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Academic Publications

ISSN 1311-8080 **IJPAM, Volume 54 No. 3 2009**

Contents

(Contiuned on back side)

ıgade, M.H. Durge

due to temperature

Ꞁanigawa, *Thermal*
k (2003), 260.

Conductions, Inter-
(1986), 135.

deflection of a thin
concentric circular
)6 (1973), 213-219.

in a circular plate,
277.

International Journal of Pure and Applied Mathematics

Volume 54 No. 3 2009, 407-427

TAKAGI'S FUNCTION REVISITED FROM AN ARITHMETICAL POINT OF VIEW

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Abstract: We revisite Takagi's peculiar function T with the aid of arithmetical techniques (instead of the more known geometrical ones). This formula simplyfies computations, and classical properties are now easily derived from it.

Among the other results, Kono's Probability Theorem, functional equations characterasing T, and Trollope summation formula are newly shown.

AMS Subject Classification: 26A09, 26A06, 26A27
Key Words: binary expansions, (local) Lipschitz condition, Hölder continuity, fixed point Banach's Theorem, Schauder's basis, selfsimilarity, fractal dimension

1. Introduction

We are getting used to work with functions in a natural way, almost without thinking on their meaning. Nowdays, the concept of function is completely defined. However, it was not so usual long time ago, and the way to establish it was complicated and tortuous. The great success of calculus (1665-1685), showing that derivation and integration of *functions* are reciprocal operations, was actually proved in the absence of an explicit and commonly accepted definition of the concept of function. It was necessary to wait until 1718 when Johan Bernouilli said that "a function of one variable is a (new) magnitude compound, one way or to another, with that variable magnitude and constants".

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However, for all relation among magnitudes, and these were the excesses, it was considered true that: (i) the Fundamental Theorem of Calculus was verified, (ii) there existed its power expansion series and (iii) it was possible to integrate and derivate on all these series (see, for example, [17]).

A modern form for the concept of function was given during the XIX century; nevertheless of these improvements, during a part of that century, a great number of mathematicians thought that continuous functions had derivatives on an "important" set of points on which they were defined (Ampère believed he had proved this fact).

Three great mathematicians found, independently, a negative answer for this question showing that there exist explicit examples of continuous functions that have not derivative on any point. They were: Bolzano, in1830 (see [14]), who published it in 1922; Cellérier (1860, aprox.), with a published paper [9] in 1890; and Weierstrass (see [6]) who gave his remarkable function

$$W(x) := \sum_{k=0}^{+\infty} a^k \cos\left(b^k \pi x\right),$$

(where $0 < a < 1, ab > 1 + \frac{3}{2}\pi$, and $b \in (2n-1)\mathbb{Z}$) on July, 1872. It was published during 1875. The first two never published their results by themselves; and the paradox is that these results were published in the reverse order that they were obtained by their authors.

Few time later, in 1903, Takagi gave an extraordinarily easy example of a continuous function without derivatives (such as it is recognized by the author in the title of his paper [20]). This function has been widely studied from a geometrical point of view (see, for example, [2], [4], [7] and [19]). It was originally defined in [20] by two different ways. We will prove in Lemma 1 below that, of course, they coincide. For $x \in [0,1]$, Takagi considers its binary expansion

$$0.x_1 x_2 ... x_n ...$$

($x_n \in \{0,1\}$ for all n's) and defines the function

$$f : [0,1] \to \mathbb{R}; \qquad f(x) := \sum_{k=1}^{+\infty} \frac{a_k}{2^k}$$

with

$$a_n := \begin{cases} \nu_n ..., & x_n = 0, \\ \pi_n ..., & x_n = 1, \end{cases}$$

where π_n and ν_n denote, respectively, the number of 0's and 1's among x_1, x_2,..., x_n (hence $\pi_n + \nu_n = n$, and $0 \le a_n \le n-1$).

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Afterwards this kind of continuous nowhere differentiable functions were rediscovered by many other authors (see [11]). During the preparation of this work we have found the paper of Allaart and Kawamura [1] where they give a version of the formula we use here.

2. Takagi's Function

We start with the following definition for Takagi's function (see [20], [2], [4] and [7]). Let $d(x)$ be the distance from each real x to the nearest integer. Let us define continuous functions

$$T_n(x) := \sum_{k=0}^{n} \frac{d\left(2^k x\right)}{2^k}, \qquad \forall x \in \mathbb{R}, \forall n \in \mathbb{N}$$

($\mathbb{N}$ denotes the set $\mathbb{Z}^+$ of positive integers). The (uniform) limit of (T_n) is the so called Takagi's function. As a consecuence, it is continuous on $\mathbb{R}$. Let us denote it by T from now on.

Hence, we have this analytic definition for T:

$$T(x) := \sum_{k=0}^{+\infty} \frac{d\left(2^k x\right)}{2^k}, \qquad \forall x \in \mathbb{R}.$$

Lemma 1. *The functions f and T are the same function.*

Proof. If x has binary expansion equals to $0.x_1x_2...x_n...$, then

$$d(x) := \begin{cases} x..., & x_1 = 0, \\ 1 - x..., & x_1 = 1. \end{cases}$$

It works as follows for binary expansions:

$$d(x) := \begin{cases} 0.x_1x_2...x_n..., & x_1 = 0, \\ 0.\overline{x}_1\overline{x}_2...\overline{x}_n..., & x_1 = 1. \end{cases}$$

with $\overline{x}_k := 1 - x_k$. Moreover,

$$\frac{d\left(2^k x\right)}{2^k} := 0.0...0\widetilde{x}_{k+1}\widetilde{x}_{k+2}\cdots, \qquad (*)$$

where

$$\widetilde{x}_j := \begin{cases} x_j..., & x_{k+1} = 0\,, \\ 1 - x_j..., & x_{k+1} = 1\,. \end{cases}$$

Finally, by addition on k in $[*]$, it follows the required equality. □

The target of this paper is to obtain another expression (the third) for the Takagi's peculiar function T, and to derive its classical properties. We begin with a uniqueness property for real numbers with two different binary expressions.

Lemma 2. *Let x be in the unit interval $[0,1]$. If it is possible to write*

$$x=\sum_{k=1}^{n}\frac{1}{2^{\alpha_k}}=\sum_{k=1}^{+\infty}\frac{1}{2^{\alpha'_k}}$$

with

$$\begin{cases}\alpha_k=\alpha'_k...., & \text{if } k=1,2,...,n-1\,,\\ k+\alpha_n=\alpha'_{n+k-1}...., & \text{if } k\in\mathbb{N}\end{cases}$$

(α_k and α'_k mean 1's at positions k for two different binary expansions of x), then

$$\sum_{k=1}^{n}\frac{\alpha_k-2\,(k-1)}{2^{\alpha_k}}=\sum_{k=1}^{+\infty}\frac{\alpha'_k-2\,(k-1)}{2^{\alpha'_k}}.$$

Proof. We can assume $x\in\,]0,1[$. Easy calculations on the series on the right give:

$$\begin{aligned}\sum_{k=1}^{+\infty}\frac{\alpha'_k-2\,(k-1)}{2^{\alpha'_k}}&=\sum_{k=1}^{n-1}\frac{\alpha'_k-2\,(k-1)}{2^{\alpha'_k}}+\sum_{k=n}^{+\infty}\frac{\alpha'_k-2\,(k-1)}{2^{\alpha'_k}}\\&=\sum_{k=1}^{n-1}\frac{\alpha'_k-2\,(k-1)}{2^{\alpha'_k}}+\sum_{k=1}^{+\infty}\frac{\alpha_n+k-2\,(n-1+k-1)}{2^{\alpha_n+k}},\end{aligned}$$

and working now on the last series of this sum we obtain:

$$\begin{aligned}\sum_{k=1}^{+\infty}\frac{\alpha_n+k-2\,(n-1+k-1)}{2^{\alpha_n+k}}&=\frac{1}{2^{\alpha_n}}\sum_{k=1}^{+\infty}\frac{\alpha_n-k-2n-2}{2^k}\\&=\frac{\alpha_n-2n}{2^{\alpha_n}}+\frac{1}{2^{\alpha_n}}\sum_{k=1}^{+\infty}\frac{4-k}{2^k}=\frac{\alpha_n-2\,(n-1)}{2^{\alpha_n}}.\end{aligned}$$ □

Theorem 3. *Let $x\in[0,1]$. If $x=\sum_{n=1}^{+\infty}\frac{1}{2^{\alpha_n}}$, then*

$$T(x)=\sum_{k=1}^{+\infty}\frac{\alpha_k-2\,(k-1)}{2^{\alpha_k}}.$$

Proof. There is no restriction if we only compute T for $x=\sum_{k=1}^{n}\frac{1}{2^{\alpha_k}}$, because arguments of continuity on a dense set. Notice that

$$d\,(2^{\alpha_{n-1}}x)=\frac{1}{2}\text{ and }d\,(2^m x)=0,\forall m\geq\alpha_n.$$

We do computatio

$$T(x)=\sum_{k=0}^{+\infty}\frac{d\,(2^k}{2^k}$$

$$=d(x)$$

$$=x+\frac{2x}{2}+\frac{(\alpha_1}{\cdots}$$

Let

$$x_1:=\frac{}{2^{\alpha}}$$

and using periodic

$$=(\alpha_1-2)\left[\frac{1}{2^{\alpha_1}}\right.$$

Doing analogous c

$$x_2:=\frac{}{2^{\alpha_3}}$$

it follows

$$T(x_1)=\sum_{k=0}^{\alpha_n-\alpha_1}$$

By substitution in

$$T(x)=\frac{\alpha_1}{2^{\alpha_1}}+(c$$

$$+\frac{1}{2^{\alpha_1}}$$

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$\frac{\alpha_n - 2(n-1)}{2^{\alpha_n}}$. ☐

$\text{ɔr } x = \sum_{k=1}^{n} \frac{1}{2^{\alpha_k}}$,

We do computations on the series expansion of $T(x)$:

$$T(x) = \sum_{k=0}^{+\infty} \frac{d\left(2^k x\right)}{2^k} = \sum_{k=0}^{\alpha_n - 1} \frac{d\left(2^k x\right)}{2^k}$$

$$= d(x) + \ldots + \frac{d\left(2^{\alpha_1 - 1} x\right)}{2^{\alpha_1 - 1}} + \frac{d\left(2^{\alpha_1} x\right)}{2^{\alpha_1}} + \ldots + \frac{d\left(2^{\alpha_n - 1} x\right)}{2^{\alpha_n - 1}}$$

$$= x + \frac{2x}{2} + \overset{(\alpha_1 - 2)}{\cdots\cdots} + \frac{2^{\alpha_1 - 2} x}{2^{\alpha_1 - 2}} + \frac{1 - 2^{\alpha_1 - 1} x}{2^{\alpha_1 - 1}} + \frac{d\left(2^{\alpha_1} x\right)}{2^{\alpha_1}} + \ldots + \frac{d\left(2^{\alpha_n - 1} x\right)}{2^{\alpha_n - 1}}$$

$$= (\alpha_1 - 2)\, x + \frac{1}{2^{\alpha_1 - 1}} + \frac{d\left(2^{\alpha_1} x\right)}{2^{\alpha_1}} + \ldots + \frac{d\left(2^{\alpha_n - 1} x\right)}{2^{\alpha_n - 1}}.$$

Let

$$x_1 := \frac{1}{2^{\alpha_2 - \alpha_1}} + \frac{1}{2^{\alpha_3 - \alpha_1}} + \cdots + \frac{1}{2^{\alpha_n - \alpha_1}}; \text{ i.e., } 2^{\alpha_1} x = 1 + x_1;$$

and using periodicity of T:

$$= (\alpha_1 - 2) \left[\frac{1}{2^{\alpha_1}} + \sum_{j=2}^{n} \frac{1}{2^{\alpha_j}} \right] + \frac{1}{2^{\alpha_1 - 1}} + \frac{1}{2^{\alpha_1}} T(x_1)$$

$$= \frac{\alpha_1}{2^{\alpha_1}} + (\alpha_1 - 2) \sum_{j=2}^{n} \frac{1}{2^{\alpha_j}} + \frac{1}{2^{\alpha_1}} T(x_1).$$

Doing analogous computations on x_1, and letting

$$x_2 := \frac{1}{2^{\alpha_3 - \alpha_2}} + \frac{1}{2^{\alpha_4 - \alpha_2}} + \cdots + \frac{1}{2^{\alpha_n - \alpha_2}}; \text{ i.e., } 2^{\alpha_2} x_1 = 1 + x_2;$$

it follows

$$T(x_1) = \sum_{k=0}^{\alpha_n - \alpha_1} \frac{d\left(2^k x_1\right)}{2^k} = (\alpha_2 - \alpha_1 - 2)\, x_1 + \frac{1}{2^{\alpha_2 - \alpha_1 - 1}} + \frac{1}{2^{\alpha_2 - \alpha_1}} T(x_2).$$

By substitution in $T(x)$ above:

$$T(x) = \frac{\alpha_1}{2^{\alpha_1}} + (\alpha_1 - 2) \sum_{j=2}^{n} \frac{1}{2^{\alpha_j}}$$

$$+ \frac{1}{2^{\alpha_1}} \left[(\alpha_2 - \alpha_1 - 2)\, x_1 + \frac{1}{2^{\alpha_2 - \alpha_1 - 1}} + \frac{1}{2^{\alpha_2 - \alpha_1}} T(x_2) \right]$$

$$= \frac{\alpha_1}{2^{\alpha_1}} + \frac{\alpha_2 - 2}{2^{\alpha_2}} + (\alpha_2 - 4) \sum_{j=3}^{n} \frac{1}{2^{\alpha_j}} + \frac{1}{2^{\alpha_2}} T(x_2).$$

Let us now suggest the validity of the following formula, with $p \in \mathbb{N}$:

$$T(x)=\sum_{j=1}^{p}\frac{\alpha_j-2\,(j-1)}{2^{\alpha_j}}+(\alpha_p-2p)\sum_{j=p+1}^{n}\frac{1}{2^{\alpha_j}}+\frac{1}{2^{\alpha_p}}T\left(x_p\right);$$

and we will prove it for $p+1$. Reasoning on x_p:

$$\begin{aligned}T(x_p) &= \sum_{k=0}^{\alpha_n-\alpha_p}\frac{d\left(2^k x_p\right)}{2^k}\\ &= \left(\alpha_{p+1}-\alpha_p-2\right)x_p+\frac{1}{2^{\alpha_{p+1}-\alpha_p-1}}+\frac{1}{2^{\alpha_{p+1}-\alpha_p}}T\left(x_{p+1}\right);\end{aligned}$$

where

$$x_{p+1}:=\frac{1}{2^{\alpha_{p+1}-\alpha_p}}+\frac{1}{2^{\alpha_{p+3}-\alpha_p}}+\cdots+\frac{1}{2^{\alpha_n-\alpha_p}};\ \text{i.e., } 2^{\alpha_{p+1}}x_p=1+x_{p+1}.$$

Hence,

$$\begin{aligned}T(x) &= \sum_{j=1}^{p}\frac{\alpha_j-2\,(j-1)}{2^{\alpha_j}}+(\alpha_p-2p)\sum_{j=p+1}^{n}\frac{1}{2^{\alpha_j}}\\ &\quad+\frac{1}{2^{\alpha_p}}\left[\left(\alpha_{p+1}-\alpha_p-2\right)x_p+\frac{1}{2^{\alpha_{p+1}-\alpha_p-1}}+\frac{1}{2^{\alpha_{p+1}-\alpha_p}}T\left(x_{p+1}\right)\right]\\ &= \sum_{j=1}^{p}\frac{\alpha_j-2\,(j-1)}{2^{\alpha_j}}+(\alpha_p-2p)\sum_{j=p+1}^{n}\frac{1}{2^{\alpha_j}}\\ &\quad+\left(\alpha_{p+1}-\alpha_p-2\right)\sum_{j=p+1}^{n}\frac{1}{2^{\alpha_j}}+\frac{2}{2^{\alpha_{p+1}}}+\frac{1}{2^{\alpha_{p+1}}}T\left(x_{p+1}\right)\\ &= \sum_{j=1}^{p}\frac{\alpha_j-2\,(j-1)}{2^{\alpha_j}}+\frac{\alpha_{p+1}-2p}{2^{\alpha_{p+1}}}\\ &\quad+\left(\alpha_{p+1}-2\,(p+1)\right)\sum_{j=p+2}^{n}\frac{1}{2^{\alpha_j}}+\frac{1}{2^{\alpha_{p+1}}}T\left(x_{p+1}\right).\end{aligned}$$

Finally, with $p=n-1$, we conclude:

$$\begin{aligned}T(x)=\sum_{j=1}^{n-1}\frac{\alpha_j-2\,(j-1)}{2^{\alpha_j}}&+\frac{\alpha_{n-1}-2(n-1)}{2^{\alpha_n}}+\frac{\alpha_n-\alpha_{n-1}}{2^{\alpha_n}}\\ &=\sum_{j=1}^{n}\frac{\alpha_j-2\,(j-1)}{2^{\alpha_j}}.\qquad\square\end{aligned}$$

This formula will provide easier computations. The next result is an explicit example of this.

Proposition
binary expansion.

Proof. In case
have periodicity. I

$x = \sum_{n=1}^{+\infty}$

$+\frac{1}{2^r}$

Applying the form

$T(x)=\sum_{j=1}^{k}\alpha_j -$

$+\Big(\frac{\beta_1-2}{2^{\beta_1}}$

where the sum of
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Moreover, the(
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The idea rests
by linear segments

Definition 5.
a Schauder basis i
such that $x=\sum_{n}^{+}$

The basis of S(
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description: $\alpha_0 :=$

$\beta_{n,k}(x)$

Proposition 4. *If $x \in \mathbb{Q}$, then $T(x) \in \mathbb{Q}$. Moreover, $T(x)$ has finite binary expansion.*

Proof. In case of finite expansions it is inmediate. On another case, we have periodicity. Let us consider

$$\begin{aligned} x &= \sum_{n=1}^{+\infty} \frac{1}{2^{\alpha_n}} = \frac{1}{2^{\alpha_1}} + \frac{1}{2^{\alpha_2}} + \cdots + \frac{1}{2^{\alpha_k}} + \frac{1}{2^{\beta_1}} + \frac{1}{2^{\beta_2}} + \cdots + \frac{1}{2^{\beta_s}} \\ &\quad + \frac{1}{2^r}\left(\frac{1}{2^{\beta_1}} + \frac{1}{2^{\beta_2}} + \cdots + \frac{1}{2^{\beta_s}}\right) + \cdots \end{aligned}$$

Applying the formula of Theorem 3 above, we will have

$$\begin{aligned} T(x) &= \sum_{j=1}^{k} \frac{\alpha_j - 2(j-1)}{2^{\alpha_j}} \\ &\quad + \left(\frac{\beta_1 - 2k}{2^{\beta_1}} + \frac{\beta_2 - 2(k+1)}{2^{\beta_2}} + \ldots + \frac{\beta_s - 2(k+s-1)}{2^{\beta_s}}\right) \sum_{t=1}^{+\infty} \frac{1}{2^{tr}} \\ &\quad + (r-2)\left(\frac{1}{2^{\beta_1}} + \frac{1}{2^{\beta_2}} + \ldots + \frac{1}{2^{\beta_s}}\right) \sum_{t=1}^{+\infty} \frac{t}{2^{tr}}, \end{aligned}$$

where the sum of each series is a rational number. Hence, the statement is true. □

Moreover, theoretical advantages will turn up. The next theorem, as an easy consequence of this formula for T, yields to a sequence of poligonals (corresponding to the partial sums T_n), which is equivalent to describe the coeficients of T for a Schauder's basis (see [13]).

The idea rests on the fact that continuous functions can be approximated by linear segments.

Definition 5. (Schauder Basis) A sequence (x_n) on a normed space X is a Schauder basis if for every x in X there is a unique sequence of scalars (a_n) such that $x = \sum_{n=0}^{+\infty} a_n x_n$; i.e.,

$$\lim_n \left\| x - \sum_{k=0}^{n} a_k x_k \right\| = 0.$$

The basis of Schauder that we will use in the space of continuous functions on the unit interval, $\mathcal{C}([0,1])$ equipped with the sup-norm, has the following description: $\alpha_0 := x$, $\alpha_1 := 1 - x$, and

$$\beta_{n,k}(x) := 2^k \left(\left| x - \frac{n}{2^k} \right| + \left| x - \frac{n+1}{2^k} \right| - \left| 2x - \frac{2n+1}{2^k} \right| \right)$$

(where $0 \leq n \leq 2^{k-1}$ and $k \geq 0$) with $\beta_{n,k}$'s taking null values out of $\left[\frac{n}{2^k}, \frac{n+1}{2^k}\right]$ and with graph, when x runs on it, given by the equal sides of the isosceles triangle determined by $\left(\frac{n}{2^k}, 0\right)$, $\left(\frac{n+1}{2^k}, 0\right)$, and $\left(\frac{2n+1}{2^{k+1}}, 1\right)$.

For the function of Takagi, we have (as in [12]):

Theorem 6. *For naturals n and k, the folowing identity is true:*

$$T\left(\frac{2n+1}{2^{k+1}}\right) = \frac{1}{2}\left[T\left(\frac{n}{2^k}\right) + T\left(\frac{n+1}{2^k}\right)\right] + \frac{1}{2^{k+1}}.$$

Proof. If $n, k \in \mathbb{N}$, then

$$\frac{n}{2^k} = \frac{1}{2^{\alpha_1}} + \frac{1}{2^{\alpha_2}} + \dots + \frac{1}{2^{\alpha_n}},$$
$$\frac{2n+1}{2^k} = \frac{1}{2^{\alpha_1}} + \frac{1}{2^{\alpha_2}} + \dots + \frac{1}{2^{\alpha_n}} + \frac{1}{2^{k+1}},$$
$$\frac{n+1}{2^k} = \frac{1}{2^{\alpha_1}} + \frac{1}{2^{\alpha_2}} + \dots + \frac{1}{2^{\alpha_n}} + \frac{1}{2^k},$$

and computations with the formula in Theorem 3 give:

$$T\left(\frac{n}{2^k}\right) = \sum_{j=1}^{n} \frac{\alpha_j - 2(j-1)}{2^{\alpha_j}},$$
$$T\left(\frac{2n+1}{2^k}\right) = \sum_{j=1}^{n} \frac{\alpha_j - 2(j-1)}{2^{\alpha_j}} + \frac{k+1-2n}{2^{k+1}},$$
$$T\left(\frac{n+1}{2^k}\right) = \sum_{j=1}^{n} \frac{\alpha_j - 2(j-1)}{2^{\alpha_j}} + \frac{k-2n}{2^k}.$$

The equality is now clear. □

3. Classical Properties of T

It is well known that T is continuous on $\mathbb{R}$. Next step is to show the non-derivability of T anywhere. A very useful tool, already used by Stieltjes, is now presented as a lemma.

Lemma 7. (see [5]) *Let us suppose that a function f has (finite) derivative on a point x. Then*

$$\lim_{u \to x^-, v \to x^+} \frac{f(u) - f(v)}{u - v} = f'(x).$$

Proof. It follows as in [5, p. 404] □

Theorem 8.
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Proof. First, l
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Hence, the quotie

diverges as $k \to +$

We now consi
in the binary exp

is infinite). We
inequalities:

and

We compute f

But, if $n \to +\infty$
$x \in]0,1[$. The cas

In fact, it is p
the calculations al

Theorem 9.
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Proof. Let $x \in$
of T on x does n
analogously (or, n

Theorem 8. (see [20] and [4]) *Takagi's function has not derivative on any point in the unit interval.*

Proof. First, let us suppose a finite expansion for x; i.e., $x = \sum_{j=1}^{n} \frac{1}{2^{\alpha_j}}$ (this is the case when the number of 1's or 0's in its binary expansion is finite). Let us consider

$$y := x + \frac{1}{2^{\alpha_n + k}} \qquad (k \in \mathbb{N}).$$

Hence, the quotient

$$\frac{T(x) - T(y)}{x - y} = \alpha_n + k - 2n$$

diverges as $k \to +\infty$, and this implies that does not exist the derivative $T'(x)$.

We now consider the case in which the number of 1's and the number of 0's in the binary expansion of x are both infinite (and hence, the set

$$\{n \in \mathbb{N} : \alpha_{n+1} > \alpha_n + 1\}$$

is infinite). We write $x = \sum_{n=1}^{+\infty} \frac{1}{2^{\alpha_n}}$, and consider two kinds of chains of inequalities:

$$u_n := \sum_{k=1}^{n} \frac{1}{2^{\alpha_k}} < x < \sum_{k=1}^{n} \frac{1}{2^{\alpha_k}} + \frac{1}{2^{\alpha_n + 1}} =: v_n$$

and

$$u_n < x < v_n + \frac{1}{2^{\alpha_n + 2}} =: \widetilde{v}_n.$$

We compute for each one of these cases:

$$\begin{aligned} \frac{T(u_n) - T(v_n)}{u_n - v_n} &= 1 + \alpha_n - 2n\,, \\ \frac{T(u_n) - T(\widetilde{v}_n)}{u_n - \widetilde{v}_n} &= \frac{2}{3} + \alpha_n - 2n\,. \end{aligned}$$

But, if $n \to +\infty$, then the lemma above says that T has not derivative on $x \in \,]0,1[$. The cases $x \in \{0,1\}$ have analogous reasonings. $\square$

In fact, it is possible to obtain much more information if we are carefull in the calculations above:

Theorem 9. (see [7]) *Takagi's function has not right sided neither left sided derivatives on any point.*

Proof. Let $x \in [0,1[$. We will prove that the right sided derivative $T'(x^+)$ of T on x does not exist. For the dual situation $T'(x^-)$ the proof will run analogously (or, may be, considering that $T(1-x) = T(x)$ for all $x \in [0,1]$).

The proof in the above theorem shows that if x has finite binary expansion, then the right sided derivative of T on x does not exist. Hence, we will reduce to the case there exist infinites 1's and infinities 0's in the (infinite) binary expansion of x. Hence

$$x = \sum_{n=1}^{+\infty} \frac{1}{2^{\alpha_n}}$$

and $\{n \in \mathbb{N} : \alpha_n + 1 < \alpha_{n+1}\}$ is an infinite set.

Let us define

$$x_n = \sum_{k=1}^{n-1} \frac{1}{2^{\alpha_k}} + \frac{1}{2^{\alpha_n - 1}} + \sum_{k=n+1}^{+\infty} \frac{1}{2^{\alpha_k}}.$$

It follows that $x_n > x$, and

$$\frac{T(x_n) - T(x)}{x_n - x} = \alpha_n - 2n.$$

But, if there exists the limit $\alpha_n - 2n \longrightarrow T'(x^+)$, then it must be an integer; and this implies a periodic expansion for x into the form

$$x = \sum_{k=1}^{r} \frac{1}{2^{\alpha_k}} + \left(\sum_{k=0}^{+\infty} \frac{1}{2^{2k+r}} \right) = \sum_{k=1}^{r} \frac{1}{2^{\alpha_k}} + \sum_{k=0}^{+\infty} \frac{1}{2^{2k+m}}.$$

Let us consider $n \geq r+1$, and define

$$y_n := x - \left(\frac{1}{2^{\alpha_n + 2}} - \frac{1}{2^{\alpha_n - 1}} \right).$$

In this case

$$\frac{T(y_n) - T(x)}{y_n - x} = \alpha_n - 2n - 2/7,$$

and we finish the proof with this contradiction. □

Hence, the peculiar function of Takagi plays an intermediate role between continuous and without sided derivatives functions.

Definition 10. (Lipschitz and Local Lipschitz Conditions) Let f be a map from a metric space (X,d) to another metric space (X',d'). We say that f verifies Lipschitz condition (or that f is Lipschitz) if

$$\exists k > 0 : x, y \in X \implies d'(f(x), f(y)) \leq k d(x,y).$$

We say that f is Lipschitz at $x \in X$ if

$$\exists k, \varepsilon > 0 : y \in X, d(x,y) < \varepsilon \implies d'(f(x), f(y)) \leq k d(x,y).$$

Proposition 11. *There exist points in which T is not Lipschitz; i.e., there*

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Lemma 12.

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exists $x \in [0,1]$ *such that if* $k, \varepsilon \in \mathbb{R}^+$, *then*

$$\{y \in [0,1] : d(x,y) < \varepsilon, |T(x) - T(y)| > k\,|x-y|\} \neq \emptyset.$$

Proof. If we consider points with finite binary expansions, then the proof is implicit in that of Theorem 9. ☐

Lemma 12. *If* (α_n) *is an increasing sequence of naturals such that* $\alpha_n - 2\,(n-1) \neq 0$, *then*

$$\sum_{k=n}^{+\infty} \frac{\alpha_k - 2\,(k-1)}{2^{\alpha_k}} \leq O\left(\frac{\alpha_n - 2\,(n-1)}{2^{\alpha_n}}\right).$$

Proof. Computations give the result:

$$\sum_{j=1}^{+\infty} \frac{\alpha_n + s_j - 2\,(n+j-1)}{2^{\alpha_n + s_j}} = \sum_{j=1}^{+\infty} \frac{\alpha_n - 2\,(n-1)}{2^{\alpha_n}} \frac{1}{2^{s_j}}$$

$$+ \frac{1}{2^{\alpha_n}} \sum_{j=1}^{+\infty} \frac{s_j - 2j}{2^{s_j}} = O\left(\frac{\alpha_n - 2\,(n-1)}{2^{\alpha_n}}\right). \quad \square$$

Proposition 13. *The function* T *is Lipschitz on a dense subset of* $[0,1]$.

Proof. Let us consider:

$$x = \sum_{n=1}^{r} \frac{1}{2^{\alpha_n}} + \sum_{n=1}^{+\infty} \frac{1}{2^{\alpha_r + 2n}},$$

and

$$y := \sum_{n=1}^{r} \frac{1}{2^{\alpha_n}} + \sum_{n=1}^{m} \frac{1}{2^{\alpha_r + 2n}} + \frac{z}{2^{\alpha_r + 2m}}; \qquad z \in [0,1[\,.$$

In this situation, there exist positive reals A and B, such that

$$A\,|x-y| \leq \frac{1}{2^{\alpha_r + 2m}} \leq B\,|x-y|$$

and

$$\begin{aligned} |T(x) - T(y)| \;&\leq\; \frac{|\alpha_{r+1} - 2r|}{2^{\alpha_{r+1}}} + \frac{|\alpha_{r+1} - 2r|}{2^{\alpha_{r+1}+2}} + \frac{|\alpha_{r+1} - 2r|}{2^{\alpha_{r+1}+4}} + \cdots \\ &+ \frac{|\alpha'_t - 2\,(t-1)|}{2^{\alpha_t}} + \cdots. \end{aligned}$$

Hence, the series in the first arrow is of order $O\left(\frac{|\alpha_{n+1} - 2n|}{2^{\alpha_{n+1}}}\right)$ and the series in the second is too, by the lemma above and the decreasing monotony of $\frac{x}{2^x}$. ☐

With a little effort, we will have more information.

Definition 14. (Hölder Continuity) A function f is said Hölder-continuous of degree β at x, if there exist positives M and δ such that $|x-y|<\delta$ implies $|f(x)-f(y)|<M\,|x-y|^{\beta}$ (in case $\beta=1$, f is Lipschitz at x).

Theorem 15. (see [19]) *The function T is Hölder-continuous of degree β, for all β in $]0,1[$.*

This result follows as a consequence of a more general result of Hata. With the formula of Theorem 3, we can prove the following continuity property:

Theorem 16. (see Hata [11]) *For $x,y\in\mathbb{R}$,*

$$|T(x)-T(y)|\leq O\left(|x-y|\ln|x-y|\right).$$

Proof. Let us consider numbers

$$x:=\sum_{k=1}^{n}\frac{1}{2^{\alpha_k}}+\frac{1}{2^{m}}+\frac{1}{2^{m+1}}+\cdots \text{ and } y:=\sum_{k=1}^{n}\frac{1}{2^{\alpha_k}}+\frac{1}{2^{m'}}+\cdots.$$

It is possible to do $m+1=m'$ or $m+1<m'$. In either case, we have that

$$\frac{1}{2^{m+2}}\leq|x-y|\leq\frac{1}{2^{m-1}},$$

and

$$\begin{aligned}|T(x)-T(y)| &\leq \left|\frac{m-2n}{2^{m}}\right|+\left|\frac{\alpha_{n+2}-2\,(n+1)}{2^{\alpha_{n+2}}}\right|+\cdots\\ &\quad+\left|\frac{m'-2n}{2^{m'}}\right|+\left|\frac{\alpha_{n+2}-2\,(n+1)}{2^{\alpha_{n+2}}}\right|+\cdots\\ &= \frac{m}{2^{m}}+\frac{m'}{2^{m'}}+2\sum_{j=1}^{+\infty}\frac{j+m}{2^{j+m}}\\ &= O\left(\frac{m}{2^{m}}\right)=O\left(|x-y|\ln|x-y|\right).\end{aligned}$$

On the other hand, if

$$x:=\sum_{k=1}^{r}\frac{1}{2^{\alpha_k}}+\frac{1}{2^{m}}+\frac{1}{2^{m_1}}+\cdots$$

and

$$y:=\sum_{k=1}^{r}\frac{1}{2^{\alpha_k}}+\frac{1}{2^{m+1}}+\ldots+\frac{1}{2^{m+n}}+\frac{1}{2^{m_2}}+\cdots$$

we will consider $m'':=\min\{m_1,m+n\}$. Hence, there exist positive reals c_1 and c_2, such that

$$\frac{c_1}{2^{m''}}\leq|x-y|\leq\frac{c_2}{2^{m''}}.$$

Finally, the corre
to $\frac{1}{2^{m+1}}+\cdots+$
result.

Proposition

Proof. If we

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then

$h =$

Hence, with $m \to$

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Proof. It is a

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Lemma 20.
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Corollary 21

Finally, the corresponding $\frac{1}{2^m}$ term in $T(x)$ operates with the corresponding one to $\frac{1}{2^{m+1}} + \cdots + \frac{1}{2^{m+n}}$ in $T(y)$, and proceeding as above, we have the desidered result. ☐

Proposition 17. *It is impossible to improve Hata's result:*

$$\exists x \in \mathbb{R} : |T(x) - T(x+h)| \neq o\left(|h| \ln |h|\right).$$

Proof. If we consider x and y with finite dyadic expansions:

$$x = \frac{1}{2^{\alpha_1}} + \frac{1}{2^{\alpha_2}} + \ldots + \frac{1}{2^{\alpha_{n-1}}} + \frac{1}{2^{\alpha_n}} \text{ and } y = x + \frac{1}{2^m},$$

then

$$h = |x - y| = \frac{1}{2^m} \text{ and } |T(x) - T(y)| = \left| \frac{m - 2n}{2^n} \right|.$$

Hence, with $m \to +\infty$, $|T(x) - T(x+h)| \neq o\left(|h| \ln |h|\right)$. ☐

4. Kôno's Theorem

If $x = \sum_{k=1}^{+\infty} 1/2^{\alpha_k}$, then we will write by $b_k(x)$ the leng of the k-th sequence of 1's in the series expansion (the two possibly different ways of definition for x are not important for us, because it occurs on a denumerable set). The b_k's are random variables.

Lemma 18. *The random variables $\{b_k; k \in \mathbb{N}\}$ are independent and identically distributed, with $p\left(b_k = n\right) = 1/2^{n+1}$.*

Lemma 19. *The set of points where $b_k \geq 2\log_2 k$ occurs infinitely many times is a zero measure set.*

Proof. It is a consequence of Borel-Cantelli Lemma (see [5]). ☐

Given a number $x = \sum_{k=1}^{+\infty} a_k/2^k$, we will study the random variables a_k's.

Lemma 20. *The random variables $\{a_k : k \in \mathbb{N}\}$ are independent and identically distributed, with $p\left(a_k = 0\right) = p\left(a_k = 1\right) = 1/2$.*

By the law of the iterated logarithm (see [5]):

Corollary 21. *Let λ denote the Lebesgue measure. Then,*

$$\lambda\left(\left\{x : \limsup_n \frac{2\sum_{k=1}^{n} a_k - n}{\sqrt{n}\sqrt{2\ln\ln n}} = 1\right\}\right) = 1$$

and

$$\lambda\left(\left\{x : \liminf_n \frac{2\sum_{k=1}^n a_k - n}{\sqrt{n}\sqrt{2\ln\ln n}} = -1\right\}\right) = 1.$$

If we now consider the subsequence of the 1's in (a_k); then we have:

Corollary 22.

$$\lambda\left(\left\{x : \limsup_n \frac{\alpha_n - 2n}{\sqrt{2\alpha_n \ln\ln \alpha_n}} = 1\right\}\right) = 1,$$

and

$$\lambda\left(\left\{x : \liminf_n \frac{\alpha_n - 2n}{\sqrt{2\alpha_n \ln\ln \alpha_n}} = -1\right\}\right) = 1.$$

Lemma 23. *If* $0 \le y = \frac{1}{2^{\alpha'_1}} + \frac{1}{2^{\alpha'_2}} + \cdots + \frac{1}{2^{\alpha'_k}} + \cdots \le 1$ *and* $\frac{s}{2^r} = \frac{1}{2^{\alpha_1}} + \frac{1}{2^{\alpha_2}} + \cdots + \frac{1}{2^{\alpha_k}}$, *then*

$$T\left(\frac{s+y}{2^r}\right) = T\left(\frac{s}{2^r}\right) + \frac{r-2k}{2^r}y + T(y).$$

Proof. The following equalities are true:

$$\begin{aligned}
T(y) &= \frac{\alpha'_1}{2^{\alpha'_1}} + \frac{\alpha'_2 - 2}{2^{\alpha'_2}} + \cdots + \frac{\alpha'_k - 2(k-1)}{2^{\alpha'_k}} + \cdots,\\
T\left(\frac{s}{2^r}\right) &= \frac{\alpha_1}{2^{\alpha_1}} + \frac{\alpha_2 - 2}{2^{\alpha_2}} + \cdots + \frac{\alpha_k - 2(k-1)}{2^{\alpha_k}},\\
\frac{s+y}{2^r} &= \frac{1}{2^{\alpha_1}} + \frac{1}{2^{\alpha_2}} + \cdots + \frac{1}{2^{\alpha_k}} + \frac{\alpha'_1}{2^{l+\alpha'_1}} + \frac{\alpha'_2 - 2}{2^{l+\alpha'_2}} + \cdots,\\
T\left(\tfrac{s+y}{2^r}\right) &= \tfrac{\alpha_1}{2^{\alpha_1}} + \tfrac{\alpha_2-2}{2^{\alpha_2}} + \cdots + \tfrac{\alpha_k-2(k-1)}{2^{\alpha_k}} + \tfrac{r+\alpha'_1-2k}{2^{r+\alpha'_1}} + \tfrac{r+\alpha'_2-2(k+1)}{2^{r+\alpha'_2}} + \cdots,
\end{aligned}$$

and, hence, the result follows. □

Theorem 24. (see Kôno, [16]) *On a set of* λ*-measure 1, we have*

$$\lambda\left(\left\{x : \limsup_h \frac{T(x+h) - T(x)}{h\sqrt{2\log_2\left(\frac{1}{h}\right)\ln\ln\log_2\left(\frac{1}{h}\right)}} = 1\right\}\right) = 1.$$

Proof. We will consider points x with infinite dyadic expansions. Let

$$x = \frac{s+y}{2^r}, x + h = \frac{s+y'}{2^r}; \qquad 0 \le y < y' \le 1.$$

We choose r as the maximum for which this relation is valid. Hence $y < \frac{1}{2} \le y'$. By the lemma above,

$$T\left(\frac{s+y}{2^r}\right) = T\left(\frac{s}{2^r}\right) + \frac{r-2k}{2^r}y + \frac{T(y)}{2^r},$$

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If we apply th rithm, then we ob

Theorem 25.

$\lim_{\substack{h\to 0\\ h>0}} \lambda\left(\left\{x \in\right.\right.$

$$T\left(\frac{s+y'}{2^r}\right) = T\left(\frac{s}{2^r}\right) + \frac{r-2k}{2^r}y' + \frac{T(y')}{2^r},$$

where $\alpha_k := r$. Hence

$$\frac{T(x+h)-T(x)}{h} = r - 2k + \frac{T(y')-T(y)}{y'-y}.$$

The last quotient is bounded unless $\frac{1}{4} < y < \frac{1}{2} \leq y' < \frac{3}{4}$. If this is the case:

$$\begin{aligned} y &= \frac{1}{2^2} + \frac{1}{2^R} + \frac{1}{2^{\beta_2}} + \cdots, \\ y' &= \frac{1}{2} + \frac{1}{2^{\gamma_1}} + \frac{1}{2^{\gamma_2}} + \cdots; \qquad \gamma_1 > R, \end{aligned}$$

then

$$\begin{aligned} T(y) &= \sum_{n=2}^{R} \frac{-n+4}{2^n} + \frac{\beta_2 - 2(R-2)}{2^{\beta_2}} + \cdots, \\ T(y') &= \frac{1}{2} + \frac{\gamma_1 - 2}{2^{\gamma_1}} + \cdots, \end{aligned}$$

and

$$\begin{aligned} \frac{T(y')-T(y)}{y'-y} &\leq \frac{\frac{1}{2} + O\left(\frac{\gamma_1}{2^{\gamma_1}}\right) - \frac{1}{2} - \frac{-4+2R}{2^{1+R}} + O\left(\frac{R}{2^R}\right)}{\frac{1}{2^R}} \\ &= O(R) = O(\ln \alpha_k) \end{aligned}$$

on a set of measure 1 (the last inequality follows from Lemma 19).

If $y' := x + h$ and $y := x$, then $\frac{T(x+h)-T(x)}{h} = r - 2k + O(\ln \alpha_k)$. Hence,

$$\limsup_{h \to 0^+} \frac{T(x+h)-T(x)}{h\sqrt{2\log_2 \frac{1}{h} \ln \ln \log_2 \frac{1}{h}}} = \limsup_{k \to +\infty} \frac{r-2k}{\sqrt{2\alpha_k \ln \ln \alpha_k}} = 1$$

on a set of λ-measure 1, by Corollary 22.

Simmetry with respect to $1/2$ implies the validity of the result for $h \to 0^-$. ☐

If we apply the Central Limit Theorem instead of the law of iterated logarithm, then we obtain the following result.

Theorem 25.

$$\lim_{\substack{h \to 0 \\ h > 0}} \lambda\left(\left\{x \in]0,1[\, : \frac{T(x+h)-T(x)}{h\sqrt{-\log_2 h}} < y\right\}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-z^2/2} dz.$$

5. Functional Equations Characterasing T and Self-Affinity

With the aid of the Banach Contractive Mapping Principle, T is characterised by functional equations (see [18]).

Theorem 26. (Functional Equations) *The function of Takagi T is the only continuous and bounded function in $[0,1]$ satisfying the functional equations*

$$\begin{cases} T\left(\frac{x}{2}\right) = \frac{x}{2} + \frac{T(x)}{2}, \\ T\left(\frac{1}{2} + \frac{x}{2}\right) = \frac{1}{2} - \frac{x}{2} + \frac{T(x)}{2}. \end{cases}$$

Proof. Let us consider the Banach space $\mathcal{C}([0,1],\mathbb{R})$ of real continuous (and bounded, a fortiori) functions defined on $[0,1]$ endowed with the supremum norm. We define the functional

$$\begin{aligned} F \quad &: \quad \mathcal{C}([0,1],\mathbb{R}) \longrightarrow \mathcal{C}([0,1],\mathbb{R}); \\ g \quad &\longrightarrow \quad F(g) : [0,1] \longrightarrow \mathbb{R} \end{aligned}$$

given by

$$F(g)(x) := \begin{cases} x + \frac{g(2x)}{2} \ldots, & 0 \leq x \leq 1/2, \\ 1 - x + \frac{g(2x-1)}{2} \ldots, & 1/2 \leq x \leq 1. \end{cases}$$

$F(g)$ is well defined, and we only have to claim the aid of the Banach Fixed Point Theorem (or contraction mapping principle): there exists one, and only one, $g \in \mathcal{C}([0,1],\mathbb{R})$ satisfying the functional equations above.

Doing manipulations on the series we obtain that T is the solution. ☐

As a consequence, we can solve the area under the graph of T.

Corollary 27. $\int_0^1 T = \frac{1}{2}$.

Proof. The functional equations above show self-affinity for T: the total area α is equal to that of a triangle of base 1, and height 1/2 and two 1/2-replica of itself; it is to say

$$\alpha = \frac{1}{4} + 2\frac{\alpha}{4},$$

and hence, the statement is true. ☐

Theorem 28. (see [2]) *Takagi's function attains its (absolute) maxima on the set A of the points whose 4-base expansion only consists on 1's and/or 2's. The maximum value of T is 2/3, and the fractal dimension of A is 1/2.*

Proof. Clearl

Hence, A is self-s

If $x \in A$, the follows

x :

Applying the

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$x := \sum_{n=1}^{k-1}$

with $a_n \in \{1,2\}$ analogous).

Then, with th

$T(x) = \sum_{n=}^{k-}$

If we now conside

$y := \sum_{n=1}^{k-1}$

then

$T(y) = \sum_{n=1}^{k-1}$

Hence, $T(x) < T($

Proof. Clearly

$$A = \left(\frac{1}{4}A + \frac{1}{4}\right) \cup \left(\frac{1}{4}A + \frac{1}{2}\right).$$

Hence, A is self-similar and consecuently, $\dim_H(A) = 1/2$.

If $x \in A$, then $x = \sum_{n=1}^{+\infty} \frac{a_n}{4^n}$, with $a_n \in \{1, 2\}$; and we can rewrite x as follows

$$x = \sum_{n=1}^{+\infty} \frac{1}{2^{2n-s_n}}, \qquad \text{with } s_n = \begin{cases} 1..., & a_n = 2, \\ 0..., & a_n = 1. \end{cases}$$

Applying the formula in Theorem 3:

$$T(x) = \sum_{n=1}^{+\infty} \frac{2n - s_n - 2(n-1)}{2^{2n-s_n}} = \sum_{n=1}^{+\infty} \frac{2 - s_n}{2^{2n-s_n}} = \sum_{n=1}^{+\infty} \frac{1}{2^{2n-1}} = \frac{2}{3}$$

(the penultimate equality does not depend on the s_n's!).

Let us consider points x with some 0 or 3 among its digits. We will prove that these points are not point of maximum for T. Let

$$x := \sum_{n=1}^{k-1} \frac{a_n}{4^n} + \frac{3}{4^k} + \sum_{n=k+1}^{+\infty} \frac{b_n}{4^n} = \sum_{n=1}^{k-1} \frac{a_n}{4^n} + \frac{3}{4^k} + \sum_{n=k+1}^{+\infty} \frac{1}{2^{2n+c_n}}$$

with $a_n \in \{1, 2\}$ and $b_n \in \{0, 1, 2, 3\}$ (for $a_k = 0$, the reasoning would be analogous).

Then, with this notation and the fact that $\frac{3}{4^k} = \frac{1}{2^{2k-1}} + \frac{1}{2^{2k}}$,

$$T(x) = \sum_{n=1}^{k-1} \frac{1}{2^{2n-1}} + \frac{1}{2^{2k-1}} + \frac{2k - 2k}{2^{2k}} + \sum_{n=1}^{+\infty} \frac{2k + c_n - 2(k+n)}{2^{2k+c_n}}.$$

If we now consider

$$y := \sum_{n=1}^{k-1} \frac{a_n}{4^n} + \frac{1}{4^k} + \sum_{n=k+1}^{+\infty} \frac{b_n}{4^n} = \sum_{n=1}^{k-1} \frac{a_n}{4^n} + \frac{1}{2^{2k}} + \sum_{n=k+1}^{+\infty} \frac{1}{2^{2n+c_n}},$$

then

$$T(y) = \sum_{n=1}^{k-1} \frac{1}{2^{2n-1}} + \frac{1}{2^{2k-1}} + \frac{2k - 2k}{2^{2k}} + \sum_{n=1}^{+\infty} \frac{2k + c_n - 2(k+n-1)}{2^{2k+c_n}}.$$

Hence, $T(x) < T(y)$, and T does not reach its maxima on points of x-type. ☐

6. Trollope's Formula

Number theory often studies asymptotic behaviour for the sum of arithmetic functions. With the help of Theorem 3, we obtain an exact formula for one of these expressions: the sum of the number of digits with binary expansion for positive integers (see [21] and [10]).

For a given $n \in \mathbb{N}$, its binary expansion is $\sum_{k=0}^{+\infty} e_k(n)\, 2^n$, with $e_k(n) \in \{0,1\}$. Let us define numbers

$$s(n) := \sum_{k=0}^{+\infty} e_k(n) \text{ and } S(N) := \sum_{n=0}^{N-1} s(n).$$

Lemma 29. *If* $1 \leq n \leq 2^m$, *then*

$$T\left(\frac{n}{2^m}\right) - T\left(\frac{n}{2^m} - \frac{1}{2^m}\right) = \frac{m - 2s(n-1)}{2^m}.$$

Proof. Case a. n is odd. Let $n = 2^{\alpha_1^*} + 2^{\alpha_2^*} + \cdots + 2^{\alpha_k^*}$, with $0 = \alpha_1^* < \alpha_2^* < \cdots < \alpha_k^*$. Hence, we can write

$$\frac{n}{2^m} = \frac{1}{2^{\alpha_1}} + \frac{1}{2^{\alpha_2}} + \cdots + \frac{1}{2^{\alpha_{k-1}}} + \frac{1}{2^m}$$

(where $m = \alpha_k$); and it inmediatly follows that

$$T\left(\frac{n}{2^m}\right) - T\left(\frac{n}{2^m} - \frac{1}{2^m}\right) = \frac{\alpha_k - 2(k-1)}{2^{\alpha_k}} = \frac{m - 2s(n-1)}{2^m}.$$

Case b. n is even. Let $n = 2^{t+\alpha_1^*} + 2^{t+\alpha_2^*} + \cdots + 2^{t+\alpha_k^*}$, with $0 = \alpha_1^* < \alpha_2^* < \cdots < \alpha_k^*$. Now,

$$\frac{n}{2^m} = \frac{1}{2^{m-t-\alpha_k^*}} + \frac{1}{2^{m-t-\alpha_{k-1}^*}} + \cdots + \frac{1}{2^{m-t-\alpha_1^*}}$$

and

$$\begin{aligned} \frac{n}{2^m} - \frac{1}{2^m} &= \frac{1}{2^{m-t-\alpha_k^*}} + \frac{1}{2^{m-t-\alpha_{k-1}^*}} + \cdots + \frac{1}{2^{m-t-\alpha_2^*}} \\ &+ \frac{1}{2^{m-t-\alpha_1^*+1}} + \frac{1}{2^{m-t-\alpha_1^*+2}} + \cdots + \frac{1}{2^m} \end{aligned}$$

give

$$T\left(\frac{n}{2^m}\right) - T\left(\frac{n}{2^m} - \frac{1}{2^m}\right) = \frac{m-t-2(k-1)}{2^{m-t}}$$

$$-\frac{m-t+1-2(k-1)}{2^{m-t+1}} - \frac{m-t+2-2k}{2^{m-t+2}} - \cdots - \frac{m-2(k+t-2)}{2^m}$$

Theorem 30

Proof. By the

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and by induction:

T

Because, $T(0) = ($

Denote $\{x\} :=$

we have:

Corollary 31

$S(n) = \frac{n\,l}{}$

Proof. Let us

It is possible to de

the class of Takag

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been studied, am

theorem is true a

$\{TW_n : n \in \mathbb{N}\}$.

Theorem 32

where $r_k :=$ Card

sum of arithmetic
formula for one of
ary expansion for

2^n, with $e_k(n) \in$

.

ith $0 = \alpha_1^* < \alpha_2^* <$

$\frac{s\,(n-1)}{2^m}.$

ith $0 = \alpha_1^* < \alpha_2^* <$

$\overline{\alpha_1^*}$

$\frac{1}{-t-\alpha_2^*}$

$+\frac{1}{2^m}$

$1)$

$\frac{2\,(k+t-2)}{2^m}$

$$= \frac{m-2\,(k+t-1)}{2^m} = \frac{m-2s\,(n-1)}{2^m}.$$ ☐

Theorem 30. (Generalised Trollope's Formula) *If* $1 \leq n \leq 2^m$, *then*

$$S(n) = \frac{nm}{2} - 2^{m-1} T\left(\frac{n}{2^m}\right).$$

Proof. By the lemma above,

$$T\left(\frac{n}{2^m}\right) = \frac{m-2s\,(n-1)}{2^m} + T\left(\frac{n}{2^m} - \frac{1}{2^m}\right);$$

and by induction:

$$T\left(\frac{n}{2^m}\right) = \frac{nm}{2^m} - \frac{s(1)+...+s(n-1)}{2^{m-1}} + T(0).$$

Because, $T(0) = 0$, the result follows. ☐

Denote $\{x\} := x - [x]$, and with the notations we have already introduced, we have:

Corollary 31. (Trollope's Formula)

$$S(n) = \frac{n \log_2 n}{2} + \frac{n\,(1-\{\log_2 n\})}{2} - n2^{-\{\log_2 n\}} T\left(\frac{1}{2^{1-\{\log_2 n\}}}\right).$$

Proof. Let us take $m = 1 + [\log_2 n]$ in the theorem above. ☐

7. Ending Proposal

It is possible to derive consequences for a more general framework if we consider the class of Takagi-van der Waerden peculiar functions:

$$TW_n(x) := \sum_{k=0}^{+\infty} \frac{d(n^k x)}{n^k} \qquad (n \in \mathbb{N})$$

(the second author found TW_{10} during 1930). This family of functions has been studied, among others, by [2], [3] and [8]. For even naturals n, the next theorem is true and it is possible to be applied in further studies of the family $\{TW_n : n \in \mathbb{N}\}$.

Theorem 32. *If* $x = \sum_{k=1}^{+\infty} \frac{x_k}{n^k} \in [0,1]$, *with* $n \in 2\mathbb{N}$, *then*

$$TW_n(x) = \sum_{k=1}^{+\infty} \frac{(2r_k - k)\,x_k - n\,(k - r_k)}{n^k},$$

where $r_k := \mathrm{Card}\{x_j \in \{0,1,2,...,\frac{n}{2}-1\} : j = 1,2,...,k\}$.

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