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Universidad Rey Juan Carlos

Université Abdelmalek Essaâdi

Faculté des Sciences Tétouan

PROCEEDINGS OF

"Seminario Internacional sobre Matemática Aplicada y su Repercusión en la Sociedad Actual"

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I.S.B.N.: 978-84-6917936-9
Depósito Legal: M-56988-2008

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1st Hispano-Moroccan Days on Applied Mathematics and Statistics
17-19 December 2008
Tetouan, Morocco

Peculiar functions built with the aid of the Cantor's function

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Key Words: *Singular functions, Hausdorff (fractal) dimension, MNI functions.*

A.M.S. Subject Classification: 26A27, 26A30, 26A45, 26A46

ABSTRACT

Monotone increasing Cantor's function is used as a "machine" of buliding examples of peculiar functions having interesting properties both from the Fractal Measure Theory and from the Singular Functions points of view.

We give four examples showing functions f where (i) f is strictly increasing with an associated Stieltjes measure df concentrated on a set with prescribed Hausdorff dimension $\alpha \in [0,1]$, (ii) f is of bounded variation with $f' = 0$ a.e. and monotone on no intervals (MNI); (iii) f is an absolutely continuous and MNI; and (iv) f is a continuous nowhere differentiable function.

1 Introduction

The work done in [1] uses a family of real functions in $[0,1]$ we denoted by T_{ab}. These functions have the following properties: they are continuous, having null derivatives on 1-measure sets, and there are not any interval where being monotone. And it has not been established if they are or not of bounded variation. With this state of things we proposed to look for an easy example of bounded variation function exhibiting the described properties.

The key for the building of the first example is the wellknown Cantor's function C. Let us remember that the ternary Cantor' set c consists on those numbers in $[0,1]$ with their 3-base representation having 0's and 2's. Geometrically it can be realized by iteration of the process of tricotomization for intervals and removing the open central ones at each step; hence, you have 2^n intervals of lenght $1/3^n$ each one of them, for the n-th iteration.

The Cantor's function is defined as follows: for each element in c, $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, a_n \in \{0,2\}$, it corresponds the image $\sum_{n=1}^{\infty} \frac{a_n/2}{2^n}$. On the other hand, each element in $[0,1] \setminus c$, which would be written as $x = \sum_{n=1}^{k} \frac{a_n}{3^n} + \frac{1}{3^{k+1}} + \frac{y}{3^{k+2}}$ with $a_n \in \{0,2\}$ and $y \in [0,1]$, its corresponding image is $\sum_{n=1}^{k} \frac{a_n/2}{2^n}$. (See, for example, [5].)

It is, in fact, a monotone increasing continuous function verifying $C(0) = 0$ and $C(1) = 1$. The ideas we use allow to find another peculiar functions. There exist methods based upon classical Fubini's Theorem concerning to convergence of series expansion with term by term derivatives that allow creating examples of strictly increasing singular functions. (See [7]). It is possible to exhibit, following similar ideas, examples of monotone increasing functions starting from initial ones that are monotone on no interval (MNI, see [3]), and with the additional property that theirs corresponding Sieltjes associated measures have their masses concentrated on null measure sets having fractal dimension 1 or 0, as you want. Up it is known by the authors, the only examples having these properties are in [1] and [2]; but the methods used here in this work are simpler than those.

The sections that follow are devoted to these examples.

2 Strictly increasing singular functions

The function C_α, where $\alpha \in]0, 1[$, proceeds in a similar way as $C := C_{1/3}$ when it was introduced: α is the ratio lenght we remove at each step. (See [9, pg.168].) It has the following properties:

i. It is monotone increasing, with an infinity of intervals where it is constant and the sum of their lengths is 1.

ii. The associated Stieltjes measure dC_α concentrates its mass on a self-similar set with Hausdorff dimension $\frac{\ln 2}{\ln 2 - \ln(1-\alpha)}$.

iii. Clearly, if $\alpha \in]0, y - x[$ then $C_\alpha(x) < C_\alpha(y)$.

Proposition 1 *The function*

$$F := \sum_{n=1}^{+\infty} \frac{1}{2^n} C_{\frac{1}{n+1}}$$

is an strictly increasing singular function which associated Stieltjes measure dF concentrates its mass on a set of measure 0, but fractal dimension 1.

For the second of the examples we previously define a non denumerable set of zero fractal dimension. We proceed as follows: on a first step we remove the centred open interval of length 1/3 (as in the Cantor ternary set building). But, on the second step we remove the centred two intervals each one of length $\left(3^3 - 2\right)/3^3$: there are four intervals each one of length $1/3^3$. The third step consists preserving eight closed subintervals each one having length $1/3^6$. Hence, on each step we preserve extreme closed subintervals with length equal to the $1/3^n$-th part of the pattern inteval we consider. At step n we will have 2^n closed intervals each one with length $1/3^{(n^2+n)/2}$. This limit process yields to a set we denote by $\tilde{c}$ having fractal zero dimension.

Lemma 2 *The set $\tilde{c}$ has Hausdorff dimension equal to zero.*

The following notation is useful for our purposes.

Notation 3 *Let $f : [0,1] \longrightarrow \mathbb{R}$ be real and bounded function. For $a, b \in [0,1], a < b$, we write*

$$^{ab}f(x) := \begin{cases} 0..., & x < a \\ f\left(\frac{x-a}{b-a}\right)..., & a \leq x \leq b \\ 1..., & x > b \end{cases} \quad and \quad \overline{f}(x) := \sum_{a,b \in \mathbb{Q} \cap [0,1]} \frac{^{ab}f(x)}{2^{n_{a,b}}}$$

where $n_{a,b}$ gives a enumeration for $(\mathbb{Q} \cap [0,1]) \times (\mathbb{Q} \cap [0,1])$.

Definition 4 *Let c_* be the limit set built as c_α but with corresponding lengths for the removed subintervals given by the sequence $1/3, 1/3^2, 1/3^3, ..., 1/3^n, ...$*

Obvious considerations give the following result.

Proposition 5 *The function $\overline{C}_*$ is a strictly increasing singular function which associated Stieltjes measure $d\overline{C}_*$ concentrates its mass on a set of fractal null dimension.*

Remark 6 *Because dC_α concentrates its mass on a set of dimension $\frac{\ln 2}{\ln 2 - \ln(1-\alpha)}$, $d\overline{C}_\alpha$ concentrates its corresponding mass on a set of equal dimension.*

If we join this fact with the previous results, we conclude that for each $\beta \in [0,1]$ it is possible the building of strictly increasing singular functions with associated Stieltjes measure concentrated on sets of dimension β, starting from singular functions with constancy intervals with sum for the total length equal to 1.

3 Bounded variation functions of MNI with null derivative a.e.

Let us remember C was defined as a constant on each segment removed at each step for the building of c. We will use two copies of C for introducing an auxiliar function.

Let be given $D_0 : [0,1] \to [0,1]$, by the formula

$$D_0(x) = \begin{cases} C(2x)..., & 0 \leq x \leq 1/2 \\ C(2-2x)..., & 1/2 \leq x \leq 1 \end{cases}$$

We call atention on the set $(c/2) \cup (1/2 + c/2)$ of points x where does not exist an interval containing x where D_0 is constant.

Notation 7 *Let us consider $f : [0,1] \longrightarrow \mathbb{R}$ bounded, and $a, b \in [0,1]$. We set*

$$f^{ab}(x) := \begin{cases} f\left(\frac{x-a}{b-a}\right)..., & a \leq x \leq b \\ 0..., & other\ cases \end{cases}$$

There exists an infinite sequence of intervals, say $\{[a_{0,n}, b_{0,n}]; n \in \mathbb{N}\}$, where D_0 is a constant function. Moreover, the total length of these intervals is 1.

Let us now consider this new function:

$$D_1(x) := D_0(x) + \sum_{n=1}^{\infty} \frac{(b_{0,n} - a_{0,n})}{2} D_0^{a_{0,n} b_{0,n}}(x).$$

and we can do it inductively: for each natural k it is possible to do

$$D_{k+1}(x) := D_k(x) + \sum_{n=1}^{\infty} \frac{(b_{k,n} - a_{k,n})}{2^{k+1}} D_0^{a_{k,n}b_{k,n}}(x);$$

and this limit process converges uniformly:

Definition 8

$$D(x) := \lim_{k\to\infty} D_k(x) = D_0(x) + \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} \frac{(b_{k,n} - a_{k,n})}{2^{k+1}} D_0^{a_{k,n}b_{k,n}}(x).$$

This just defined function D has the previously announcced properties.

Theorem 9 *The function D is (i) of bounded variation and has null derivatives on set of measure 1; (ii) D reaches its maxima on a set of rationals which is dense in $[0,1]$; and (iii) D reaches its minima on a set of rationals which is dense too in $[0,1]$. Moreover, there is no interval where D be a monotone function.*

4 A continuous nowhere diffentiable function

Definition 10 *Let us consider the interval $[a_{k,n}, b_{k,n}]$ having length $2^{-(k+1)}3^{-r_{k,n}}$. We consider the function G given by:*

$$G(x) := D_0(x) + \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} \frac{1}{2^{r_{n,k}}} D_0^{a_{k,n}b_{k,n}}(x).$$

Let us define the new function $F(x) := G(x/2)$.

The way we have built this function shows that it verifies the equations:

$$\begin{cases} F\left(\frac{x}{3}\right) = \frac{F(x)}{2} \\ F\left(\frac{x+2}{6}\right) = \frac{1}{2} + \frac{F(x)}{2} \\ F\left(\frac{x+3}{6}\right) = \frac{1}{2} + \frac{F(1-x)}{2} \\ F\left(\frac{x+2}{3}\right) = \frac{1}{2} + \frac{F(x)}{2} \end{cases}$$

And the contraction mapping principle of Banach fulfills its unicity.

Theorem 11 *The function F is a continuous nowhere differentiable function.*

Selfsimilarity arguments yield to the following result.

Proposition 12

$$\int_0^1 F(x)dx = 3/4.$$

5 An absolutely continuous function Monotone on No Interval (MNI)

Following previous ideas we will build an absolutely continuous function of MNI. We will newly use a Cantor type set. In the unit interval $[0,1]$, we drop out the central open interval of length $1/2^2$ (i.e., $]3/8, 5/8[$); and, proceeding by iteration on each new pair of closed intervals, we removed in each one of them the respective central open subinterval of length $1/2^{2n}$.

The process yields to a set of measure $1/2$, which we will denote by h. The corresponding function H can be obtained as limit from a sequence of polinomial aproximations being constant on each removed subinterval; but we prefer introducing it in a more adequate way for our purpouses.

Definition 13 *Let be*

$$H(x) := 2\int_0^x \chi_h(t)dt$$

(where χ_h denotes characteristic or indicator function for the set h).

It is absolutely continuous in an obvious way, monotone increasing, there exists a family of intervals whose union is dense in $[0,1]$ with h being constant on each interval of this family; and $H(0) = 0$, $H(1) = 1$.

Let us now proceed with the construction. We introduce a function $K_0 : [0,1] \to [0,1]$ given by

$$K_0(x) = \begin{cases} H(2x)..., & 0 \leq x \leq 1/2 \\ H(2-2x)..., & 1/2 \leq x \leq 1 \end{cases}$$

This function K_0 is constant on an infinity of intervals we numerate by $[c_{0,n}, d_{0,n}]$. If we proceed inductively, we have:

$$K_{s+1}(x) := K_s(x) + \sum_{n=1}^{\infty} \frac{(d_{s,n} - c_{s,n})}{2^{s+1}} K^{c_{s,n}d_{s,n}}(x).$$

Definition 14

$$K(x) := \lim_{s\to\infty} K_s(x) = K_0(x) + \sum_{s=0}^{\infty}\sum_{n=1}^{\infty} \frac{(d_{s,n} - c_{s,n})}{2^{s+1}} K^{c_{s,n}d_{s,n}}(x).$$

Now, the Lebesgue's dominated convergence theorem helps us to establish the following result.

Theorem 15 *The function K is an absolutely continuous function of MNI. Points where it reaches its relative maxima and minima values are a set of rationals dense in $[0,1]$.*

5.1 Another way to proceed

Finding an absolutely continuous function monotone on no interval type is guaranteed if we build a decomposition of the unit interval $[0,1]$ in two measurable subsets A and B verifying that $\lambda(J \cap A) > 0$ and $\lambda(J \cap B) > 0$, for each open interval J. This is the function to consider:

$$g(x) := \int_0^x [\chi_A(t) - \chi_B(t)]\, dt.$$

These sets A and B cannot be "homogeneously" distributed; for example: it is impossible to find a number α verifying $\lambda(J \cap A) = \alpha\lambda(J)$ and $\lambda(J \cap B) = (1-\alpha)\lambda(J)$. If it would be so, the function $g(x) = \int_0^x \chi_A(t)dt = \alpha x$ would have null derivatives on a set of positive measure $\lambda(B)$, which is impossible.

The sets A and B can be realised in two ways.

a) Via removing centrered subintervals we build a positive measure Cantor set, say S. Let us denote $A_1 := S \cap [0, 1/2]$ and $B_1 := S \cap [1/2, 1]$. We proceed by induction with the middle points for each removed interval: by A_n and B_n denote, respectively, the points in the first or in the second halves for the intevals removed in the n-th step. (Clearly, this is a free process in which you can select which elements will be in each set. The only we need is sets having union of measure 1.) We can do $A := \cup_n A_n$ and $B := \cup_n B_n$. With these sets we get the function g.

b) Another way, we can be helped by strictly increasing singular functions. Let $f : [0,1] \to [0,1]$ be one of such functions (having $f(0) = 0$, $f(1) = 1$), A^* a zero measure set concentrating the total mass, and B^* its complement; i.e.: $df(A^*) = 1$, $dx(A^*) = 0$, $df(B^*) = 0$, $dx(B^*) = 1$.

On the other hand, let $g : [0,1] \to [0,1]$, with $g(0) = 0$, $g(1) = 1$, be an absolutely continuous and strictly increasing function. If we define $h := \frac{f+g}{2}$, the corresponding sets are $A = h(A^*)$ and $B = h(B^*)$. For an interval J, let us write $J^* := h^{-1}(J)$. Hence, $\lambda(J \cap A) = dh(J^* \cap A^*) = \frac{df(J^*)+dg(J^* \cap A^*)}{2} = \frac{df(J^*)}{2} > 0$. (Last equality holds via Banach-Zaretzki's theorem in [6, pág. 167]). For B, it proceeds in a similar way.

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