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The Hausdorff dimension of the level sets of Takagi's function

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ABSTRACT

Recently, Maddock (2006) [12] has conjectured that the Hausdorff dimension of each level set of Takagi's function is at most 1/2. We prove this conjecture using the self-affinity of the function of Takagi and the existing relationship between the Hausdorff and box-counting dimensions.

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1. Introduction

The existence of continuous nowhere differentiable functions was an open problem during part of the 19th century. A great number of mathematicians thought that continuous functions had derivatives on a considerable set of points on which they were defined. (For example, Ampère believed that he had proved this.)

However, three great mathematicians, independently, found a negative answer for this question, showing that there exist explicit examples of continuous functions that have no derivative at any point. They are: Bolzano (1830, not published until 1922—see [1]); Cellérier (1860, approximately, not published until 1890—see [2]); and Weierstrass, who produced his remarkable function

$$W(x) := \sum_{k=0}^{+\infty} a^k \cos(b^k \pi x), \quad 0 < a < 1, \; ab > 1 + \frac{3}{2}\pi, \; b + 1 \in 2\mathbb{Z},$$

published it in 1875 (see [3]).

Later, in 1903, Takagi [4] gave an extraordinarily easy example of a continuous nowhere differentiable function on the unit interval, as follows:

$$T(x) := \sum_{k=0}^{+\infty} \frac{d(2^k x)}{2^k}, \quad \forall x \in \mathbb{R}, \tag{1}$$

where $d(x)$ denotes the distance from each real number x to the nearest integer. The graph of T in $[0, 1]$ is shown in Fig. 1.

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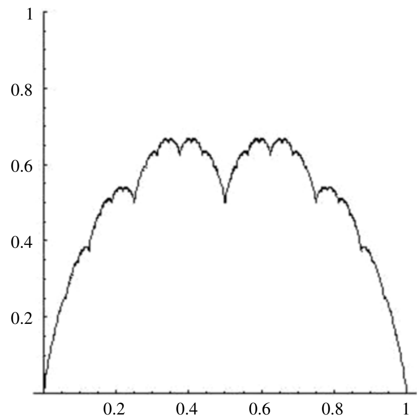

Fig. 1. Takagi function.

Due to the non-differentiability of T, its values fluctuate greatly even on the smallest intervals, and properties of the graph and the level sets of this function, i.e. $L_y = \{x : T(x) = y\}$, have been the focus of some study. The function T and its properties have been studied by various authors. Among these properties, we should note two well-known facts. The first is that

$$T(x + y) - T(x) = O(|y| \ln |y|), \quad \text{if } y \to 0$$

(see [5]), and the second is that its graph has a Hausdorff dimension of 1 (see [6] or [7]).

Kahane [8], in 1959, established that the maximum value of T is $2/3$, and he described the points where this extreme is attained; specifically, they are the points x in $[0, 1]$ such that the binary expansion $x = \sum_{k=0}^{+\infty} \frac{x_k}{2^k}$ satisfies $x_{2k+1} + x_{2k+2} = 1$ for $k = 0, 1, 2, \ldots$. This condition is equivalent to having only digits 1 and/or 2 in their base-4 expansion (see [9]). Therefore, the set where T takes its maximum value is a self-similar Cantor-type set.

Self-similarity allows us to obtain the Hausdorff and box-counting dimensions of the level set where the Takagi function reaches its maximum. This number is exactly equal to $1/2$ (see also [9,10]).

The above result can be stated, in other words, as the Hausdorff dimension of $L_{2/3}$ of T being equal to $1/2$ (see [9]). Some special cases of level sets for T are studied in [11,12]. Maddock [12] conjectured that the Hausdorff dimension for the level set $L_{2/3}$ of the Takagi function is a maximum; that is,

$$\dim_{\mathcal{H}}(L_{2/3}) \le 1/2$$

for all $y \in [0, 1]$. In a recent paper by Maddock himself [10], he establishes the number $\alpha = 0.668$ as an upper bound for the Hausdorff dimensions of the intersection of the graph of T with any line with integer slope. The interest in the study of the level sets of the Takagi function has continued in several papers (see for instance [13–15]). In [13], the author states that any level set containing only countably many local level sets has Hausdorff dimension at most $1/2$. But as Theorem 4.7(ii) in [13] shows, this still leaves out uncountably many level sets. Besides, the same author says that "it may be very difficult to prove Maddock's conjecture that the Hausdorff dimension of each level set is at most $1/2$".

Our paper is aimed at proving the conjecture posed by Maddock. The technique that we use is based on the self-affinity of the function T.

The contents of the paper are as follows. Section 2 is devoted to preliminaries, and we give a useful self-affinity property that allows us to work at different scales. The self-affinity given by the functional equations describing T will be used in depth, both in Section 2 (Lemmas 2.4 and 2.5) and in Section 3 (Lemmas 3.1 and 3.2).

Section 3 contains the main result of this paper; that is, all the level sets of Takagi's function have a Hausdorff dimension not greater than $1/2$ (Theorem 3.4). Therefore, Maddock's conjecture is answered affirmatively.

In Section 4, we end with some conclusions and an open question.

2. Preliminaries

Definition 2.1. Let A be a subset of $\mathbb{R}^n$, and $0 \le s \le n$, $\delta > 0$. We define the s-dimensional Hausdorff measure of A by

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \left\{ \inf \sum_i |U_i|^s \right\},$$

the infimum being taken over all countable covers $\{U_i\}$ of A by sets in $\mathbb{R}^n$ with $0 < |U_i| \le \delta$, where we write

$$|U| = \sup\{\|x - y\| : x, y \in U\}$$

for the diameter of the set U.

The Hausdorff dimension $\dim_{\mathcal{H}}(A)$ is the parameter s_0 such that $\mathcal{H}^s(A) = \infty$ for $s < s_0$, and $\mathcal{H}^s(A) = 0$ for $s > s_0$.

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Definition 2.2. Let A be a bounded subset of $\mathbb{R}^n$. For each $\delta > 0$ we denote by $N_\delta(A)$ the minimum number of sets of diameter less than δ that are needed to cover the set A. The lower and upper box-counting dimensions are defined as $\underline{\dim}_B(A) = \underline{\lim}_{\delta\to 0} \frac{\log N_\delta(A)}{-\log\delta}$ and $\overline{\dim}_B(A) = \overline{\lim}_{\delta\to 0} \frac{\log N_\delta(A)}{-\log\delta}$. If these real numbers are equal we call the common value the box-counting or Minkowski dimension.

Now, there is one result that we shall require, which can be found in [16, Prop. 4.1].

Proposition 2.3. *Suppose that $A \subset \mathbb{R}^n$ can be covered by n_k sets of diameter at most δ_k, with $\delta_k \to 0$ as $k \to \infty$; then*

$$\dim_{\mathcal{H}}(A) \le \underline{\dim}_B(A) \le \overline{\dim}_B(A) \le \underline{\lim}_{k\to\infty} \frac{\log n_k}{-\log \delta_k}.$$

One of the main properties of the Takagi function is self-affinity. We remark that the Takagi function has a self-affine structure described by two functional equations (see [11, Sec. 7.2]). For our purposes, we use the self-affinity of T in the form that the next lemma describes. Its proof follows from (1), and can be found in [17, Prop. 2.1].

Let r_m denote the number of ones in the binary representation of m : $r_m = \sum_{i=1}^{\infty} \epsilon_i$, where $m = \sum_{i=1}^{\infty} 2^i \epsilon_i$ with $\epsilon_i \in \{0, 1\}$. Then, let $p_{k,m} = k - 2r_m$.

Lemma 2.4. *For given $k \in \mathbb{Z}^+$, $0 \le m \le 2^k - 1$, and $x \in [0, 1]$, the Takagi function satisfies the relation*

$$T\left(\frac{m}{2^k} + \frac{x}{2^k}\right) = p_{k,m}\frac{x}{2^k} + T\left(\frac{m}{2^k}\right) + \frac{T(x)}{2^k}.$$

Next, we can state the following results.

Lemma 2.5. *The Takagi function satisfies the relations that follow for $s \in \mathbb{Z}^+ \cup \{0\}$:*

$$T\left(\frac{j}{2^{2s+2}}\right) = \frac{2js+2}{2^{2s+2}}, \quad \text{for } j = 1, 2, 3, \tag{2}$$

$$\max\left\{T(x) : x \in \left[0, \frac{1}{2^{2s+2}}\right]\right\} = \frac{2s+2}{2^{2s+2}} + \frac{1}{3}\frac{2}{4^{2s+2}}, \tag{3}$$

and

$$\max\left\{T(x) : x \in \left[\frac{1}{2^{2s+2}}, \frac{2}{2^{2s+2}}\right]\right\} = \frac{4s+2}{2^{2s+2}} + \frac{1}{3}\frac{2}{4^{2s+1}}. \tag{4}$$

Proof. Let us calculate $T(\frac{1}{2^k})$. Set $m = 0$ and $x = 1$ in Lemma 2.4. In this case, $r = 0$ and $p = k$, which implies that $T(\frac{1}{2^k}) = \frac{k}{2^k}$. For the particular cases of the points $\frac{1}{2^{2s+2}}$ and $\frac{1}{2^{2s+1}}$, the above reasoning gives $T(\frac{1}{2^{2s+2}}) = \frac{2s+2}{2^{2s+2}}$ and $T(\frac{2}{2^{2s+2}}) = \frac{4s+2}{2^{2s+2}}$ respectively.

In the case of $T(\frac{3}{2^{2s+2}})$, we use $k = 2s + 2, m = 2, x = 1$ and the result obtained for $T(\frac{2}{2^{2s+2}})$. Consequently, $T(\frac{3}{2^{2s+2}}) = \frac{6s+2}{2^{2s+2}}$.

The bounds (3) and (4) are particular cases of Proposition 4.3 in [17]. □

Lemma 2.6. (a) *The number $p_{2k,m}$ is an even integer.*

(b) *If $p_{2k,m} = 0$, and m is even, then we have*

$$(p_{2k,m}, p_{2k,m+1}) = (0, -2),$$

and if $m =$ is odd, then we have

$$(p_{2k,m-1}, p_{2k,m}) = (2, 0).$$

Proof. (a) It is immediate because $p_{2k,m} = 2k - 2r_m$.

(b) Four cases have to be distinguished: m is $4n$, $4n + 1$, $4n + 2$ or $4n + 3$.

Let us recall that $p_{2k,4n} = 2k - 2r_{4n}$. In this case, the dyadic expansion for $4n$ has r_{4n} ones. Then,

$$\begin{cases} r_{4n+1} = r_{4n} + 1 \\ r_{4n+2} = r_{4n} + 1 \\ r_{4n+3} = r_{4n} + 2. \end{cases}$$

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Therefore, we get

$$\begin{cases} p_{2k,4n} = 2k - 2r_{4n} \\ p_{2k,4n+1} = 2(k-1) - 2r_{4n} = p_{2k,4n} - 2 \\ p_{2k,4n+2} = 2(k-1) - 2r_{4n} = p_{2k,4n} - 2 \\ p_{2k,4n+3} = 2(k-2) - 2r_{4n} = p_{2k,4n} - 4. \end{cases}$$

From this, part (b) of the lemma follows easily. □

Alternatively, we could have obtained a proof for this result using the expression given in (1).

3. The main result

Let $\mathcal{F}_k(y)$ denote the collection of intervals $J = [\frac{j}{2^{2k}}, \frac{j+1}{2^{2k}}]$, such that $T(J)$ intersects the interval $[\frac{s}{2^{2k-1}}, \frac{s+1}{2^{2k-1}}[$, with $s \in \mathbb{Z}$, which contains y.

For an arbitrary interval $I \subset [0, 1]$, let $N_k(I, y)$ denote the number of intervals in $\mathcal{F}_k(y)$ which intersect I. If $I = [0, 1]$, we use the notation $N_k(y)$.

We introduce some technical results.

Lemma 3.1. *If $J = [\frac{j}{2^{2k}}, \frac{j+1}{2^{2k}}]$ and $p_{2k,j} \neq 0$, then*

$$N_{k+1}(J, y) \leq 2N_k(J, y). \quad (5)$$

Proof. Let us suppose that $p_{2k,j} = 2s > 0$. If $N_k(J, y) = 0$, then $N_{k+1}(J, y) = 0$ as well and inequality (5) is trivially satisfied. We study the case $N_k(J, y) = 1$. The self-affinity of T gives that we can restrict our study to the interval $[0, 1/2^{2s}]$. If we set $h : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$h(x, y) = \left(\frac{2^{2k}x - j}{2^{2s}}, 2^{2k-2s}\left(y - T\left(\frac{j}{2^{2k}}\right)\right)\right),$$

then, the part of the graph of T corresponding to J, is mapped onto the corresponding piece for $J^* = [0, 1/2^{2s}]$.

Let us denote as y^* the second coordinate in $h(0, y)$. We have that

$$N_k(J, y) = N_s(J^*, y^*), \qquad N_{k+1}(J, y) = N_{s+1}(J^*, y^*).$$

We can consider four possibilities. For each one, the values in (2) and the bounds in (3) and (4), allow us to obtain:

(i) If $y^* \in [0, \frac{2s+2}{2^{2s+2}}[$, then $N_{s+1}(J^*, y^*) = 1$.
(ii) If $y^* \in [\frac{2s+2}{2^{2s+2}}, \frac{4s+2}{2^{2s+2}}[$, then $N_{s+1}(J^*, y^*) = 2$.
(iii) If $y^* \in [\frac{4s+2}{2^{2s+2}}, \frac{6s+2}{2^{2s+2}}[$, then $N_{s+1}(J^*, y^*) = 2$.
(iv) If $y^* \geq \frac{6s+2}{2^{2s+2}}$, then $N_{s+1}(J^*, y^*) = 2$.

In summary, (5) is true in all cases.

The case $p_{2k,j} = 2s < 0$ proceeds as above by symmetry. □

The pictures in Fig. 2 illustrate the reasoning in the above lemma when $s = 1$. The case (i) corresponds to the top left picture. In this picture, the grey rectangle corresponds to the interval with (non-empty) intersection. The top right picture is the case (ii). The cases (iii) and (iv) are drawn in the other two pictures.

Lemma 3.2. *If $J = [\frac{j}{2^{2k}}, \frac{j+1}{2^{2k}}]$ and $p_{2k,j} = 0$, then*

$$N_{k+1}(J', y) \leq 2N_k(J', y), \quad (6)$$

where

$$J' = \begin{cases} \left[\frac{j-1}{2^{2k}}, \frac{j+1}{2^{2k}}\right], & \text{if } j \text{ is odd,} \\ \left[\frac{j}{2^{2k}}, \frac{j+2}{2^{2k}}\right], & \text{if } j \text{ is even.} \end{cases} \quad (7)$$

Proof. If $p_{2k,j} = 0$ and j is odd, then, by virtue of Lemma 2.6, we have $p_{2k,j-1} = 2$ in the interval $[\frac{j-1}{2^{2k}}, \frac{j}{2^{2k}}]$. The self-affinity of T means that the study can be reduced to the case $(p_{2,0}, p_{2,1}) = (2, 0)$, where the intervals considered are $[0, 1/4]$ and $[1/4, 1/2]$. Once again, we denote by y^* the second coordinate in the image of the point $(0, y)$ via the similarity

$$h(x, y) = \left(\frac{2^{2k}x - j + 1}{2^2}, 2^{2k-2}\left(y - T\left(\frac{j-1}{2^{2k}}\right)\right)\right).$$

Set $J^* = [0, 1/2]$; then $N_k(J', y) = N_1(J^*, y^*)$, $N_{k+1}(J', y) = N_2(J^*, y^*)$.

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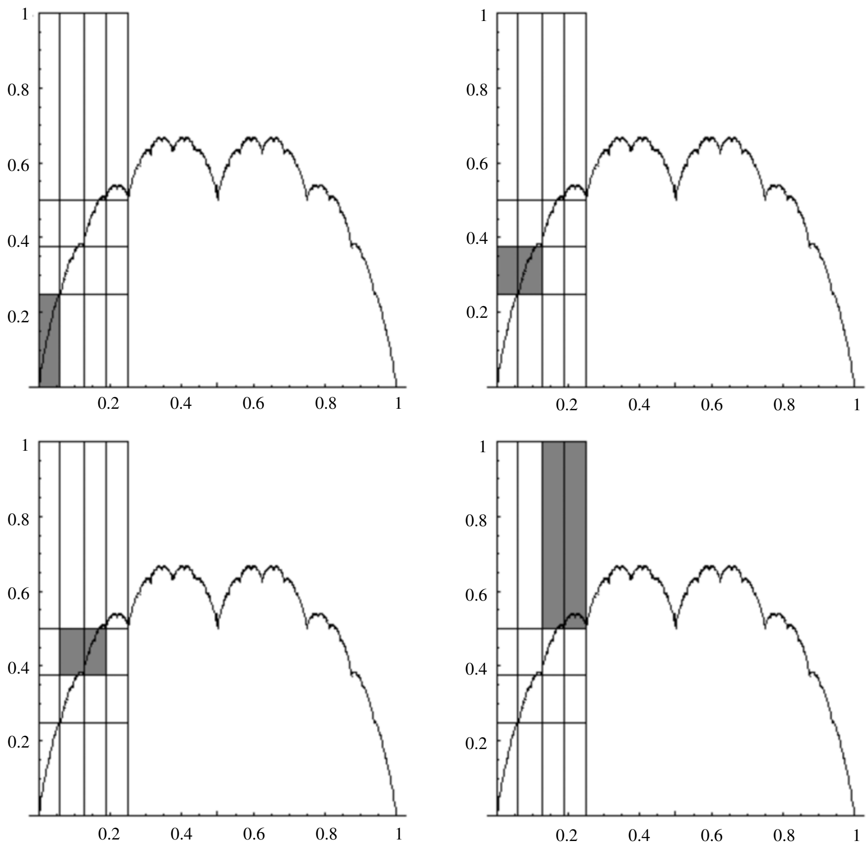

Fig. 2. Case $p_{2k,j} = 2$.

Six possibilities exist in this case:

(i) If $y^* \in [5/8, 3/4[$, then $N_1(J^*, y^*) = 2$ and $N_2(J^*, y^*) = 4$.
(ii) If $y^* \in [1/2, 5/8[$, then $N_1(J^*, y^*) = 2$ and $N_2(J^*, y^*) = 4$.
(iii) If $y^* \in [3/8, 1/2[$, then $N_1(J^*, y^*) = 1$ and $N_2(J^*, y^*) = 2$.
(iv) If $y^* \in [1/4, 3/8[$, then $N_1(J^*, y^*) = 1$ and $N_2(J^*, y^*) = 2$.
(v) If $y^* \in [1/8, 1/4[$, then $N_1(J^*, y^*) = 1$ and $N_2(J^*, y^*) = 1$.
(vi) If $y^* \in [0, 1/8[$, then $N_1(J^*, y^*) = 1$ and $N_2(J^*, y^*) = 1$.

Therefore, the inequality (6) is fulfilled.

The symmetry of T with respect to $x = 1/2$ allows us to deduce the statement where $p_{2k,j} = 0$ and j is even. □

The reasoning in the above lemma for the two first cases rests on the geometric ideas shown in the pictures in Fig. 3.

The numbers $N_k(y)$ enjoy the following remarkable property for our purposes.

Proposition 3.3. *If $k \in \mathbb{Z}^+$ and $y \in [0, 1]$, then*

$$N_{k+1}(y) \leq 2N_k(y). \tag{8}$$

Proof. Each interval $J = [\frac{j}{2^{2k}}, \frac{j+1}{2^{2k}}]$ either satisfies $N_{k+1}(J, y) \leq 2N_k(J, y)$ itself or is part of an interval J' given by (7) satisfying $N_{k+1}(J', y) \leq 2N_k(J', y)$. From this, we deduce inequality (8). □

We now proceed with the proof of the conjecture by Maddock.

Theorem 3.4. *The Hausdorff and box-counting dimensions of the level sets L_y of Takagi's function are, at most,* $1/2$.

Proof. First, reasoning by induction, and using inequality (8), if $k \in \mathbb{Z}^+$ and $y \in [0, 1]$, then we have $N_k(y) \leq 2^{k+1}$.

Using this bound and the upper box-counting definition,

$$\overline{\dim}_B L_y \leq \overline{\lim_k} \frac{\log_2 N_k(y)}{2k} \leq \overline{\lim_k} \frac{k+1}{2k} = 1/2.$$

The above bound and Proposition 2.3 allow us to write

$$\dim_{\mathcal{H}} L_y \leq 1/2. \quad \square$$

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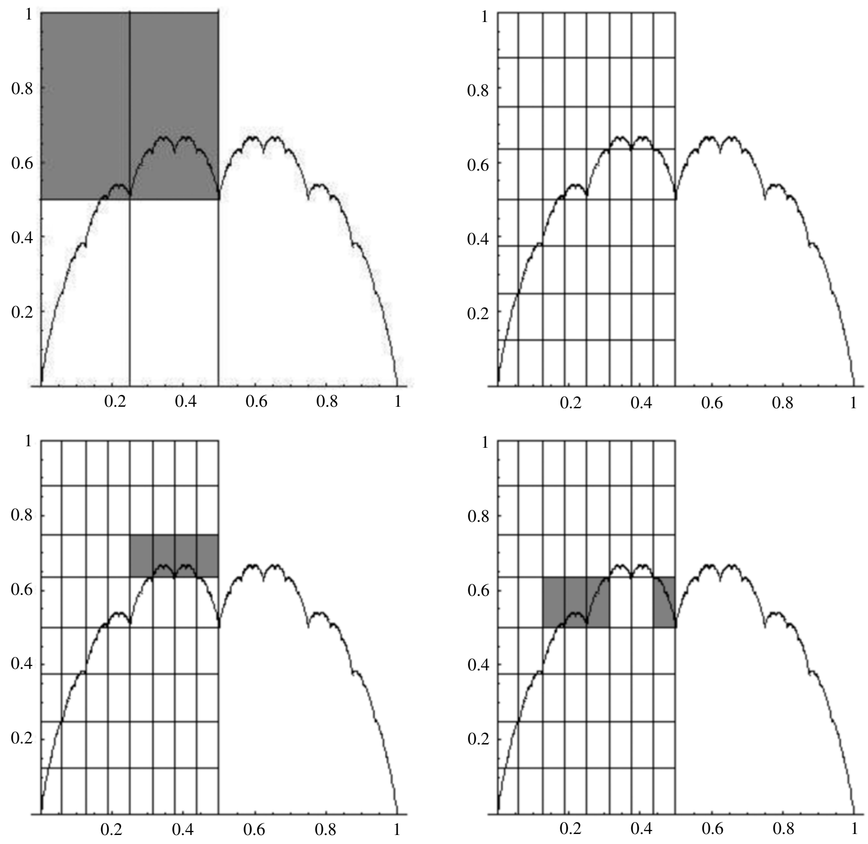

Fig. 3. Case $p_{2k,j} = 0$.

4. Conclusions and open question

1. If $y \in [0, 1]$, then $\dim_{\mathcal{H}}(L_y) \in [0, 1/2]$.
2. For $x \in [0, 1]$,

$$T(x) = 2/3 \iff x = \sum_{n=0}^{+\infty} \frac{x_n}{4^n}, \quad \text{with } x_n \in \{1, 2\},$$

   and, hence, $\dim_{\mathcal{H}} L_y = 1/2$, if $y = 2/3$.
   The reverse is false: if y is a point that is a relative (that is, local) maximum for T, then $\dim_{\mathcal{H}} L_y = 1/2$, too. Therefore, there is a dense set of points satisfying this. Moreover, according to Proposition 4.5 in [13], the level sets for dyadic rational numbers have null Hausdorff dimension. Therefore, there exists a dense set of points satisfying $\dim_{\mathcal{H}} L_y = 0$.
3. Question: for each $0 < \alpha < 1/2$, is there a corresponding level set A such that $\dim_{\mathcal{H}}(A) = \alpha$?

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References

[1] V. Jarník, On Bolzano's functions, Časopis Pěst. Mat. 51 (1922) 248–266.
[2] M.Ch. Cellérier, Note sur les principies fondamentaux de l'analyse, Darboux Bull. 14 (1980) 142–160.
[3] P. du Bois-Reymond, Versuch einer Classification der willkürlichen Functionen reeller Argumente nach ihren Aenderungen in des kleisten Intervallen, J. Reine Angew Math. 79 (1875) 21–37.
[4] T. Takagi, A simple example of the continuous function without derivative, Proc. Phys. Soc. Tokyo. Ser. II 1 (1903) 176–177.
[5] M. Hata, Topological aspects of selfsimilar sets and singular functions, in: J. Bélais, S. Dubuc (Eds.), Fractal Geometry and Analysis, Kluwer Acad. Publ., 1991, pp. 255–276.
[6] P.C. Allaart, K. Kawamura, Extreme values of some continuous nowhere differentiable functions, Math. Proc. Cambridge Philos. Soc. 140 (2) (2006) 269–295.
[7] R.D. Mauldin, S.C. Williams, On the Hausdorff dimension of some graphs, Trans. Amer. Math. Soc. 298 (2) (1986) 793–803.
[8] J.-P. Kahane, Sur l'exemple, donné par M. de Rham, d'une fonction continue sans dérivée, Enseig. Math. 5 (1959) 53–57.
[9] Y. Baba, On maxima of Takagi–van der Waerden functions, Proc. Amer. Math. Soc. 91 (1984) 373–376.
[10] Z. Maddock, Level sets of the Takagi function: Hausdorff dimension, Monatsh. Math. 160 (2) (2010) 167–186.

Author's personal copy

[11] D.E. Knuth, The Art of Computer Programming, Vol. 4, Pearson Education, Upper Saddle River, 2005, fasc. 3.
[12] Z. Maddock, Properties of the Takagi function, internet-preprint, 2006.
[13] P.C. Allaart, How large are the level sets of the Takagi function? 8 Feb 2011. arXiv:1102.1616v1 [math.CA].
[14] J.C. Lagarias, Z. Maddock, Level sets of the Takagi function: local level sets, 2010. arXiv:1009.0855.
[15] J.C. Lagarias, Z. Maddock, Level sets of the Takagi function: generic level sets, 2010. arXiv:1011.3183.
[16] K.J. Falconer, Fractal Geometry. Mathematical Foundations and Applications, John Wiley & Sons, Chichester, 1990.
[17] M. Krüppel, On the extrema and the improper derivatives of Takagi's continuous nowhere differentiable function, Rostock. Math. Kolloq. 62 (2007) 41–59.