x is a sort of replica of X itself.

We now consider two properties which are stronger than ergodicity. They are the mixing and the one-sided Bernoulli space properties.

A measure-preserving transformation F is said to be *mixing* (or *strongly mixing*, for other authors), if $\lim_{n\to\infty} \mu(A \cap F^{-n}B) = \mu(A)\mu(B)$ holds for every pair of sets $A, B \in \Omega$; or equivalently, for all $f, g \in L^2(X, \Omega, \mu)$,

$$\lim_{n\to\infty} \int_X f\bigl(F^n(x)\bigr) g(x)\, d\mu(x) = \int_X f(x)\, d\mu(x) \int_X g(x)\, d\mu(x).$$

Let us note that, if the set B is invariant, then $\mu(A \cap B) = \mu(A)\mu(B)$, and if we take $A = B$, then it follows that $\mu(B)$ is 0 or 1. Therefore, mixing implies ergodicity (see [20]).

Let $k \geqslant 2$ be an integer and let $p_0, p_1, \ldots, p_{k-1}$ be positive real numbers satisfying the relation $\sum_{i=0}^{k-1} p_i = 1$. Let $K = \{0, 1, \ldots, k-1\}$ and let $P = 2^K$ be its power set. The triple (K, P, μ) is the probability space where $\mu(i) = p_i$. The space $\prod_{j=1}^{\infty} (K, P, \mu)$ jointly with the transformation $\sigma((x_0, x_1, x_2, \ldots)) = (x_1, x_2, \ldots)$ is called the *one-side Bernoulli space*. It can be verified that one-side Bernoulli implies mixing (see [26, Sec. 4.9]).

Let us recall (see [20, Chap. 8]) that for given (X_i, Ω_i, μ_i), $i = 1, 2$, probability spaces, an *isomorphism between measure-preserving transformations* $F_i : X_i \to X_i$ is a map $\phi : X_1 \to X_2$ such that

(a) ϕ is a bijection (after removing sets of zero measure, if necessary),
(b) both ϕ and ϕ^{-1} are measurable maps (i.e. $\phi^{-1}(\Omega_2) \subset \Omega_1$ and $\phi(\Omega_1) \subset \Omega_2$),
(c) $\mu_1(\phi^{-1}B) = \mu_2(B)$, for $B \in \Omega_2$ (also $\mu_2(\phi B) = \mu_1(B)$, for $B \in \Omega_1$), and
(d) $\phi \circ F_1 = F_2 \circ \phi$.

If a space X is isomorphic to another one being one-side Bernoulli, then X will be called *Bernoulli*.

The *entropy of the system* (X, Ω, μ, F), denoted by $h(F)$, is one of the main invariants under isomorphisms. However, throughout this paper we do not use the definition of this concept properly, and we refer the interested reader to, for example, [3,12,20,26]. We here describe a result we use below.

As is usual, for a given partition α of X, $\bigvee_{i=0}^{n-1} F^{-i}\alpha$ is the set whose elements are sets of the form $A_{i_0} \cap F^{-1}A_{i_1} \cap \cdots \cap F^{-n+1}A_{i_{n-1}}$, consisting of all points such that $x \in A_{i_0}, F(x) \in A_{i_1}, \ldots, F^{n-1}(x) \in A_{i_{n-1}}$. We denote by $\bigvee_{i=0}^{\infty} F^{-i}\alpha$ the σ-algebra generated by the families $F^{-i}\alpha$. If a finite partition α exists, satisfying that $\bigvee_{i=0}^{\infty} F^{-i}\alpha = \Omega$, then we say that α is a *strong generator* with respect to F.

The next result is a restricted version for the Shannon–McMillan–Breiman theorem (see [3,20,26]):

Let $F : X \to X$ be a measure-preserving transformation of a probability space (X, B, μ). If $\alpha = \{X\}$ is a strong generator with respect to F, then for μ-almost all $x \in X$, we have that

$$\lim_{n\to\infty} \frac{-\ln(\mu(\Delta_n(x)))}{n} = h(F),$$

with $x \in \Delta_n$, $\Delta_n(x) \in \bigvee_{i=0}^{n-1} F^{-i}\alpha$, and Δ_n being the nth *cylinder* of x.

We use Mandelbrot's original definition of *fractal set*, that is, a set whose topological dimension is less than its Hausdorff dimension $\dim_{\mathcal{H}}$. For basic properties concerning Hausdorff dimension and other notions that are useful to express fractal properties of sets, the reader is referred to [9,10].

We recall here (see for example [10, p. 192] or [24]) that if μ is a probability measure on a metric space X, then the *Hausdorff dimension* of μ is defined by

$$\dim_{\mathcal{H}}(\mu) = \inf\bigl\{\dim_{\mathcal{H}}(K)\colon\ K \subset X \text{ and } \mu(K) = 1\bigr\}.$$

If the probability measure μ induces a copula C, then we write $\dim_{\mathcal{H}}(C) = \dim_{\mathcal{H}}(\mu)$. When K is a set of measure 1 we say that μ is *concentrated in* K.

Finally, a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is called a *contracting similarity* (or a *similarity transformation of ratio* r) if there is r, $0 < r < 1$, such that $|f(x) - f(y)| = r|x - y|$, for all $x, y \in \mathbb{R}^n$. A similarity transforms subsets of $\mathbb{R}^n$ into geometrically similar sets. The invariant set (or attractor) for a finite family of similarities is said to be a self-similar set. Theorem 4 in [11] shows that the support of copula C_T is the invariant set for a system of similarities obtained from the partitions of $\mathbb{I}^2$ determined by T.

For an introduction to the techniques of representation of some fractals via iterated function systems (IFS) see [9,10].

3. Self-similar copulas

3.1. A ergodic study

We now study families of self-similar copulas C given in [11]. In this first subsection, we are interested in the development of their properties by methods from Probability and Ergodic Theory. We use their self-similarity and the self-similarity of their fractal supports.

Let T be a transformation matrix which determines a self-similar copula $C : \mathbb{I}^2 \to \mathbb{I}$, as described in Definition 2.1 and Condition 2.2. Let us denote by S_C (or simply by S) the support of the copula C and by μ_C (or simply by μ, if there is no confusion) the measure associated to the copula C.

Definition 3.1. Let us denote by S_1, S_2, S_3, S_4, and S_5, respectively, the intersection of S with the following squares, for $r \in]0, \frac{1}{2}[$:

$$\begin{cases} Q_1 \text{ of vertices } (0,0), (0,r), (r,0), (r,r); \\ Q_2 \text{ of vertices } (0,1-r), (0,1), (r,1-r), (r,1); \\ Q_3 \text{ of vertices } (r,r), (1-r,r), (r,1-r), (1-r,1-r); \\ Q_4 \text{ of vertices } (1-r,0), (1-r,r), (1,0), (1,r); \\ Q_5 \text{ of vertices } (1-r,1-r), (1-r,1), (1,1-r), (1,1). \end{cases}$$

Then, we define the function $F : S \to S$, given by:

$$F(x,y) = \begin{cases} (\frac{x}{r}, \frac{y}{r}), & \text{if } (x,y) \in S_1, \\ (\frac{x}{r}, \frac{y-1+r}{r}), & \text{if } (x,y) \in S_2, \\ (\frac{x-r}{1-r}, \frac{y-r}{1-r}), & \text{if } (x,y) \in S_3^*, \\ (\frac{x-1+r}{r}, \frac{y}{r}), & \text{if } (x,y) \in S_4, \\ (\frac{x-1+r}{r}, \frac{y-1+r}{r}), & \text{if } (x,y) \in S_5, \end{cases} \tag{1}$$

where $S_3^* = S_3 \setminus \{(r,r), (1-r,r), (r,1-r), (1-r,1-r)\}$. Set $F_i : S_i \to S$ for the corresponding restriction of F to S_i.

Remark 3.2. Let us note that, for the above definition, we have taken into account the points in the intersection of two subsets (e.g. the set S_3^*). For the rest of examples in other sections we do not use this distinction in an explicit form, because they are sets of measure zero and they have no consequences for the results.

With respect to the results in [11], we use parameters $r \in]0, s[$, with $s \leqslant \frac{1}{2}$, to denote the copulas C_r (C for short) we will study.

In the following S_i denotes the intersection of the support S of the copula C with the corresponding square Q_i.

Proposition 3.3. *The function F is μ measure-preserving.*

Proof. Let D be a Borel set. Therefore, $F^{-1}(D)$ is given by five self-similar sets. (It is possible to have some fewer points for the third, but it is not important for D.) Each one of these sets is included in the corresponding S_i, and satisfies that the measure for each one of the first four squares is $\frac{r}{2}\mu(D)$, and for the third it is $(1-2r)\mu(D)$. Therefore, the sum of all of them gives $\mu(F^{-1}(D)) = \mu(D)$. $\square$

We introduce a key to identify points as intersections of sequences of nested squares which will be very useful in the following:

Definition 3.4. For each point $(x,y) \in S_i$, we define the function $a((x,y)) = i$, with i denoting the index of the set S_i to which (x,y) belongs. Let us denote $b_n = a(F^n(x,y))$, for all $n \in \mathbb{Z}_+$. (These variables will be denoted in a similar way in the sections that follow.)

Therefore, the point (x,y) is determined by the sequence $\{b_n\}_{n \in \mathbb{Z}_+}$.

Let us denote by $S_{i_0,i_1,\ldots,i_k}$ the set $F_{i_0}^{-1} \circ F_{i_1}^{-1} \circ \cdots \circ F_{i_k}^{-1}(S)$. For each point (x, y) we say that the set $\Delta_{k+1}((x, y)) = S_{b_0,b_1,\ldots,b_k}$ is its *cylinder of* $k+1$*-order*. Because $(x, y) \in S_{b_0,b_1,\ldots,b_k}$, then

$$(x, y) = \bigcap_{n\in\mathbb{Z}_+} F_{i_0}^{-1} \circ F_{i_1}^{-1} \circ \cdots \circ F_{i_k}^{-1}(S).$$

As stated above, the points are uniquely determined.

In many situations, especially, in those involving computations of Hausdorff dimensions, we will use the fact that the composition $G_{i_0} \circ G_{i_1} \circ \cdots \circ G_{i_k}(\mathbb{I}^2)$ is a square containing (x, y), where the functions are in the form:

$$\begin{cases} G_1 : \mathbb{I}^2 \longrightarrow Q_1, & (u, v) \to (ru, rv), \\ G_2 : \mathbb{I}^2 \longrightarrow Q_2, & (u, v) \to (ru, rv + 1 - r), \\ G_3 : \mathbb{I}^2 \longrightarrow Q_3, & \big((1-2r)u + r, (1-2r)v + r\big), \\ G_4 : \mathbb{I}^2 \longrightarrow Q_4, & (u, v) \to (ru + 1 - r, rv), \\ G_5 : \mathbb{I}^2 \longrightarrow Q_5, & (u, v) \to (u, v) \to (ru + 1 - r, rv + 1 - r) \end{cases} \tag{2}$$

for $r \in \,]0, \frac{1}{2}[$.

We shall now show the ergodicity for the system built above, and some of its applications. We make use of Lemma 3.5 in [2, §24]. Let us recall that a *field* is a class of subsets containing X, and closed under complements and finite unions.

Lemma 3.5. *(See [2, §24, Lem. 2].) Let (X, Ω, μ, F) be a dynamical system. Let us suppose that $P \subset B_0 \subset \Omega$, where B_0 is a field, every set in B_0 is a finite or countable disjoint union of P-sets (i.e. sets in P), and B_0 generates Ω. Let us suppose, in addition, that there exists a positive number c with this property: For each A in P there is an integer n_A such that*

$$\mu\big(A \cap F^{-n_A}(D)\big) \geqslant c\mu(A)\mu(D),$$

for all D in P. Then $F^{-1}(E) = E$, implies that $\mu(E)$ is 0 or 1.

Theorem 3.6. *The dynamical system $(S, \mathcal{B}(\mathbb{I}^2), \mu, F)$ is ergodic.*

Proof. We claim that it is a consequence of the lemma above if we choose P as the class whose elements are sets $S_{i_0,i_1,\ldots,i_k}$, with $i_j \in \{0, 1, 2, 3, 4\}$ and $k \in \mathbb{Z}_+$.

If A is the element in P given by $S_{i'_0,i'_1,\ldots,i'_k}$, we then consider $n_A = k$. For a set M we have that $F^{-k}(M)$ is the union of 5^k subsets of S, each one being a copy, at different scales, of M; in particular, each one is contained in a set $S_{i_0,i_1,\ldots,i_k}$, with $i_j \in \{1, 2, 3, 4, 5\}$.

When M is in P in the form $S_{i^*_0,i^*_1,\ldots,i^*_s}$, we have that

$$A \cap F^{-k}(M) = S_{i'_0,i'_1,\ldots,i'_k,i^*_0,i^*_1,\ldots,i^*_s}.$$

This expression together with the self-similarity of μ, gives that

$$\mu\big(A \cap F^{-k}(M)\big) = \mu(A)\mu(M),$$

and then we can apply Lemma 3.5 to $c = 1$. As a consequence, F is ergodic. □

Remark 3.7. It is interesting to note that the self-similarity for the support of the copula, implies that the dynamical system $(S, \mathcal{B}(\mathbb{I}^2), \mu, F)$ is a one-sided Bernoulli space and, therefore, mixing. This fact would allow us to prove results in this paper in a different fashion. But, we prefer to work in this manner because this technique, as far as we know, has not been used before in the same way as we have used it and, moreover, it can be applied in other contexts.

Now, using the ergodicity of F, we can prove some results (with respect to the measure μ associated to the copula C_r) on points in the support S:

Corollary 3.8. *The set of points in S satisfying*

$$\begin{cases} \lim_{n\to\infty} \dfrac{\mathrm{Card}\{k\colon\ b_k = 1,\ k = 0,\dots,n\}}{n} = r/2, \\ \lim_{n\to\infty} \dfrac{\mathrm{Card}\{k\colon\ b_k = 2,\ k = 0,\dots,n\}}{n} = r/2, \\ \lim_{n\to\infty} \dfrac{\mathrm{Card}\{k\colon\ b_k = 3,\ k = 0,\dots,n\}}{n} = 1 - 2r, \\ \lim_{n\to\infty} \dfrac{\mathrm{Card}\{k\colon\ b_k = 4,\ k = 0,\dots,n\}}{n} = r/2, \\ \lim_{n\to\infty} \dfrac{\mathrm{Card}\{k\colon\ b_k = 5,\ k = 0,\dots,n\}}{n} = r/2 \end{cases}$$

is a set of μ-measure 1.

Proof. The proof is a consequence of the Ergodic Theorem (see for instance [20, Th. 10.2]), when it is applied to the characteristic function of the sets S_i. In fact, the announced subset of S is the intersection of those sets produced by this theorem when it is applied to each one of the characteristic functions. □

Corollary 3.9. *The set of points in S satisfying*

$$\lim_{n\to\infty} \frac{b_0 + \cdots + b_n}{n} = 3, \tag{3}$$

is a set of μ-measure 1.

Let us note that equality (3) can be obtained as a consequence of the Ergodic Theorem applied to the function a introduced in Definition 3.4.

Another consequence of applying the Ergodic Theory shows that the copula C can be obtained as a limit in the weak convergence of distribution functions of atomic probabilities:

Corollary 3.10. *Let $(\alpha,\beta) \in \mathbb{I}^2$. For almost all $(x,y) \in S$, we have*

$$C(\alpha,\beta) = \lim_{n\to\infty} \frac{\mathrm{Card}\{F^k(x,y)\colon\ \pi_1(F^k(x,y)) \leqslant \alpha,\ \pi_2(F^k(x,y)) \leqslant \beta,\ k \leqslant n\}}{n}$$

(*where the π's are the corresponding projections on each axis*).

Corollary 3.11. *For μ-almost every point $(x,y) \in S$, with cylinders of order n, $\Delta_n = \Delta_n((x,y))$, we have*:

$$\lim_{n\to\infty} \frac{\ln \mu(\Delta_n)}{n} = -2r \ln r - (1-2r)\ln(1-2r) + 2r\ln 2.$$

Proof. Corollary 3.8 implies that for elements in certain set of measure 1, they satisfy that, for the n first coefficients b_k, the number of times they are 1 is $\frac{rn}{2} + o(n)$. The same is true for 2, 4 or 5.

For the digit 3, it takes the value $(1-2r)n + o(n)$, with associated probability equal to $1-2r$. The μ-probability for cylinder Δ_n is $(\frac{r}{2})^{2rn+o(n)}(1-2r)^{(1-2r)n+o(n)}$, which implies the result. □

On the other hand, as a consequence of the Shannon–McMillan–Breiman theorem (see [2,20,26]), we can obtain the entropy for the considered system we are working on:

Corollary 3.12. *The entropy for the dynamical system $(S, \mathcal{B}(\mathbb{I}^2), \mu, F)$ is*

$$h(F) = -2r\ln r - (1-2r)\ln(1-2r) + 2r\ln 2.$$

We recall that the copulas C_r we have studied are supported by sets whose fractal dimensions are between 1 and 2. A natural question arises: whether or not sets with μ-measure equal to 1, but with Hausdorff dimension less than the Hausdorff dimension of the support of C_r, exist. In the next corollary we obtain an affirmative answer, and prove that for this family of copulas, the measure μ is concentrated in a set whose Hausdorff dimension is less than the one corresponding to its support.

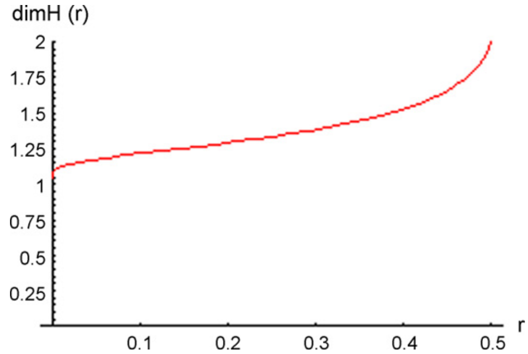

Fig. 1. The graph of Hausdorff dimension.

Therefore, we hereby provide, in the body of ideas of [11], the first explicit example of a copula C with fractal support S_C such that the dimension of the associated measure μ_C does not coincide with the Hausdorff dimension of the copula support; that is, specifically:

$$\dim_{\mathcal{H}}(\mu_C) < \dim_{\mathcal{H}}(S_C).$$

Corollary 3.13. *The measure μ is concentrated on a set with a Hausdorff dimension equal to*

$$\frac{2r\ln r + (1-2r)\ln(1-2r) - 2r\ln 2}{2r\ln r + (1-2r)\ln(1-2r)}.$$

Proof. The desired set is that of the points where

$$\lim_{n\to\infty}\frac{\ln\mu(\Delta_n)}{n} = -2r\ln r - (1-2r)\ln(1-2r) + 2r\ln 2.$$

Now, the proof follows from [10, Prop. 4.9]: we adjust it for squares in the form $G_{i_0}\circ G_{i_1}\circ\cdots\circ G_{i_k}(\mathbb{I}^2)$ (see (2)). □

Fig. 1 shows the graph of the function of these dimensions with respect to r.

If we make use of the fact that the system is mixing, then we obtain a result of the type of Gauss–Kuzmin–Lévy (see [21]):

Corollary 3.14. *Let m be a probability satisfying $m \ll \mu$ (i.e. $\mu(A)=0$ implies $m(A)=0$). If $(\alpha,\beta)\in\mathbb{I}^2$, then:*

$$\lim_{n\to\infty} m\big(\big\{(x,y):\ F^n\big((x,y)\big)\leqslant(\alpha,\beta)\big\}\big) = \mu\big([0,\alpha]\times[0,\beta]\big).$$

Proof. Set $h := \frac{dm}{d\mu}$, the derivative of Radon–Nikodým (of m with respect to μ). Then,

$$\begin{aligned}\lim_{n\to\infty} m\big(\big\{(x,y):\ F^n\big((x,y)\big)\leqslant(\alpha,\beta)\big\}\big) &= \lim_{n\to\infty}\int_{\mathbb{I}^2}\big(\chi_{[0,\alpha]\times[0,\beta]}\circ F^n\big)\big((x,y)\big)\,dm \\ &= \lim_{n\to\infty}\int_{\mathbb{I}^2}\big(\chi_{[0,\alpha]\times[0,\beta]}\circ F^n\big)\big((x,y)\big)h(x,y)\,d\mu\end{aligned}$$

(by using the mixing property)

$$\begin{aligned}&= \int_{\mathbb{I}^2}\chi_{[0,\alpha]\times[0,\beta]}\big((x,y)\big)\,d\mu\int_{\mathbb{I}^2}h\big((x,y)\big)\,d\mu \\ &= \mu\big([0,\alpha]\times[0,\beta]\big). \qquad\square\end{aligned}$$

Remark 3.15. The functions b_n, introduced in Definition 3.4, are identically distributed independent random variables. This fact, jointly with the Strong Law of Large Numbers, provides new proofs for Corollaries 3.8 and 3.9. Moreover, the application of the Law of Iterated Logarithm improves the result from the viewpoint of adding an optimal error in this form:

$$\begin{cases} \dfrac{\operatorname{Card}\{k\colon\ b_k=1,\ k=0,\dots,n\}}{n} = r/2 + O\left(\sqrt{\dfrac{\ln\ln n}{n}}\right), \\ \dfrac{\operatorname{Card}\{k\colon\ b_k=2,\ k=0,\dots,n\}}{n} = r/2 + O\left(\sqrt{\dfrac{\ln\ln n}{n}}\right), \\ \dfrac{\operatorname{Card}\{k\colon :b_k=3,\ k=0,\dots,n\}}{n} = 1-2r + O\left(\sqrt{\dfrac{\ln\ln n}{n}}\right), \\ \dfrac{\operatorname{Card}\{k\colon\ b_k=4,\ k=0,\dots,n\}}{n} = r/2 + O\left(\sqrt{\dfrac{\ln\ln n}{n}}\right), \\ \dfrac{\operatorname{Card}\{k\colon\ b_k=5,\ k=0,\dots,n\}}{n} = r/2 + O\left(\sqrt{\dfrac{\ln\ln n}{n}}\right), \\ \dfrac{b_0+\dots+b_n}{n} = 3 + O\left(\sqrt{\dfrac{\ln\ln n}{n}}\right). \end{cases}$$

Now, applying the Central Limit Theorem with the Lindeberg–Lévy condition (see [2, p. 357]) to the identically distributed independent random variables b_n with a mean of 3 and a variance of $5r$, we obtain the following result.

Corollary 3.16. *For random variables $\{b_k\}$, and given real numbers a and b:*

$$\lim_{n\to\infty} \mu\left(a < \frac{\sum_{k=1}^n b_k - 3n}{\sqrt{5rn}} < b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}}\,dx.$$

3.2. Generalization: A two-parameter family of self-similar copulas

Instead of the matrix

$$T_r = \begin{bmatrix} r/2 & 0 & r/2 \\ 0 & 1-2r & 0 \\ r/2 & 0 & r/2 \end{bmatrix}$$

(see Definition 2.1 and [11]), we introduce a new matrix which provides a new family of copulas with the same support as before, but with a different mass distribution.

We consider, for $0<a<r$ and $a+b=r$, the matrix

$$T_{r,a} = \begin{bmatrix} b & 0 & a \\ 0 & 1-2r & 0 \\ a & 0 & b \end{bmatrix},$$

with $r \in\ ]0, \frac{1}{2}[$. In this case, we obtain a new two-parameter family $C_{r,a}$ of copulas, whose support coincides with the support of C_r, and a new associated measure $\mu'_{r,a}$ (μ' for brevity).

For these measures we can carry out a study similar to that done for μ (see Definitions 3.1 and 3.4, and Remark 3.2). Therefore, we deduce the next results whose proofs go along the similar ideas used above (here F is defined by (1)):

Theorem 3.17. *The function F preserves μ', and the dynamical system $(S, \mathcal{B}(\mathbb{I}^2), \mu', F)$ is ergodic.*

This result is again a key to apply consequences of the Ergodic Theory, and to obtain properties on points on the support S of the two-parameter family of copulas $C_{r,a}$ with respect to the measure μ'.

Corollary 3.18. *The set of points in S satisfying*

$$\begin{cases} \lim\limits_{n\to\infty} \dfrac{\operatorname{Card}\{k\colon\ b_k = 1,\ k = 0, \ldots, n\}}{n} = a, \\ \lim\limits_{n\to\infty} \dfrac{\operatorname{Card}\{k\colon\ b_k = 2,\ k = 0, \ldots, n\}}{n} = b, \\ \lim\limits_{n\to\infty} \dfrac{\operatorname{Card}\{k\colon\ b_k = 3,\ k = 0, \ldots, n\}}{n} = 1 - 2r, \\ \lim\limits_{n\to\infty} \dfrac{\operatorname{Card}\{k\colon\ b_k = 4,\ k = 0, \ldots, n\}}{n} = b, \\ \lim +_{n\to\infty} \dfrac{\operatorname{Card}\{k\colon\ b_k = 5,\ k = 0, \ldots, n\}}{n} = a \end{cases}$$

is a set of μ'-measure 1.

Corollary 3.19. *The set of points in S satisfying*

$$\lim_{n\to\infty} \frac{b_0 + \cdots + b_n}{n} = 3$$

is a set of μ'-measure 1.

Using techniques from Ergodic Theory, we can now obtain the analogous to Corollary 3.11:

Corollary 3.20. *If $(\alpha, \beta) \in \mathbb{I}^2$, then*

$$C_{r,a}(\alpha, \beta) = \lim_{n\to\infty} \frac{\operatorname{Card}\{F^k(x, y)\colon\ \pi_1(F^k(x, y)) \leqslant \alpha,\ \pi_2(F^k(x, y)) \leqslant \beta,\ k \leqslant n\}}{n},$$

for μ'-almost all $(x, y) \in S$.

Corollary 3.21. *For almost all $(x, y) \in S$, if $\Delta_n = \Delta_n((x, y))$ denotes its cylinder of order n, then*

$$\lim_{n\to\infty} \frac{\ln \mu'(\Delta_n)}{n} = -2a \ln a - 2b \ln b - (1 - 2r) \ln(1 - 2r).$$

Corollary 3.22. *The entropy of the dynamical system $(S, \mathcal{B}(\mathbb{I}^2), \mu', F)$ is*

$$h(F) = -2a \ln a - 2b \ln b - (1 - 2r) \ln(1 - 2r).$$

Corollary 3.23. *The measure μ' is concentrated on a set of Hausdorff dimension equal to*

$$\frac{2a \ln a + 2b \ln b + (1 - 2r) \ln(1 - 2r)}{2r \ln r + (1 - 2r) \ln(1 - 2r)}.$$

We end this section with a result showing that the two-parameter family of copulas $C_{r,a}$ studied here, provides, to the best of our knowledge, the first example of copulas having the same support, but their associated measures are mutually singular. Moreover, by virtue of Corollary 3.23, the support of the copula and the set where the mass is concentrated, have different Hausdorff dimensions.

Proposition 3.24. *Let $\mu_1 := \mu'_{r,a_1}$ and $\mu_2 := \mu'_{r,a_2}$ be two probability measures defined on $\mathcal{B}(\mathbb{I}^2)$ both associated to the family of copulas $C_{r,a}$. Then, they induce copulas that are supported by the same set. Moreover, if $\mu_1 \neq \mu_2$, then the induced copulas are mutually singular.*

Proof. The set of points with an average of 1s in b_n's equal to a_1 has μ_1-measure 1. On the other side, this set is μ_2-null.

The reverse situation appears in the case of points whose corresponding average of 1s is a_2. $\square$

4. Copulas with full support and Hausdorff dimension less than two

In [11] it is established that if T is a transformation matrix with all entries non-zero, then the support of C_T is $\mathbb{I}^2$ (i.e. it is of full support). Using the above methods, in this section we give an unknown example of a copula whose support fills the unit square, and its associated measure is concentrated on a set with a Hausdorff dimension less than two.

For this goal, we build a partition of $\mathbb{I}^2$ and a suitable 3×3 matrix generating the type of copulas with the announced property.

Definition 4.1. Let the nine squares that follow be:

$$\begin{cases} Q_1 \text{ has vertices } (0,0), (0,1/3), (1/3,0), (1/3,1/3); \\ Q_2 \text{ has vertices } (0,1/3), (0,2/3), (1/3,1/3), (1/3,2/3); \\ Q_3 \text{ has vertices } (0,2/3), (0,1), (1/3,2/3), (1/3,1); \\ Q_4 \text{ has vertices } (1/3,0), (1/3,1/3), (2/3,0), (2/3,1/3); \\ Q_5 \text{ has vertices } (1/3,1/3), (1/3,2/3), (2/3,1/3), (2/3,2/3); \\ Q_6 \text{ has vertices } (1/3,2/3), (1/3,1), (2/3,2/3), (2/3,1); \\ Q_7 \text{ has vertices } (2/3,0), (2/3,1/3), (1,0), (1,1/3); \\ Q_8 \text{ has vertices } (2/3,1/3), (2/3,2/3), (1,1/3), (1,2/3); \\ Q_9 \text{ has vertices } (2/3,2/3), (2/3,1), (1,2/3), (1,1). \end{cases}$$

Let us consider the matrix

$$T'_r = \begin{pmatrix} \frac{1/3-r}{2} & \frac{1/3-r}{2} & r \\ \frac{1/3-r}{2} & r & \frac{1/3-r}{2} \\ r & \frac{1/3-r}{2} & \frac{1/3-r}{2} \end{pmatrix},$$

for $r \in]0, \frac{1}{3}[$. Let us now define $M : \mathbb{I}^2 \to \mathbb{I}^2$ by:

$$M(x,y) = \begin{cases} (3x, 3y), & \text{if } (x,y) \in Q_1, \\ (3x, 3(y-1/3)), & \text{if } (x,y) \in Q_2, \\ (3x, 3(y-2/3)), & \text{if } (x,y) \in Q_3, \\ (3(x-1/3), 3y), & \text{if } (x,y) \in Q_4, \\ (3(x-1/3), 3(y-1/3)), & \text{if } (x,y) \in Q_5, \\ (3(x-1/3), 3(y-2/3)), & \text{if } (x,y) \in Q_6, \\ (3(x-2/3), 3y), & \text{if } (x,y) \in Q_7, \\ (3(x-2/3), 3(y-1/3)), & \text{if } (x,y) \in Q_8, \\ (3(x-2/3), 3(y-2/3)), & \text{if } (x,y) \in Q_9. \end{cases}$$

We obtain a family of self-similar copulas C'_r once again. Its associated measure is denoted by δ_r (δ for brevity).

Next, we give analogous statements to those in the preceding section, and we conclude (in Proposition 4.8), by showing that, for these copulas, their associated measure is concentrated on a set with Hausdorff dimension lesser than two.

Theorem 4.2. *The function M preserves δ, and the dynamical system $(\mathbb{I}^2, \mathcal{B}(\mathbb{I}^2), \delta, M)$ is ergodic.*

Corollary 4.3. *The subset of points in $\mathbb{I}^2$ satisfying*

$$\begin{cases} \lim\limits_{n\to\infty} \dfrac{\operatorname{Card}\{k\colon b_k = 1,\ k = 0,\dots,n\}}{n} = r, \\ \lim\limits_{n\to\infty} \dfrac{\operatorname{Card}\{k\colon b_k = 2,\ k = 0,\dots,n\}}{n} = \dfrac{1/3-r}{2}, \\ \lim\limits_{n\to\infty} \dfrac{\operatorname{Card}\{k\colon b_k = 3,\ k = 0,\dots,n\}}{n} = \dfrac{1/3-r}{2}, \\ \lim\limits_{n\to\infty} \dfrac{\operatorname{Card}\{k\colon b_k = 4,\ k = 0,\dots,n\}}{n} = \dfrac{1/3-r}{2}, \\ \lim\limits_{n\to\infty} \dfrac{\operatorname{Card}\{k\colon b_k = 5,\ k = 0,\dots,n\}}{n} = r, \\ \lim\limits_{n\to\infty} \dfrac{\operatorname{Card}\{k\colon b_k = 6,\ k = 0,\dots,n\}}{n} = \dfrac{1/3-r}{2}, \\ \lim\limits_{n\to\infty} \dfrac{\operatorname{Card}\{k\colon b_k = 7,\ k = 0,\dots,n\}}{n} = \dfrac{1/3-r}{2}, \\ \lim\limits_{n\to\infty} \dfrac{\operatorname{Card}\{k\colon b_k = 8,\ k = 0,\dots,n\}}{n} = \dfrac{1/3-r}{2}, \\ \lim\limits_{n\to\infty} \dfrac{\operatorname{Card}\{k\colon b_k = 9,\ k = 0,\dots,n\}}{n} = r \end{cases}$$

is a set of δ-measure 1.

Corollary 4.4. *The set of points in $\mathbb{I}^2$ satisfying*

$$\lim_{n\to\infty} \frac{b_0 + \cdots + b_n}{n} = 5,$$

is a set of δ-measure 1.

Corollary 4.5. *Let $(\alpha, \beta) \in \mathbb{I}^2$. Then,*

$$C'_r(\alpha, \beta) = \lim_{n\to\infty} \frac{\operatorname{Card}\{F^k(x, y)\colon \pi_1(F^k(x, y)) \leqslant \alpha,\ \pi_2(F^k(x, y)) \leqslant \beta,\ k \leqslant n\}}{n}$$

for almost all $(x, y) \in \mathbb{I}^2$.

Corollary 4.6. *For almost all (x, y) in $\mathbb{I}^2$, on cylinders Δ_n of n-order, we have*:

$$\lim_{n\to\infty} \frac{\ln \delta(\Delta_n)}{n} = 3r \ln r + 6\left(\frac{1/3 - r}{2}\right) \ln\left(\frac{1/3 - r}{2}\right).$$

Corollary 4.7. *The entropy of the system $(\mathbb{I}^2, \mathcal{B}(\mathbb{I}^2), \delta, M)$ is*

$$h(M) = 3r \ln r + 6\left(\frac{1/3 - r}{2}\right) \ln\left(\frac{1/3 - r}{2}\right).$$

Proposition 4.8. *The measure δ is concentrated on a set with Hausdorff dimension equal to*

$$\frac{3r \ln r + 6\left(\frac{1/3-r}{2}\right) \ln\left(\frac{1/3-r}{2}\right)}{-\ln 3}.$$

Finally, the property analogous to that in Proposition 3.24, remains true in this context:

Proposition 4.9. *Let two different probability measures δ_{r_1} and δ_{r_2} defined on $\mathcal{B}(\mathbb{I}^2)$, be both associated to the family of copulas C'_r, $r \in]0, \frac{1}{3}[$. Then, they induce mutually singular copulas supported in $\mathbb{I}^2$.*

5. On the connectedness of supports

We remark that all the copulas we have studied in the above sections are of connected support. This last section is devoted to building, with the same methods as above, explicit examples of copulas with non-connected support and copulas with totally disconnected support.

5.1. Copulas with non-connected support

Let us consider the matrix given by

$$T^*_r = \begin{pmatrix} r/2 & 0 & 0 & r/2 \\ 0 & 0 & \frac{1-2r}{2} & 0 \\ 0 & \frac{1-2r}{2} & 0 & 0 \\ r/2 & 0 & 0 & r/2 \end{pmatrix},$$

where $r \in]0, \frac{1}{2}[$. The measure associated to the copula C^*_r will be denoted by μ^*.

We recall that, in all that follows, S_i denotes the intersection of the support S of the copula with the set defined in each case.

Definition 5.1. Let us consider, for $r \in]0, \frac{1}{2}[$, the squares that follow:

$$\begin{cases} Q_1 \text{ has vertices } (0,0), (0,r), (r,0), (r,r); \\ Q_2 \text{ has vertices } (0,1-r), (0,1), (r,1-r), (r,1); \\ Q_3 \text{ has vertices } (1-r,0), (1-r,r), (1,0), (1,r); \\ Q_4 \text{ has vertices } (1-r,1-r), (1-r,1), (1,1-r), (1,1); \\ Q_5 \text{ has vertices } (r,r), (r,1/2), (1/2,r), (1/2,1/2); \\ Q_6 \text{ has vertices } (1/2,1/2), (1/2,1-r), (1-r,1/2), (1-r,1-r). \end{cases}$$

And we define the function $H^*: S \to S$ given by

$$H^*(x,y)=\begin{cases} (\frac{x}{r},\frac{y}{r}), & \text{if } (x,y)\in S_1,\\ (\frac{x}{r},\frac{y-1+r}{r}), & \text{if } (x,y)\in S_2,\\ (\frac{x-1+r}{r},\frac{y}{r}), & \text{if } (x,y)\in S_3,\\ (\frac{x-1+r}{r},\frac{y-1+r}{r}), & \text{if } (x,y)\in S_4,\\ (\frac{2x-2r}{1-2r},\frac{2y-2r}{1-2r}), & \text{if } (x,y)\in S_5,\\ (\frac{x-1/2}{1-2r},\frac{y-1/2}{1-2r}), & \text{if } (x,y)\in S_6. \end{cases}$$

It is clear as an immediate consequence that the support of this copula is not-connected, because S_2 and $S_1\cup S_3\cup S_4\cup S_5\cup S_6$ are respectively contained in two mutually disjoint open sets (see Remark 3.2).

Theorem 5.2. *The function H^* preserves measure μ^* and the dynamical system $(S,\mathcal{B}(\mathbb{I}^2),\mu^*,H^*)$ is ergodic.*

Now, the ergodicity of the defined system jointly with similar techniques used above, produces analogous results in properties of sets of points in the support of S^*. We specially call attention to the next result.

Corollary 5.3. *The measure μ^* is concentrated in a set with Hausdorff dimension*

$$\frac{2r\ln r+(1-2r)\ln(1-2r)-\ln 2}{2r\ln r+(1-2r)\ln(1-2r)}.$$

5.2. Copulas with totally disconnected support

This last example of copula remembers us the shuffles of Min (see [17,16]), because the intersection of its support and any vertical or horizontal line reduces to a point (but in a denumerable set it will reduce to two points). However, its support is totally disconnected; which is an example unknown in the literature we have examined up to now. It comes from the matrix,

$$T_r^{**}=\begin{bmatrix} r & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & r & 0\\ 0 & 0 & 1-4r & 0 & 0\\ 0 & r & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & r \end{bmatrix},$$

with $r\in\,]0,1/4[$, and the associated measure for the copula C^{**} will be denoted by μ^{**}.

Definition 5.4. Let us consider the following squares, with $r\in\,]0,\frac{1}{4}[$:

$$\begin{cases} Q_1 \text{ has vertices } (0,1-r),(0,1),(r,1-r),(r,1);\\ Q_2 \text{ has vertices } (r,r),(r,2r),(2r,r),(2r,2r);\\ Q_3 \text{ has vertices } (2r,2r),(2r,1-2r),(2r,1-2r),(1-2r,1-2r);\\ Q_4 \text{ has vertices } (1-2r,1-2r),(1-2r,1-r),(1-r,1-2r),(1-r,1-r);\\ Q_5 \text{ has vertices } (1-r,0),(1-r,r),(1,0),(1,r). \end{cases}$$

Let us define the function $H^{**}: S\to S$, given by

$$H^*(x,y)=\begin{cases} (\frac{x}{r},\frac{y-1+r}{r}), & \text{if } (x,y)\in S_1,\\ (\frac{x-r}{r},\frac{y-r}{r}), & \text{if } (x,y)\in S_2,\\ (\frac{x-2r}{1-4r},\frac{y-2r}{1-4r}), & \text{if } (x,y)\in S_3,\\ (\frac{x-1+2r}{r},\frac{y-1+2r}{r}), & \text{if } (x,y)\in S_4,\\ (\frac{x-1+r}{r},\frac{y}{r}), & \text{if } (x,y)\in S_5. \end{cases}$$

Theorem 5.5. *The function H^{**} preserves μ^{**} and the dynamical system $(S,\mathcal{B}(\mathbb{I}^2),\mu^{**},H^{**})$ is ergodic.*

Then, analogous results to those in Sections 3 and 4 can be proved. We emphasize the following:

Corollary 5.6. *The measure μ^{**} is concentrated on a set with Hausdorff dimension equal to* 1.

Proposition 5.7. *The support S of the copula is totally disconnected.*

Proof. We introduce variables b_k defined in a similar way as in Definition 3.4. They determine the points in the support uniquely. For two points (x, y) and (x', y') in the support, with $b_k((x, y)) = b_k((x', y'))$, for $1 \leqslant k \leqslant n$, and $b_{n+1}((x, y)) \neq b_{n+1}((x', y'))$, if we define

$$\begin{cases} G_1 : \mathbb{I}^2 \longrightarrow Q_1, & (u, v) \to (ru, rv + 1 - r), \\ G_2 : \mathbb{I}^2 \longrightarrow Q_2, & (u, v) \to (ru + r, rv + r), \\ G_3 : \mathbb{I}^2 \longrightarrow Q_3, & (u, v) \to \big((1 - 4r)u + 2r, (1 - 4r)v + 2r\big), \\ G_4 : \mathbb{I}^2 \longrightarrow Q_4, & (u, v) \to (ru + 1 - 2r, rv + 1 - 2r), \\ G_5 : \mathbb{I}^2 \longrightarrow Q_5, & (u, v) \to (ru + 1 - r, rv), \end{cases}$$

then, the squares $G_{i_0} \circ G_{i_1} \circ \cdots \circ G_{i_k} \circ G_{i_{k+1}} \circ G_{i_{k+2}}(\mathbb{I}^2)$ and $G_{i_0} \circ G_{i_1} \circ \cdots \circ G_{i_k} \circ G_{i'_{k+1}} \circ G_{i'_{k+2}}(\mathbb{I}^2)$ are disjoint. Therefore, we can separate them by open subsets in $\mathbb{I}^2$, and points (x, y) and (x', y') will belong to different components (i.e. maximal connected sets), and as a consequence, the support S is totally disconnected. □

Proposition 5.8. *The intersection of S with any vertical line reduces to a point, but there is a denumerable number of cases where this intersection is of two points. The same is true for horizontal lines.*

Proof. We present the reasoning for the case of vertical lines. (The other case can be done by symmetry on the diagonal.) The intersection of a straight line and a support S, is included in the intersection of the k-order squares $G_{i_0} \circ G_{i_1} \circ \cdots \circ G_{i_k}(\mathbb{I}^2)$. This intersection is a segment whose length tends to zero; therefore, the intersection is a point.

In such cases when it is not a segment, then it composed by two segments; each one of them corresponding to one side of a square in the form $G_{i_0} \circ G_{i_1} \circ \cdots \circ G_{i_k}(\mathbb{I}^2)$. Then, in this case, the intersection reduces to two points. □

If we remove a denumerable set of points in the support of these copulas in such a way that each intersection of vertical with horizontal lines be a unique point, then we obtain the graph of a measure-preserving bijection. It is a particular case of the functions that Durante et al. used in [7] to define the generalized shuffles of a copula.

Finally, we emphasize that this result is the best we can obtain:

1. In the case when the intersection is a point on each vertical line, then the support would coincide with the graph of some function. Because the support has to be a compact set, the graph is compact as well. However, this implies continuity for the function, and therefore, its graph is connected.
2. When the intersection is a point on each vertical line, but on a finite number of them, the graph will contain, at least, the (connected) graph of some continuous function in an interval $[a, b] \subset \mathbb{I}$. Therefore, it is not totally disconnected.

6. Conclusions

In this paper we introduce a two-parameter family of copulas. The method we have used is via transformation matrices T with different entries, but determining the same subdivision of the unit square $\mathbb{I}^2$ into subrectangles R_{ij}. The copulas of this family have the same support, and their masses are concentrated on sets having a Hausdorff dimension less than the respective Hausdorff dimension of their support. Although the support is the same for all the copulas in the family, the corresponding distributions of masses are very different; let us note that the associated doubly stochastic measures are mutually singular.

We show examples of copulas whose support is the unit square $\mathbb{I}^2$, and the masses are concentrated on a set with a Hausdorff dimension less than 2. For all $\alpha > 1$, there exists a copula in this family whose mass is concentrated on a set with a Hausdorff dimension less than α.

In every case, we study random variables and ergodic properties related to these copulas. Therefore, the paper moves along a line that directly connects Copula Theory with techniques from Ergodic Theory. This shows, in particular, the interest in studying the self-similarity of the supports and the doubly stochastic measures associated to these copulas (because in this case, the corresponding dynamical system is a one-sided Bernoulli space).

This framework concentrates some topics of our interest for future investigations. Among them, we can list the study of homomorphisms between supports of copulas; applications of the study of copulas with given support to approximations of copulas, using recent ideas related to the role that the shuffles of Min play in these approximations; or to analyzing conditions under which copulas with “strange” support are extreme copulas.

References

[1] E. de Amo, M. Díaz Carrillo, J. Fernández-Sánchez, Measure-preserving functions and the independence copula, Mediterr. J. Math. 8 (3) (2011) 445–465.
[2] P. Billingsley, Probability and Measure, 3rd edn., Wiley Ser. Probab. Math. Statist., John Wiley & Sons Inc., New York, 1995.

[3] P. Billingsley, Ergodic Theory and Information, John Wiley & Sons Inc., New York, London, Sydney, 1965.
[4] G. Birkhoff, Lattice Theory, revised edition, Amer. Math. Soc. Colloq. Publ., vol. 25, Amer. Math. Soc., New York, 1948.
[5] J.R. Brown, Doubly stochastic measures and Markov operators, Michigan Math. J. 12 (1965) 367–375.
[6] F. Durante, E.P. Klement, J.J. Quesada-Molina, P. Sarkoci, Remarks on two product-like constructions for copulas, Kybernetika (Prague) 43 (2007) 235–244.
[7] F. Durante, P. Sarkoci, C. Sempi, Shuffles of copulas, J. Math. Anal. Appl. 352 (2) (2009) 914–921.
[8] F. Durante, C. Sempi, Copula theory: an introduction, in: P. Jaworski, F. Durante, W. Härdle, T. Rychlik (Eds.), Workshop on Copula Theory and Its Applications, in: Lecture Notes in Statist. Proc., Springer, Berlin, Heidelberg, 2010.
[9] G.A. Edgar, Measure, Topology, and Fractal Geometry, Undergrad. Texts in Math., Springer, Berlin, Heidelberg, New York, 1990.
[10] K.J. Falconer, Fractal Geometry. Mathematical Foundations and Applications, 2nd edn., John Wiley, 2003.
[11] G.A. Fredricks, R.B. Nelsen, J.A. Rodríguez-Lallena, Copulas with fractal supports, Insurance Math. Econom. 37 (1) (2005) 42–48.
[12] P.R. Halmos, Ergodic Theory, Chelsea, New York, 1956.
[13] A. Kolesárová, R. Mesiar, C. Sempi, Measure-preserving transformations, copula and compatibility, Mediterr. J. Math. 5 (2008) 325–339.
[14] J. Lindenstrauss, A remark on extreme doubly stochastic measures, Amer. Math. Monthly 72 (1965) 379–382.
[15] M. McClure, The Hausdorff dimension of Hilbert's coordinate functions, Real Anal. Exchange 24 (1998/1999).
[16] P. Mikusiński, H. Sherwood, M.D. Taylor, Shuffles of Min, Stoch. Anal. Appl. 13 (1992) 61–74.
[17] R.B. Nelsen, An Introduction to Copulas, 2nd edn., Springer Ser. Statist., Springer, New York, 2006.
[18] E.T. Olsen, W.F. Darsow, B. Nguyen, Copulas and Markov operators, in: Distributions with Fixed Marginals and Related Topics, Seattle, WA, 1993, in: IMS Lecture Notes Monogr. Ser., vol. 28, Inst. Math. Statist., Hayward, CA, 1996, pp. 244–259.
[19] J.E.L. Peck, Doubly stochastic measures, Michigan Math. J. 6 (1959) 217–220.
[20] M. Pollicott, M. Yuri, Dynamical Systems and Ergodic Theory, London Math. Soc. Stud. Texts, vol. 40, Cambridge University Press, Cambridge, 1998.
[21] A.M. Rockett, P. Szüsz, Continued Fractions, World Sci. Publishing Co., River Edge, NJ, 1992.
[22] W. Rudin, Real and Complex Analysis, McGraw–Hill, New York, 1966.
[23] H. Sagan, Space Filling Curves, Springer-Verlag, New York, 1994.
[24] M. Urbański, The Hausdorff dimension of the graphs of continuous self-affine functions, Proc. Amer. Math. Soc. 108 (4) (1990) 921–930.
[25] R.A. Vitale, Parametrizing doubly stochastic measures, in: Distributions with Fixed Marginals and Related Topics, Seattle, WA, 1993, in: IMS Lecture Notes Monogr. Ser., vol. 28, Inst. Math. Statist., Hayward, CA, 1996, pp. 358–364.
[26] P. Walters, An Introduction to Ergodic Theory, Springer-Verlag, New York, Heidelberg, Berlin, 1980.