PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 139, Number 4, April 2011, Pages 1441–1444
S 0002-9939(2010)10565-8
Article electronically published on September 30, 2010

ANOTHER PROOF OF EULER'S FORMULA FOR $\zeta(2k)$

E. DE AMO, M. DÍAZ CARRILLO, AND J. FERNÁNDEZ-SÁNCHEZ

(Communicated by Matthew A. Papanikolas)

Abstract. We give a new proof of Euler's formula related to the sum of the inverses of even powers of positive integers.

The problem of summing the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ can be traced back to the 17th century. James Bernoulli claimed that he had computed its value in 1691, but the first proof for the formula

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

was published by Euler in 1734. Later, in 1740, this great mathematician gave a more general result by summing $\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}$ (Riemann zeta function) for even positive integers:

$$\zeta(2k)=(-1)^{k+1}\,\frac{B_{2k}2^{2k-1}\pi^{2k}}{(2k)!}. \tag{1}$$

Some historical remarks can be found in [2] or [12]. The numbers B_n are rational, the so-called Bernoulli numbers. They are the coefficients in the series expansion

$$B(z):=\frac{z}{e^z-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}z^n, \qquad |z|<2\pi. \tag{2}$$

The formula $\zeta(2)=\frac{\pi^2}{6}$ has been proved by different methods. Most of them are elementary (but not easy). A compilation of fourteen of these proofs can be found in [6].

After Euler's proof of (1), a number of new proofs have appeared. (See [1], [3]-[5], [7]-[11], [13]-[20].)

The aim of this paper is to give a new proof of (1). This proof, we believe, is simpler than those cited above. We use the Taylor series expansion for the tangent function:

$$\tan x=\sum_{n=1}^{\infty}\frac{B_{2n}(-4)^n(1-4^n)}{(2n)!}x^{2n-1},\ \ |x|<\pi/2. \tag{3}$$

Received by the editors March 29, 2010 and, in revised form, April 21, 2010 and April 29, 2010.
2000 *Mathematics Subject Classification.* Primary 40C15; Secondary 40C99.

For completeness we derive the series expansion for the tangent function from that for (2) defining the Bernoulli numbers. The equation

$$g(z) := \frac{z}{e^z - 1} + \frac{z}{2} = 1 + \sum_{n=2}^{\infty} \frac{B_n}{n!} z^n$$

implies that $g(z) = g(-z)$, so that $B_n = (-1)^n B_n$ $(n \geq 2)$; and thus $B_{2n+1} = 0$ if $n \geq 1$. Therefore, we can rewrite the relation (2) in the form

$$B(x) = \frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} z^{2n}, \quad |z| < 2\pi.$$

From this equation and the relations

$$\begin{aligned} \cot z &= i + \frac{B(i2z)}{z}, \\ \tan z &= \cot z - 2\cot 2z, \end{aligned}$$

we deduce the series expansion for the tangent in (3).

Let us use the formula in (3) for $\tan xy$ with $|x| \leq \pi/2$ and $|y| < 1$:

$$\begin{aligned} \int_0^{\pi/2} \tan xy \; dx &= \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} \int_0^{\pi/2} x^{2n-1} dx \; y^{2n-1} \\ &= \sum_{n=1}^{\infty} \frac{B_{2n}(-1)^n(1-4^n)}{(2n)!} \frac{\pi^{2n}}{2n} y^{2n-1} \end{aligned}$$

(where we have used Fubini's theorem for changing the order of integration).

On the other hand, integration by parts in the above integral, together with the infinite product formula for the cosine and the Taylor series expansion for $\ln(1+x)$, gives

$$\begin{aligned} \int_0^{\pi/2} \tan xy \; dx &= \left. \frac{-\ln(\cos(xy))}{y} \right]_0^{\pi/2} = \frac{-\ln(\cos(\pi y/2))}{y} \\ &= \frac{-\ln\left(\prod_{n=1}^{\infty}\left(1 - \frac{(\pi y/2)^2}{\pi^2 (n-1/2)^2}\right)\right)}{y} \\ &= \frac{-\ln\left(\prod_{n=1}^{\infty}\left(1 - \left(\frac{y}{2n-1}\right)^2\right)\right)}{y} \\ &= -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{(-1)^{2k+1}}{k} \frac{y^{2k-1}}{(2n-1)^{2k}} \\ &= \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2k}} y^{2k-1}. \end{aligned}$$

Again, Fubini's theorem allows the order of the sums to be interchanged.

If we identify coefficients in the series expansions, then

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2k}} = (-1)^k \frac{B_{2k}(1-4^k)}{(2k)!} \frac{\pi^{2k}}{2}. \tag{4}$$

Because

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} - \sum_{n=1}^{\infty} \frac{1}{(2n)^{2k}} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2k}},$$

we have

$$\zeta(2k) = \left(1 - \frac{1}{4^k}\right) \sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2k}}.$$

By substitution in (4), we obtain Euler's formula in (1).

Acknowledgements

The authors thank the anonymous referee for helpful suggestions, and Dr. Habib Muzaffar for providing them with a very complete list of references, thereby making the list below more comprehensive.

References

[1] Apostol, T. M., *Another elementary proof of Euler's formula for* $\zeta(2n)$, Amer. Math. Monthly **80** (1973) 425-431. MR0314780 (47:3330)

[2] Ayoub, R., *Euler and the Zeta Function*, Amer. Math. Monthly **81** (1974) 1067-1086. MR0360116 (50:12566)

[3] Bényi, Á., *Finding the sums of harmonic series of even order*, College Math. J. **36** (2005) 44-48. MR2106352

[4] Berndt, B. C., *Elementary evaluation of* $\zeta(2n)$, Math. Mag. **48** (1975) 148-154. MR0366832 (51:3078)

[5] Carlitz, L., *A recurrence formula for* $\zeta(2n)$, Proc. Amer. Math. Soc. **12** (1961) 991-992. MR0133306 (24:A3140)

[6] Chapman, R., *Evaluating* $\zeta(2)$, preprint, http://www.maths.ex.ac.uk /~rjc/etc/zeta2.pdf

[7] Chen, X., *Recursive formulas for* $\zeta(2k)$ *and* $L(2k-1)$, College Math. J. **26** (1995) 372-376.

[8] Chen, M. P., *An elementary evaluation of* $\zeta(2n)$, Chinese J. Math. **3** (1975) 11-15. MR0441889 (56:280)

[9] Chungang, J. and Yonggao, C., *Euler's formula for* $\zeta(2k)$*, proved by induction on* k, Math. Mag. **73** (2000) 154-155. MR1573450

[10] Estermann, T., *Elementary Evaluation of* $\zeta(2n)$, J. London Math. Soc. **22** (1947) 10-13. MR0022613 (9:234d)

[11] Hovstad, R. M., *The series* $\sum_{k=1}^{\infty} k^{-2p}$*, the area of the unit circle and Leibniz' formula*, Nordisk Mat. Tidskr. **20** (1972) 92-98. MR0323067 (48:1425)

[12] Kline, M., *Euler and infinite series*, Math. Mag. **56** (5) (1983) 307-314. MR720652 (86a:01020)

[13] Kuo, H. T., *A recurrence formula for* $\zeta(2n)$, Bull. Amer. Math. Soc. **55** (1949) 573-574. MR0029937 (10:683d)

[14] Osler, T.J., *Finding* $\zeta(2p)$ *from a product of sines*, Amer. Math. Monthly **111** (2004) 52-54. MR2026315

[15] Robbins, N., *Revisiting an old favorite,* $\zeta(2m)$, Math. Mag. **72** (1999) 317-319. MR1573407

[16] Stark, E. L., *The series* $\sum_{k=1}^{\infty} k^{-s}, s = 2, 3, 4 \cdots$, *once more*, Math. Mag. **47** (1974) 197-202. MR0352775 (50:5261)

[17] Tsumura, H., *An elementary proof of Euler's formula for* $\zeta(2m)$, Amer. Math. Monthly **111** (5) (2004) 430-431. MR2057393

[18] Williams, G. T., *A new method of evaluating* $\zeta(2n)$, Amer. Math. Monthly **60** (1953) 19-25. MR0051856 (14:536j)

[19] Williams, K. S., *On* $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$, Math. Mag. **44** (1971) 273-276. MR0294929 (45:3997)

[20] Zhang, N. Y. and Williams, K. S., *Application of the Hurwitz zeta function to the evaluation of certain integrals*, Canad. Math. Bull. **36** (1993) 373-384. MR1245823 (94j:11083)

Departamento de Algebra y Análisis Matemático, Universidad de Almería, 04120-Almeria, Spain

E-mail address: edeamo@ual.es

Departamento de Análisis Matemático, Universidad de Granada, 18071-Granada, Spain

E-mail address: madiaz@ugr.es

Departamento de Algebra y Análisis Matemático, Universidad de Almería, 04120-Almeria, Spain