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On the duality of aggregation operators and k-negations

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Abstract

In this paper we study a class of duality functions given by the solution of a system of functional equations related to the De Rham system. With the aid of a generalized dyadic representation system in the unit interval, we study a negation N which is a duality function for pairs of operators satisfying certain boundary conditions. New properties of N are investigated, including its singularity and fractal dimensions for several related sets. As an application we obtain an explicit expression for k-negations.

Keywords: De Rham system; Singular function; Aggregation operator; Generalized dyadic representation system; k-negation; Hausdorff dimension

1. Introduction and preliminaries

In [18], a duality relation is studied for pairs of binary operations on the unit interval $\mathbb{I} = [0, 1]$, involving members of a class of aggregation operators which satisfy certain boundary conditions. More precisely, from the solution of a system of functional equations related to the De Rham system, it is established that for any F in the above class, there exist exactly two functions G and N such that the pair (F, G) is N-dual.

In this paper we seek an explicit description of N, and we study new properties for this function. Specifically, N is a singular function, i.e. it is a decreasing function whose derivative vanishes almost everywhere. As a consequence, new properties of k-negations can be described by N.

In this section we briefly introduce the general background.

Aggregation operators are mathematical objects that have the goal of reducing a set of numbers into a unique representative (or meaningful) number. Let $\mathbb{I}^2$ be the unit square. An *aggregation operator* is defined as a function $F : \mathbb{I}^2 \to \mathbb{I}$ that satisfies:

(i) $F(0, 0) = 0$ and $F(1, 1) = 1$ (*boundary conditions*); and
(ii) $F(x_1, y_1) \leq F(x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ (*non-decreasing monotonicity*).

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One of the most common aggregation operators are t-norms. A *t-norm* (or *triangular norm*) T is defined as an associative (i.e. $T(x, T(y, z)) = T(T(x, y), z)$) and commutative aggregation operator $T : \mathbb{I}^2 \to \mathbb{I}$ satisfying $T(1, x) = x$ for all $x \in \mathbb{I}$ (with neutral element 1). A *t-conorm* S is defined as an associative and commutative aggregation operator $S : \mathbb{I}^2 \to \mathbb{I}$ satisfying $S(0, x) = x$ for all $x \in \mathbb{I}$ (with neutral element 0).

A *negation* N is defined as a non-increasing function $N : \mathbb{I} \to \mathbb{I}$ with boundary conditions $N(0) = 1, N(1) = 0$. If N is involutive, i.e. if $N(N(x)) = x$ holds for all $x \in \mathbb{I}$, we say that N is a *strong negation*.

For any given t-conorm S and any strong negation N, the function

$$(x, y) \longrightarrow N(S(N(x), N(y))), \quad \forall (x, y) \in \mathbb{I}^2$$

is a t-norm.

It is well known that t-norms and t-conorms have been used to define the intersection and union of fuzzy sets, and to model the logic "*and*" and "*or*" in fuzzy logic as a generalization of the Boolean logic connectives to multi-valued logic (see for example [10]). In [25, Theorem 2.1] strong negations are characterized in the context of the theory of fuzzy sets by means of some $\mathbb{I}$-automorphism (order-preserving bijection). In [15], a general framework for characterizing self-dual aggregation operators has been presented. Another interesting development in aggregation operator theory is copulas. For instance, commutative and associative copulas are t-norms, and the 1-Lipschitz (binary) aggregation operator with neutral element 1, is a quasi-copula (see [1,9,20]). For an overview of results in the field of aggregation operators we refer the reader to [5].

We will denote by Φ the subclass of commutative aggregation operators F that satisfy the relations:

$$F(x, 0) = F(1, 0)x \text{ and } F(x, 1) = (1 - F(1, 0))x + F(1, 0)$$

for all $x \in \mathbb{I}$, with $F(1, 0) \in]0, 1[$. Let us observe that t-norms and t-conorms are in Φ. The convex linear hull of elements in Φ remains in Φ.

Let T, S be in Φ and let N be a negation function. N is said to be a *duality function for the pair* (T, S) (or that *the pair* (T,S) *is N-dual*), if $N(T(x, y) = S(N(x), N(y))$ for all x, y in $\mathbb{I}$.

For a deeper discussion on duality, negations and automorphisms, see [11–15], and the references therein.

The following result is proved in [18]. We state it with the notation we will use in this paper.

Theorem 1.1 (*Mayor and Torrens [18, Theorem 2]*). *Let F be in Φ. Given $k, k', 0 < k, k' < 1$, there exists a unique $G_{F,k'}$ in Φ, with $G_{F,k'}(1, 0) = k'$, and a unique negation function $N_{k,k'} : \mathbb{I} \to \mathbb{I}$ such that the pair $(F, G_{F,k'})$ is $N_{k,k'}$-dual.*

Moreover, it is possible to give expressions for $N_{k,k'}$ and $G_{F,k'}$. They are related to the solution of the following system of functional equations (known as the De Rham system):

$$\begin{cases} R\left(\frac{x}{2}\right) = kR(x) \\ R\left(\frac{x+1}{2}\right) = (1-k)R(x) + k \end{cases}$$

This system just admits only one bounded solution; it is, in fact, a strictly increasing continuous bijection $R_k : \mathbb{I} \to \mathbb{I}$. Other properties of R_k can be found in [3,4,6,21–24]. The graph of $R_{.3}$ is illustrated in Fig. 1.

We can use the following equalities involving functions R_k, $G_{F,k'}$ and $N_{k,k'}$:

$$N_{k,k'} = R_{k'}(1 - R_k^{-1}) \tag{1}$$

and

$$G_{F,k'}(x, y) = N_{k,k'} F(N_{k,k'}^{-1}(x), N_{k,k'}^{-1}(y))$$

where R_k and $R_{k'}$ are the solutions of respective De Rham's systems for k and k' in $]0, 1[$.

As well as the above, function $N_{k,k'}$ is characterized as the only solution in the unit interval $\mathbb{I}$ of a new system of functional equations that is closely related to De Rham's system:

$$\begin{cases} f(kx) = k' + (1-k')f(x) \\ f(k + (1-k)x) = k'f(x) \end{cases}$$

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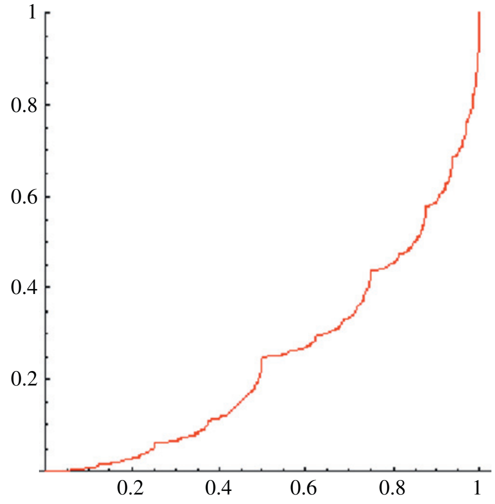

Fig. 1. The graph of $R_{.3}$.

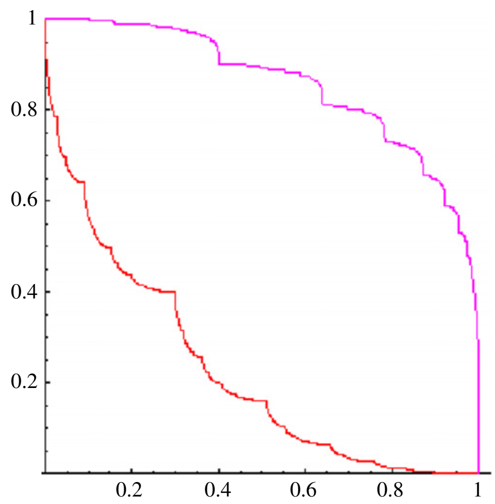

Fig. 2. The graphs of $N_{.3,.4}$ and $N_{.4,.9}$.

More details can be found in [6,16–19]. The graphs corresponding to $N_{.3,.4}$ (the lower) and $N_{.4,.9}$ (the upper) are shown in Fig. 2.

We have seen above that the two-parameter functions $N_{k,k'}$ defined on $\mathbb{I}$, can be used in order to consider duality relations for a class of binary operations on $\mathbb{I}$. Although an explicit expression for $R_k(x)$ is known when we consider the dyadic representation for x, an explicit expression for $N_{k,k'}(x)$ is still unknown. The best result we know in this direction is shown in [18, Theorem 3], where such expression is given for a denumerable set of points in $\mathbb{I}$. Moreover, since R_k is an increasing continuous bijection on $\mathbb{I}$, (1) shows that $N_{k,k'}$ is a continuous and decreasing bijection on $\mathbb{I}$ but any other properties have not been studied.

In this paper, we deal with duality functions for pairs of aggregation operators on $\mathbb{I}$, and we give a description and new properties of $N_{k,k'}$ which complete some previous studies on this topic.

This paper is structured as follows: in Section 2 we present a brief overview of a generalized dyadic number system that will be used later.

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Section 3 is devoted to the study of two-parameter functions defined on $\mathbb{I}$. With the aid of such a generalized dyadic representation system for real numbers in $]0,1]$, an explicit expression of $N_{k,k'}$ is given. After this, if $k \neq 1-k'$, we prove that the function $N_{k,k'}$ is not only strictly increasing and continuous, but also singular. If $k \neq 1-k'$, then there exists a set of λ-measure one in which the derivative of $N_{k,k'}$ vanishes (with λ the standard Lebesgue measure). We also show additional properties for this function: $N_{k,k'}$ maps a set of λ-measure 0 onto a set of λ-measure 1, and vice versa. The Hausdorff dimensions of these sets are computed. Moreover, functions $N_{k,k'}$ are characterized as the unique solution of a system of functional equations.

In Section 4, the above results are applied to k-negations which are special types of strong negations.

We use standard terms of measure theory and fractal geometry whose definitions and properties may be found, for example, in the classic of Falconer [7].

2. Generalized dyadic number system

In this section, the reader is referred to [2, Section 3] for a wider and more detailed study. Actually, a representation system for numbers in $]0,1]$ is introduced in [2, Section 3]. This system generalizes the dyadic (or binary) one. As it is known, the dyadic system permits the expression of any real number in $]0,1]$ through a series in the form $x=\sum_{n=1}^{+\infty} 1/2^{\alpha_n}$, where (α_n) is a strictly increasing sequence of positive integers. For this new representation, we introduce two numbers k and $1-k$ (with $k\in]0,1[$), and we obtain expansions in the form $x=\sum_{n=1}^{+\infty}(1-k)^n k^{m_n}$, with $m_n \leq m_{n+1}$. These series expansions are similar to those of dyadic representation. The expansion is unique except for a denumerable set of numbers x for which there are exactly two representations, one of which is finite. We call the new representation the *generalized dyadic number* system (of k *parameter*).

Note that in the case $k=1-k=1/2$, both representations are the same for numbers in $]0,1]$. We recall the following result.

Proposition 2.1 (*de Amo and Fernández-Sánchez [2, Proposition 3]*). *Let* $k\in]0,1[$. *If* $x\in]0,1]$, *then there exists a unique increasing sequence of positive integers* $1\leq m_0\leq m_1\leq\cdots\leq m_d\leq\cdots$, *such that* $x=\sum_{d=0}^{+\infty}(1-k)^d k^{m_d}$.

The expansion $x=\sum_{d=0}^{+\infty}(1-k)^d k^{m_d}$ is unique, but in the stationary case (i.e. $m_j>m_{j-1}$ and $m_d=m_j$, if $d\geq j$), we have the equality $x=\sum_{d=0}^{+\infty}(1-k)^d k^{m_d}=\sum_{d=0}^{j-1}(1-k)^d k^{m_d}+(1-k)^j k^{m_j-1}$. This finite expression can be considered as a second expansion in this system for x.

3. Functions $N_{k,k'}$

We introduce a class of two-parameter functions defined on $\mathbb{I}$ in order to consider some fundamental properties of the duality function $N_{k,k'}$.

Definition 3.1. For each pair $k,k'\in]0,1[$, let us consider the function $f_{k,k'}:\mathbb{I}\to\mathbb{I}$, given as follows:

(i) If x has a non-stationary infinite expansion, i.e. there exist $1\leq t_0<t_1<\cdots<t_d<\cdots$, such that

$$x=k^{t_0}+\cdots+k^{t_0}(1-k)^{s_0}+k^{t_1}(1-k)^{s_0+1}+\cdots+k^{t_1}(1-k)^{s_1}+\cdots$$
$$+k^{t_d}(1-k)^{s_{d-1}+1}+\cdots+k^{t_d}(1-k)^{s_d}+\cdots$$

then

$$f_{k,k'}(x)=k'+k'(1-k')+\cdots+k'(1-k')^{t_0-2}+k'^{s_0+2}(1-k')^{t_0-1}+\cdots+k'^{s_0+2}(1-k')^{t_1-2}$$
$$+k'^{s_1+2}(1-k')^{t_1-1}+\cdots+k'^{s_1+2}(1-k')^{t_2-2}$$
$$+\cdots k'^{s_{d-1}+2}(1-k')^{t_{d-1}-1}+\cdots+k'^{s_{d-1}+2}(1-k')^{t_d-2}+\cdots$$

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If $t_0 = 1$, then

$$\begin{aligned} f_{k,k'}(x) = & k'^{s_0+2} + k'^{s_0+2}(1-k') + \cdots + k'^{s_0+2}(1-k')^{t_1-2} \\ & + k'^{s_1+2}(1-k')^{t_1-1} + \cdots + k'^{s_1+2}(1-k')^{t_2-2} \\ & + \cdots k'^{s_{d-1}+2}(1-k')^{t_{d-1}-1} + \cdots + k'^{s_{d-1}+2}(1-k')^{t_d-2} + \cdots \end{aligned}$$

(ii) If x has a finite expansion, that is, in the stationary case,

$$x = k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + \cdots + k^{t_d}(1-k)^{s_{d-1}+1} + \cdots + k^{t_d}(1-k)^{s_d}$$

then

$$\begin{aligned} f_{k,k'}(x) = & k' + k'(1-k') + \cdots + k'(1-k')^{t_0-2} \\ & + \cdots k'^{s_{d-1}+2}(1-k')^{t_{d-1}-1} + \cdots + k'^{s_{d-1}+2}(1-k')^{t_d-2} k'^{s_d+1}(1-k')^{t_d-1} \end{aligned}$$

Several examples are included below illustrating how Definition 3.1 works.

Example 3.2. In case of a non-stationary infinite expansion, let

$$\begin{aligned} x = & k^2 + k^2(1-k) + \cdots + k^2(1-k)^5 + k^4(1-k)^6 + \cdots + k^4(1-k)^{11} \\ & + k^7(1-k)^{12} + \cdots + k^7(1-k)^{16} + k^{11}(1-k)^{17} + \cdots + k^{11}(1-k)^{22} \\ & + k^{25}(1-k)^{23} + \cdots + k^{25}(1-k)^{30} + \cdots \end{aligned}$$

Then the values for t_i and s_i are given by

$$\begin{array}{ll} t_0 = 2 & s_0 = 5 \\ t_1 = 4 & s_1 = 11 \\ t_2 = 7 & s_2 = 16 \\ t_3 = 11 & s_3 = 22 \\ t_4 = 25 & s_4 = 30 \\ \ldots & \ldots \end{array}$$

and the first terms for the series expansion of $f_{k,k'}(x)$ are

$$\begin{aligned} f_{k,k'}(x) = & k' + k'^7(1-k') + k'^7(1-k')^2 + k'^{13}(1-k')^3 + k'^{13}(1-k')^4 + k'^{13}(1-k')^5 \\ & + k'^{18}(1-k')^6 + \cdots + k'^{18}(1-k')^9 \\ & + k'^{24}(1-k')^{10} + \cdots + k'^{24}(1-k')^{23+} + \cdots \end{aligned}$$

Example 3.3. The case

$$\begin{aligned} x = & k^1 + k^1(1-k) + \cdots + k^1(1-k)^7 + k^5(1-k)^8 + \cdots + k^5(1-k)^{13} \\ & + k^6(1-k)^{14} + \cdots + k^6(1-k)^{19} + k^{17}(1-k)^{20} + \cdots + k^{17}(1-k)^{26} \\ & + k^{28}(1-k)^{27} + k^{28}(1-k)^{28} + \cdots \end{aligned}$$

has these corresponding values for t_i and s_i:

$$\begin{array}{ll} t_0 = 1 & s_0 = 7 \\ t_1 = 5 & s_1 = 13 \\ t_2 = 6 & s_2 = 19 \\ t_3 = 17 & s_3 = 26 \\ t_4 = 28 & s_4 = 28 \\ \ldots & \ldots \end{array}$$

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Then $t_0 = 1$, and the first terms in the series expansion for $f_{k,k'}(x)$ are given by

$$\begin{aligned} f_{k,k'}(x) = {} & k'^9 + k'^9(1-k') + k'^9(1-k')^2 + k'^9(1-k')^3 + k'^{15}(1-k')^4 \\ & + k'^{21}(1-k')^5 + \cdots + k'^{21}(1-k')^{15} \\ & + k'^{28}(1-k')^{16} + \cdots + k'^{28}(1-k')^{26} + \cdots \end{aligned}$$

Example 3.4. In the finite case,

$$\begin{aligned} x = {} & k^2 + k^2(1-k) + \cdots + k^2(1-k)^5 + k^4(1-k)^6 + \cdots + k^4(1-k)^{11} \\ & + k^7(1-k)^{12} + \cdots + k^7(1-k)^{16} + k^{11}(1-k)^{17} + \cdots + k^{11}(1-k)^{22} \\ & + k^{25}(1-k)^{23} + \cdots + k^{23}(1-k)^{30} \end{aligned}$$

the values for t_i and s_i are

$$\begin{aligned} t_0 &= 2 & s_0 &= 5 \\ t_1 &= 4 & s_1 &= 11 \\ t_2 &= 7 & s_2 &= 16 \\ t_3 &= 11 & s_3 &= 22 \\ t_4 &= 25 & s_4 &= 30 \end{aligned}$$

and $f_{k,k'}(x)$ has this expansion:

$$\begin{aligned} f_{k,k'}(x) = {} & k' + k'^7(1-k') + k'^7(1-k')^2 + k'^{13}(1-k')^3 + k'^{13}(1-k')^4 + k'^{13}(1-k')^5 \\ & + k'^{18}(1-k')^6 + \cdots + k'^{18}(1-k')^9 + k'^{24}(1-k')^{10} + \cdots + k'^{24}(1-k')^{23} + k'^{31}(1-k')^{24} \end{aligned}$$

We will prove that this new $f_{k,k'}$ family consists of functions which are singular and that verify a theorem similar to Theorem 1 in [18].

Proposition 3.5. *$f_{k,k'}$ is strictly decreasing and continuous.*

Proof. Strict monotonicity is proved by considering four cases. In fact, given any x, y in $\mathbb{I}$, $y < x$, we have:

Case a: x and y have both a non-stationary infinite expansions that coincide for the first d values of t_i and s_i. For the $(d+1)$th value, we have two possibilities:

(a.i)

$$x = k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + \cdots + k^{t_d}(1-k)^{s_d} + k^{t_{d+1}}(1-k)^{s_d+1} + \cdots + k^{t_{d+1}}(1-k)^{s_{d+1}} + \cdots$$

and

$$y = k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + \cdots + k^{t_d}(1-k)^{s_d} + k^{t_{d+1}}(1-k)^{s_d+1} + \cdots + k^{t_{d+1}}(1-k)^{s'_{d+1}} + \cdots$$

(Note that $s'_{d+1} < s_{d+1}$).

Applying $f_{k,k'}$:

$$\begin{aligned} f_{k,k'}(x) = {} & k' + k'(1-k') + \cdots + k'(1-k')^{t_0-2} + \cdots \\ & + k'^{s_{d-1}+2}(1-k')^{t_{d-1}-1} + \cdots + k'^{s_{d-1}+2}(1-k')^{t_d-2} \\ & + k'^{s_d+2}(1-k')^{t_d-1} + \cdots + k'^{s_d+2}(1-k')^{t_{d+1}-2} + k'^{s_{d+1}+2}(1-k')^{t_{d+1}-1} + \cdots \end{aligned}$$

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and

$$f_{k,k'}(y) = k' + k'(1-k') + \cdots + k'(1-k')^{t_0-2} + \cdots$$
$$+k'^{s_{d-1}+2}(1-k')^{t_{d-1}-1} + \cdots + k'^{s_{d-1}+2}(1-k')^{t_d-2}$$
$$+k'^{s_d+2}(1-k')^{t_d-1} + \cdots + k'^{s_d+2}(1-k')^{t_{d+1}-2} + k'^{s'_{d+1}+2}(1-k')^{t_{d+1}-1} + \cdots$$

which yields to $f_{k,k'}(x) < f_{k,k'}(y)$.

(a.ii)

$$x = k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + \cdots + k^{t_d}(1-k)^{s_d} + k^{t_{d+1}}(1-k)^{s_d+1} + \cdots + k^{t_{d+1}}(1-k)^{s_{d+1}} + \cdots +$$

and

$$y = k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + \cdots + k^{t_d}(1-k)^{s_d} + k^{t'_{d+1}}(1-k)^{s_d+1} + \cdots + k^{t_{d+1}}(1-k)^{s'_{d+1}} + \cdots$$

(Note that $t_{d+1} < t'_{d+1}$.)

Writing $f_{k,k'}(x)$ and $f_{k,k'}(y)$ as in (a.i), we have $f_{k,k'}(x) < f_{k,k'}(y)$.

Similar reasoning applies to the following cases.

Case b: x has a non-stationary infinite expansion and y has finite expansion.

Case c: x has a finite expansion and y has a non-stationary infinite expansion.

Case d: x and y have both finite expansions.

Next, in order to prove the continuity, given any $x \in \mathbb{I}$ we need to consider two cases:

(a′) x has a non-stationary infinite expansion, i.e.:

$$x = k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + \cdots + k^{t_d}(1-k)^{s_{d-1}+1} + \cdots + k^{t_d}(1-k)^{s_d} + \cdots$$

Here, for any sequence (x_n) converging to x there exists a positive integer m such that if $n \geq m$, then

$$x_n = k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + \cdots + k^{t_d}(1-k)^{s_{d-1}+1} + \cdots + k^{t_d}(1-k)^{s_d}$$
$$+k^{t'_{d+1}}(1-k)^{s_d+1} + \cdots + k^{t'_{d+1}}(1-k)^{s'_d+1} + \cdots$$

that is, x_n and x coincide on the first $d+1$ values of t_i and s_i, if $n \geq m$. Therefore, $f_{k,k'}(x)$ and $f_{k,k'}(x_n)$ have coinciding expansions of the form

$$k' + k'(1-k') + \cdots + k'(1-k')^{t_0-2} + \cdots$$
$$+k'^{s_{d-1}+2}(1-k')^{t_{d-1}-1} + \cdots + k'^{s_{d-1}+2}(1-k')^{t_d-2} + k'^{s_d+2}(1-k')^{t_d-1} + \cdots$$

which implies that

$$\begin{aligned} |f_{k,k'}(x) - f_{k,k'}(x_n)| &\leq 2k'^{s_d+2}[(1-k')^{t_d-1} + (1-k')^{t_d} + \cdots] \\ &= 2k'^{s_d+1}(1-k')^{t_d-1} \end{aligned}$$

Now, if $n \to +\infty$ then $d, t_d, s_d \to +\infty$. Hence, $f_{k,k'}(x_n) \to f_{k,k'}(x)$.

(b′) x has finite expansion. In this case, we consider sequences (x_n) one-side converging to x.

If $x_n \searrow x$, since

$$x = k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + \cdots + k^{t_d}(1-k)^{s_{d-1}+1} + \cdots + k^{t_d}(1-k)^{s_d}$$

as we have seen above, the sequence can be given by

$$x_n = k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + \cdots k^{t_d}(1-k)^{s_{d-1}+1} + \cdots + k^{t_d}(1-k)^{s_d} + k^{t'_{d+1}}(1-k)^{s_d+1} + \cdots$$

where $t'_{d+j} = t'_{d+j}(n)$ for all j, and $t'_{d+1} \to +\infty$, if n does.

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Next, applying $f_{k,k'}$:

$$f_{k,k'}(x) = k' + k'(1-k') + \cdots + k'(1-k')^{t_0-2} + \cdots$$
$$+k'^{s_{d-1}+2}(1-k')^{t_{d-1}-1} + \cdots + k'^{s_{d-1}+2}(1-k')^{t_d-2} + k'^{s_d+1}(1-k')^{t_d-1}$$

and

$$f_{k,k'}(x_n) = k' + k'(1-k') + \cdots + k'(1-k')^{t_0-2} + \cdots$$
$$+k'^{s_{d-1}+2}(1-k')^{t_{d-1}-1} + \cdots + k'^{s_{d-1}+2}(1-k')^{t_d-2}$$
$$+k'^{s_d+2}(1-k')^{t_d-1} + \cdots + k'^{s_d+2}(1-k')^{t'_{d+1}-2} + \cdots$$

Finally, note that if $t'_{d+1} \to +\infty$, then

$$k'^{s_d+2}(1-k')^{t_d-1} + \cdots + k'^{s_d+2}(1-k')^{t'_{d+1}-2} + \cdots \to k'^{s_d+1}(1-k')^{t_d-1}$$

Then, it follows that $f_{k,k'}(x_n) \to f_{k,k'}(x)$.

If $x_n \nearrow x$ the proof is similar to (a′).

Whether $x_n \to x$, both cases (a′) and (b′) together give the convergence $f_{k,k'}(x_n) \to f_{k,k'}(x)$ as $n \to +\infty$; so that $f_{k,k'}$ is continuous on $\mathbb{I}$. □

The next lemma is useful in what follows.

Lemma 3.6 (*de Amo and Fernández-Sánchez [2, Theorem 13]*). *The set*

$$\left\{ x = \sum_{n=1}^{+\infty} (1-k)^n k^{m_n} : \exists \lim_{n\to\infty} \frac{m_n}{n} = \frac{k}{1-k} \right\}$$

is a set of λ-measure 1.

Definition 3.7. A point $x = \sum_{n=1}^{+\infty}(1-k)^n k^{m_n}$ is said to be normal in the generalized dyadic representation system (of k parameter) if $\lim_{n\to\infty} m_n/n = k/(1-k)$.

Let us now prove the singularity of these functions seeing that for all normal numbers where $f'_{k,k'}(x)$ exists, it has to vanish necessarily.

Theorem 3.8. *If $k \neq 1-k'$, then there exists a set of measure 1 in which the derivative of $f_{k,k'}$ vanishes.*

Proof. Let us consider x a normal number. Observe first that x can be represented uniquely by a non-stationary infinite expansion

$$x = k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + \cdots + k^{t_d}(1-k)^{s_{d-1}+1} + \cdots + k^{t_d}(1-k)^{s_d} + \cdots$$

Now, we consider two sequences (x_d) and (y_d):

$$x_d = k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + \cdots + k^{t_d}(1-k)^{s_{d-1}+1} + \cdots + k^{t_d}(1-k)^{s_d} \tag{2}$$

and

$$y_d = k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + \cdots + k^{t_d}(1-k)^{s_{d-1}+1} + \cdots + k^{t_d}(1-k)^{s_d} + k^{t_d}(1-k)^{s_d+1} \tag{3}$$

satisfying $x_d \le x \le y_d$, for all $d \in \mathbb{Z}^+$.

Applying $f_{k,k'}$:

$$f_{k,k'}(x_d) = k' + k'(1-k') + \cdots + k'(1-k')^{t_0-2} + \cdots$$
$$+k'^{s_{d-1}+2}(1-k')^{t_{d-1}-1} + \cdots + k'^{s_{d-1}+2}(1-k')^{t_d-2} + k'^{s_d+1}(1-k')^{t_d-1}$$

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and

$$f_{k,k'}(y_d) = k' + k'(1-k') + \cdots + k'(1-k')^{t_0-2} + \cdots$$
$$+k'^{s_{d-1}+2}(1-k')^{t_{d-1}-1} + \cdots + k'^{s_{d-1}+2}(1-k')^{t_d-2} + k'^{s_d+2}(1-k')^{t_d-1}$$

Next, using "big O and little o notation" (*Landau notation*), and taking into account that the relation $\lim_{d\to\infty} t_d/s_d = k/(1-k)$ is valid for normal numbers, the expression

$$\frac{f_{k,k'}(y_d) - f_{k,k'}(x_d)}{y_d - x_d}$$

can be written under the form

$$\frac{k'^{s_d+2}(1-k')^{t_d-1} - k'^{s_d+1}(1-k')^{t_d-1}}{k^{t_d}(1-k)^{s_d+1}}$$
$$= \frac{-k'^{s_d+1}(1-k')^{t_d}}{k^{t_d}(1-k)^{s_d+1}} = O\left(\left(\frac{-k'(1-k')^{k/(1-k)+o(1)}}{k^{k/(1-k)+o(1)}(1-k)}\right)^{s_d+1}\right)$$

Now, using the fact that $k'(1-k')^{k/(1-k)}/k^{k/(1-k)}(1-k)$ attains its maximum 1 whenever $k' = 1-k$, it follows that

$$\lim_{d\to\infty} \frac{f_{k,k'}(y_d) - f_{k,k'}(x_d)}{y_d - x_d} = 0$$

Therefore, derivative $f'_{k,k'}(x)$ vanishes for all x normal.

But function $f_{k,k'}$ is monotonically decreasing, whence it is differentiable almost everywhere. Accordingly, there exists a set A of the Lebesgue measure 1 such that $f'_{k,k'}(x)$ exists for all x in A (see [22, p. 5]). Hence, $f'_{k,k'}$ must vanish in a set of measure 1. □

Proposition 3.9. *If $k \neq 1-k'$, then $f_{k,k'}$ does not admit a non-zero derivative at any $x \in \mathbb{I}$.*

Proof. We distinguish the following two cases:

If x has a finite expansion:

$$x = k^{t_0} + \cdots + k^{t_d}(1-k)^{s_d}$$

we define the sequence

$$x_n^* = k^{t_0} + \cdots + k^{t_d}(1-k)^{s_d} + k^n(1-k)^{s_d+1}$$

For each n:

$$\frac{f_{k,k'}(x_n^*) - f_{k,k'}(x)}{x_n^* - x} = \frac{-k'^{s_d+1}(1-k')^n}{k^n(1-k)^{s_d+1}}$$
$$= -\left(\frac{k'}{1-k}\right)^{s_d+1}\left(\frac{1-k'}{k}\right)^n$$

and when $n \to +\infty$, this sequence tends to 0 or to $-\infty$, depending on the value of $(1-k')/k$. Hence, if the limit exists, it must be zero.

If x has a non-stationary infinite expansion, then let us consider sequences (x_d) and (y_d), as in the above theorem. In addition to

$$y'_d = k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + \cdots + k^{t_d}(1-k)^{s_{d-1}+1} + \cdots + k^{t_d}(1-k)^{s_d}$$
$$+k^{t_d}(1-k)^{s_d+1} + k^{t_d}(1-k)^{s_d+2}$$

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Applying $f_{k,k'}$, we have the following formulas:

$$\frac{f_{k,k'}(y_d)-f_{k,k'}(x_d)}{y_d-x_d}=\frac{-k'^{s_d+1}(1-k')^{t_d}}{k^{t_d}(1-k)^{s_d+1}}$$

and

$$\frac{f_{k,k'}(y'_d)-f_{k,k'}(x_d)}{y'_d-x_d}=\frac{1+k'}{2-k}\frac{-k'^{s_d+1}(1-k')^{t_d}}{k^{t_d}(1-k)^{s_d+1}}$$

Finally, if the derivative exists, the quotient of the limits of the above formulas $(1+k')/(2-k)$ must be equal to 1; but this is not the case when $k \neq 1-k'$. Consequently, if the derivative exists, it must be equal to zero. □

Theorem 3.10. *$f_{k,k'}$ is the unique bounded solution of the system of functional equations*

$$\begin{cases} f(kx)=k'+(1-k')f(x) \\ f(k+(1-k)x)=k'f(x) \end{cases}$$

Proof. First, we show that $f_{k,k'}$ satisfies this system of functional equations. Let us consider the case when x has a non-stationary infinite expansion. The finite case is similar. Let us consider

$$x=k^{t_0}+\cdots+k^{t_0}(1-k)^{s_0}+k^{t_1}(1-k)^{s_0+1}+\cdots+k^{t_d}(1-k)^{s_d}+\cdots$$

then

$$\begin{aligned} f_{k,k'}(x)=&\,k'+k'(1-k')+\cdots+k'(1-k')^{t_0-2} \\ &+k'^{s_0+2}(1-k')^{t_0-1}+\cdots+k'^{s_0+2}(1-k')^{t_1-2} \\ &+k'^{s_1+2}(1-k')^{t_1-1}+\cdots+k'^{s_1+2}(1-k')^{t_2-2}+\cdots \end{aligned}$$

For these formulas, writing kx and $f_{k,k'}(kx)$, the first equation follows by direct verification.

For the second equation the reasoning is as follows: if $t_0 \neq 1$, then

$$k+(1-k)x=k+k^{t_0}(1-k)+\cdots+k^{t_0}(1-k)^{s_0+1}+k^{t_1}(1-k)^{s_0+2}+\cdots+k^{t_1}(1-k)^{s_1+1}+\cdots$$

and

$$\begin{aligned} f_{k,k'}(k+(1-k)x)=&\,k'^2+k'^2(1-k')+\cdots+k'^2(1-k')^{t_0-2} \\ &+k'^{s_0+3}(1-k')^{t_0-1}+\cdots+k'^{s_0+3}(1-k')^{t_1-2}+\cdots \end{aligned}$$

When $t_0=1$,

$$f_{k,k'}(x)=k'^{s_0+2}+\cdots+k'^{s_0+2}(1-k')^{t_1-2}+\cdots$$

and

$$f_{k,k'}(k+(1-k)x)=k'^{s_0+3}+\cdots+k'^{s_0+3}(1-k')^{t_1-2}+\cdots$$

For both cases, the second of the equations is fulfilled.

Now, let us prove the uniqueness of $f_{k,k'}$. Let us consider $\mathcal{B}(\mathbb{I})$ the Banach space of real bounded functions defined on $\mathbb{I}$ endowed with the sup-norm, and the functional operator introduced by the formula

$$F:\mathcal{B}(\mathbb{I})\longrightarrow\mathcal{B}(\mathbb{I}),\quad g\to F(g)$$

where $F(g)$ acts in the following way:

$$F(g)(y):=\begin{cases} k'+(1-k')g\left(\frac{y}{k}\right)\ldots, & 0\leq y\leq k \\ k'g\left(\frac{y-k}{1-k}\right)\ldots, & k\leq y\leq 1 \end{cases}$$

Hence, F is a contraction of ratio $b := \max\{k', 1-k'\}$. In fact, if we denote $h_i := F(g_i)$ for given $g_i \in \mathcal{B}(\mathbb{I})$, $i = 1, 2$, then, an easy computation shows that

$$h_2(y) - h_1(y) = \begin{cases} (1-k')\left[g_2\left(\frac{y}{k}\right) - g_1\left(\frac{y}{k}\right)\right] \ldots, & 0 \le y \le k \\ k'\left[g_2\left(\frac{y-k}{1-k}\right) - g_1\left(\frac{y-k}{1-k}\right)\right] \ldots, & k \le y \le 1 \end{cases}$$

and we conclude that $\|h_2 - h_1\| \le b\|g_2 - g_1\|$.

Finally, the contraction mapping theorem of the Banach ensures the existence of one (and only one) fixed point for F. Hence, we have the uniqueness of $f_{k,k'}$, and the proof is finished. □

Corollary 3.11. *$N_{k,k'}$ and $f_{k,k'}$ coincide.*

Corollary 3.12. *$N_{k',k}$ is the inverse function of $N_{k,k'}$.*

The following result is a Frostman-type lemma, which is useful for our purposes (see [7, pp. 60–61]).

Lemma 3.13. *Let F be a Borel set in $\mathbb{R}$, and let μ be a finite measure on F. Let $B_r(x)$ denote the ball with center x and radius r. Under these assumptions,*

(i) *If the upper limit* $\limsup \mu(B_r(x))/r^s$ *is bounded on F, then* $\dim_{\mathcal{H}} F \le s$.
(ii) *If there is a positive real c such that* $\liminf \mu(B_r(x))/r^s > c > 0$ *on F, then* $\dim_{\mathcal{H}} F \ge s$.

We shall now use cylinders (of the generalized dyadic system in [2]) instead of balls.

We need a refinement of Lemma 3.6. Its proof follows in a similar way to that given in [2, Theorem 13].

Lemma 3.14. *The set*

$$\left\{x = k^{t_0} + \cdots + k^{t_d}(1-k)^{s_d} + \cdots : \exists \lim_{d\to\infty} \frac{t_d}{s_d} = \frac{k}{1-k} \text{ and } \lim_{d\to\infty} \frac{t_d}{t_{d-1}} = 1\right\}$$

is a set of λ-measure 1.

Theorem 3.15. *If $k' \ne 1-k$, then the function $N_{k,k'}$ applies a set of λ-measure 0 onto a set of λ-measure 1. The Hausdorff dimension of the first set is* $\ln[k'^{k'}(1-k')^{1-k'}]/\ln[k^{1-k'}(1-k)^{k'}]$.

Proof. The set $\{x = k^{t_0} + \cdots + k^{t_d}(1-k)^{s_d} + \cdots : \exists \lim_{d\to\infty} t_d/s_d = (1-k')/k'\}$ is λ-null (by applying Lemma 3.6), therefore

$$\left\{x = k^{t_0} + \cdots + k^{t_d}(1-k)^{s_d} + \cdots : \exists \lim_{d\to\infty} \frac{t_d}{s_d} = \frac{1-k'}{k'} \text{ and } \lim_{d\to\infty} \frac{s_d}{s_{d-1}} = 1\right\}$$

is also λ-null. This set is applied by $N_{k,k'}$ onto

$$\left\{x = k'^{t_0} + \cdots + k'^{t_d}(1-k')^{s_d} + \cdots : \exists \lim_{d\to\infty} \frac{t_d}{s_d} = \frac{k'}{1-k'} \text{ and } \lim_{d\to\infty} \frac{t_d}{t_{d-1}} = 1\right\}$$

whose λ-measure is 1. To obtain the Hausdorff dimension of the former set, note that the explicit expression of $N_{k,k'}$ leads to the fact that the interval $[x_d, y_d]$, with x_d and y_d given by (2) and (3), is mapped onto another interval of length $(1-k')^{t_d}(k')^{s_d}$. Specifically, it is the interval

$$[N_{k,k'}(y_d), N_{k,k'}(x_d)]$$

since $N_{k,k'}$ is a bijective decreasing function.

Therefore, the Hausdorff dimension is given by the number

$$\sup\left\{\beta > 0 : \frac{(1-k')^{t_d}(k')^{s_d+1}}{[k^{t_d}(1-k)^{s_d+1}]^\beta} < +\infty\right\}$$

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Taking logarithms

$$\lim_{d\to\infty} \frac{\ln(1-k')^{t_d} + \ln(k')^{s_d+1} - \ln k^{\beta t_d} - \ln(1-k)^{\beta(s_d+1)}}{s_d} s_d < +\infty$$

and since $t_d/s_d \to (1-k')/k'$ as $d \to +\infty$, this gives $\beta = \ln[k'^{k'}(1-k')^{1-k'}]/\ln[k^{1-k'}(1-k)^{k'}]$, which is the desired Hausdorff dimension. □

Because $N_{k,k'}$ is the inverse function of $N_{k',k}$, we have the following result:

Theorem 3.16. *If $k' \neq 1-k$, then $N_{k,k'}$ applies a set of λ-measure 1 onto a set of λ-measure 0 whose Hausdorff dimension is* $\ln[k^k(1-k)^{1-k}]/\ln[k'^{1-k}(1-k')^k]$.

4. k-Negations

The class of strictly continuous Archimedean t-norms (and t-conorms) is of particular interest because its elements can be represented by means of a single function t. We call them the *additive generators*. It can be shown that T is a strictly continuous Archimedean t-norm if and only if there is a continuous and strictly decreasing function t such that

$$T(x,y) = t^{[-1]}(t(x)+t(y))$$

with $t : \mathbb{I} \to [0,+\infty]$, satisfying $t(1)=0$, where $t^{[-1]}$ is the pseudo-inverse of t, that is, it is defined by $t^{[-1]}(x) = t^{-1}(x)$ if $x \in [0, t(0)]$ and $t^{[-1]}(x) = 0$ if $x \in]t(0), +\infty]$. An equivalent theorem exists for t-conorms (see for example [1]).

The *associated negation* to T is the function $N_T : \mathbb{I} \to \mathbb{I}$ given by

$$N_T(x) = t^{[-1]}(t(0)-t(x))$$

and the t-conorm $T^*(x,y) = N_T(T(N_T(x), N_T(y)))$ is called the *dual t-conorm* of T.

Clearly, $F = (1-k)T + kT^* \in \Phi$. Moreover, F is idempotent (i.e. $F(x,x) = x, \forall x \in \mathbb{I}$) if and only if the normalized additive generator ($t(0)=1$) is $t = 1 - R_k^{-1}$. Therefore, N_T is among the $N_{k,k'}$ functions. Actually, with $k' = 1-k$, they are the so-called k-negations, and they are noted by N_k.

Function N_k is characterized as the unique bounded function in $\mathbb{I}$ satisfying the system of functional equations:

$$\begin{cases} g(kx) = k + (1-k)g(x) \\ g^2(x) = x \end{cases}$$

For more details on negations and k-negations, see [1,8,19].

The results in Section 3 are directly applied to the k-negation N_k. An explicit expression of this function can be seen below:

Theorem 4.1. *For each $k \in]0,1[$, let us consider the k-negation function $N_k : \mathbb{I} \to \mathbb{I}$. Then, N_k is given by*

(i) *If x has a non-stationary infinite expansion, i.e. there exist $1 \leq t_0 < t_1 < \cdots < t_d < \cdots$, such that*

$$x = k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + k^{t_1}(1-k)^{s_0+1} + \cdots + k^{t_1}(1-k)^{s_1} + \cdots$$
$$+ k^{t_d}(1-k)^{s_{d-1}+1} + \cdots + k^{t_d}(1-k)^{s_d} + \cdots$$

then

$$N_k(x) = k + k(1-k) + \cdots + k(1-k)^{t_0-2} + k^{s_0+2}(1-k)^{t_0-1} + \cdots + k^{s_0+2}(1-k)^{t_1-2}$$
$$+ k^{s_1+2}(1-k)^{t_1-1} + \cdots + k^{s_1+2}(1-k)^{t_2-2} + \cdots$$
$$+ k^{s_{d-1}+2}(1-k)^{t_{d-1}-1} + \cdots + k^{s_{d-1}+2}(1-k)^{t_d-2} + \cdots$$

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If $t_0 = 1$, *then*

$$\begin{aligned}N_k(x) = &k^{s_0+2} + k^{s_0+2}(1-k) + \cdots + k^{s_0+2}(1-k)^{t_1-2}\\ &+k^{s_1+2}(1-k)^{t_1-1} + \cdots + k^{s_1+2}(1-k)^{t_2-2} + \cdots\\ &+k^{s_{d-1}+2}(1-k)^{t_{d-1}-1} + \cdots + k^{s_{d-1}+2}(1-k)^{t_d-2} + \cdots\end{aligned}$$

(ii) *In the stationary case, that is, when x has a finite expansion*

$$x = k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + \cdots + k^{t_d}(1-k)^{s_{d-1}+1} + \cdots + k^{t_d}(1-k)^{s_d}$$

then

$$\begin{aligned}N_k(x) = &k + k(1-k) + \cdots + k(1-k)^{t_0-2} + \cdots\\ &+k^{s_{d-1}+2}(1-k)^{t_{d-1}-1} + \cdots + k^{s_{d-1}+2}(1-k)^{t_d-2} + k^{s_d+1}(1-k)^{t_d-1}\end{aligned}$$

By applying the above theorem, we have the following properties for the k-negation N_k.

Theorem 4.2. *For each $k \in]0,1[$, let us consider the k-negation function $N_k : \mathbb{I} \to \mathbb{I}$ (under the above expression). Then,*

(i) *N_k is continuous.*
(ii) *For each $k \in]0,1[\backslash 1/2$, there is a set of λ-measure 1 in which N_k vanishes.*
(iii) *For each $k \in]0,1[\backslash 1/2$, N_k does not admit non-zero derivatives.*
(iv) *N_k is the unique solution for the system of functional equations given by*

$$\begin{cases} f(kx) = k + (1-k)f(x) \\ f(k + (1-k)x) = kf(x) \end{cases}$$

(v) *N_k maps a set of λ-measure 0 with Hausdorff dimension* $\ln[k^k(1-k)^{1-k}]/\ln[k^{1-k}(1-k)^k]$, *onto a set of λ-measure 1.*
(vi) *N_k maps a set of λ-measure 1 onto a set of λ-measure 0 with the Hausdorff dimension* $\ln[k^k(1-k)^{1-k}]/\ln[k^{1-k}(1-k)^k]$.

5. Conclusions

We have given several new results in the theory of aggregation operators completing previous studies on this domain. From a new representation system for the real numbers in $]0, 1]$, that generalizes the dyadic one, we obtain an explicit expression of duality functions for a class of binary operations on the unit interval $\mathbb{I}$. Observe that our method of representation can be used to prove singularity for duality functions and to analyze fractal dimensions for some distinguished sets. The particular case of k-negations is specially studied.

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