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ABSOLUTE CONTINUITY THEOREMS FOR ABSTRACT RIEMANN INTEGRATION

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Abstract. Absolute continuity for functionals is studied in the context of proper and abstract Riemann integration examining the relation to absolute continuity for finitely additive measures and giving results in both directions: integrals coming from measures and measures induced by integrals.

To this end, we look for relations between the corresponding integrable functions of absolutely continuous integrals and we deal with the possibility of preserving absolute continuity when extending the elemental integrals.

Keywords: finitely additive integration, abstract Riemann integration, absolute continuity

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1. Introduction

It is well known that there are two classical ways of developing an Integration Theory:

On the one hand, there is the set theoretic starting point, which we will denote as (μ/Ω): X is a non empty set, Ω is a σ-algebra of the power set of X and μ is a measure on Ω. In this context, standard and classical methods lead to the $L_1(\Omega,\mu)$ class of the Lebesgue integrable functions (see [11]).

On the other hand, there exists a functional setting which we will denote as (I/B): The starting point here is a Daniell Loomis system, that is a triple (X,B,I) where B is a vector lattice of real functions defined on X and I is a Daniell integral on B ($\{h_n\}\subseteq B$, $h_n\downarrow 0\Rightarrow I(h_n)\to 0$). In this case we get the corresponding class $L_1(B,I)$ of Daniell integrable functions. For a recent account of the functional extension procedures we refer the reader to [6].

Given (X, Ω, μ) with μ a finite finitely additive measure and Ω a ring, we call (X, B_Ω, I_μ) the *induced Loomis system*, where B_Ω is the vector lattice of μ-simple functions,

$$B_\Omega := \left\{ h \in \mathbb{R}^X : \ h = \sum_{i=1}^{n} a_i \chi_{A_i}, \ a_i \in \mathbb{R}, \ A_i \in \Omega, \ \mu([h \neq 0]) < +\infty \right\},$$

and I_μ is its canonical elemental integral given by

$$I_\mu(h) := \sum_{i=1}^{n} a_i \mu(A_i), \ \forall h \in B_\Omega.$$

3. Proper and abstract Riemann integration

Let (X, B, I) be a Loomis system. For $f \in \overline{\mathbb{R}}^X$, following Loomis in [14] we define by

$$I^-(f) := \inf\{I(h) : \ h \in B, h \geqslant f\},$$
$$I^+(f) := \sup\{I(h) : \ h \in B, h \leqslant f\}$$

the corresponding upper and lower integrals of f, which verify $-\infty \leqslant I^+(f) \leqslant I^-(f) \leqslant +\infty$, $\forall f \in \overline{\mathbb{R}}^X$, I^- is subadditive, I^+ is superadditive, and both are positively homogeneous.

The class of the *properly Riemann integrable functions* is defined by

$$R_{\text{prop}}(B, I) := \{f \in \mathbb{R}^X : \ I^+(f) = I^-(f) \in \mathbb{R}\},$$

or, equivalently, by

$$R_{\text{prop}}(B, I) = \{f \in \mathbb{R}^X : \forall \varepsilon > 0, \ \exists h, g \in B, \ h \leqslant f \leqslant g \text{ and } I(g - h) < \varepsilon\}$$

and it is a vector lattice where the functional $I := I^+ = I^-$ is linear and increasing, i.e., it is an integral which extends the original I.

For this class there are no satisfactory Lebesgue convergence type theorems to make a consistent Integration Theory. Therefore, it is necessary to introduce a "local convergence" to ensure this kind of results.

The *local I-convergence* for sequences of functions $\{f_n\}$ in $\overline{\mathbb{R}}^X$ to a function f in $\overline{\mathbb{R}}^X$, denoted by $\{f_n\} \longrightarrow f(I^-)$, means that $\{I^-(|f_n - f| \wedge h)\} \to 0$, $\forall h \in +B$, and it

Definition 3.1. A Loomis system (X, B, I) is called $C_{+\infty}$ or *upper continuous* if

$$\lim_{r \to +\infty} I^*(f - f \wedge r) = 0, \ \forall f \in +B.$$

Upper continuity on B is hereditary for the class $R_1(B, I)$; that is:

Lemma 3.2. *If (X, B, I) is $C_{+\infty}$, then so is $R_1(B, I)$.*

In general $R_1(B, I)$ need not be closed under multiplication, but we will use the following two facts which can be easily checked.

Lemma 3.3. *If $BB \subseteq B$, $f \in R_1(B, I)$ and $k \in B$ is bounded, then $fk \in R_1(B, I)$.*

Lemma 3.4. *If (X, B, I) is a $C_{+\infty}$ Loomis system and h and χ_A are in $R_1(B, I)$ then so is $h\chi_A$.*

There are three basic theorems to obtain a good Measure and Integration Theory: Lebesgue, Fubini and Radon-Nikodym type theorems. For the class $R_1(B, I)$, Lebesgue theorems were given by Díaz-Carrillo and Muñoz-Rivas in [9] and Fubini type theorems were found by de Amo and Díaz-Carrillo in [3]. Partial attempts in order to obtain Radon-Nikodym type theorems were done by de Amo, Chiţescu and Díaz-Carrillo (see [1] and [2]). We will now study the notion of absolute continuity in this functional setting of proper and abstract integration and its relations to the notion of absolute continuity for finitely additive measures.

4. Absolute continuity

We recall that, given two finitely additive measures μ and ν on a ring Ω, ν is said to be *absolutely continuous* with respect to μ, and is denoted by $\nu \ll \mu$, if

$$\forall \, \varepsilon > 0, \ \exists \, \delta > 0 \colon \ A \in \Omega, \ \mu(A) < \delta \Rightarrow \nu(A) < \varepsilon$$

(see Bochner [5, p. 778], Fefferman [12, p. 35], Dunford-Schwartz [11, p. 131]).

This definition clearly implies the classical one,

$$\mu(A) = 0 \ \Rightarrow \ \nu(A) = 0, \ \forall A \in \Omega,$$

and both are, in fact, equivalent when μ and ν are measures such that $\nu(A) < +\infty$ for all $A \in \Omega$ with $\mu(A) < +\infty$.

Theorem 4.4. *Let μ and ν be finitely additive measures such that $\nu(A) < +\infty$ for all $A \in \Omega$ with $\mu(A) < +\infty$. If $\nu \ll \mu$ then $I_\nu \ll I_\mu$.*

Proof. Assume that $\nu \ll \mu$, let $\varepsilon > 0$ and $f \in +B_\Omega$. There are $a_i > 0$ and pairwise disjoint $A_i \in \Omega$ such that $f = \sum_{i=1}^{n} a_i \chi_{A_i}$. Set $A := \bigcup_{i=1}^{n} A_i \in \Omega$ and $\beta := \sup\{a_i\colon i = 1, \dots, n\} > 0$. Note that $\mu(A) < +\infty$, since $\mu([f \neq 0]) < +\infty$.

If $\nu(A) = 0$, then $I_\nu(f) \leqslant \beta\nu(A) = 0$ and therefore $I_\nu(h) \leqslant I_\nu(f) = 0 < \varepsilon$, $\forall h \in +B_\Omega$ with $h \leqslant f$.

Assume that $\nu(A) > 0$ and let $\alpha := \frac{1}{2}\varepsilon/\nu(A) > 0$. Since $\nu \ll \mu$, there exists $\varrho > 0$ such that

$$\forall E \subseteq \Omega,\ \mu(E) < \varrho \Rightarrow \nu(E) < \frac{\varepsilon}{2\beta}.$$

Let $\delta := \alpha\varrho > 0$ and $h \in +B_\Omega$ with $h \leqslant f$ and $I_\mu(h) < \delta$. There are $e_i > 0$ and pairwise disjoint $E_i \in \Omega$ such that $h = \sum_{j=1}^{m} e_j \chi_{E_j}$. Moreover, since $h \leqslant f$ we have $E := \bigcup_{j=1}^{m} E_j \subseteq \bigcup_{i=1}^{n} A_i = A$ and $e_j \leqslant \beta$, $\forall j = 1, \dots, m$.

Let us now consider sets

$$T := \{t \in \mathbb{N}\colon 1 \leqslant t \leqslant m,\ e_t < \alpha\},$$
$$S := \{t \in \mathbb{N}\colon 1 \leqslant t \leqslant m,\ e_t \geqslant \alpha\},$$

which are disjoint with $S \cup T = \{1, \dots, m\}$, and define functions

$$h_1 := \sum_{t \in T} e_t \chi_{E_t} \quad \text{and} \quad h_2 := \sum_{s \in S} e_s \chi_{E_s}.$$

Evidently $h_1, h_2 \in +B_\Omega$ and $h = h_1 + h_2$. Furthermore,

$$I_\nu(h_1) < \alpha \sum_{t \in T} \nu(E_t) \leqslant \alpha\nu(A) = \frac{\varepsilon}{2}$$

and an easy computation shows that $\mu\Big(\bigcup_{s \in S} E_s\Big) < \delta/\alpha = \varrho$. Hence, $\nu\Big(\bigcup_{s \in S} E_s\Big) < \frac{1}{2}\varepsilon/\beta$ and, consequently,

$$I_\nu(h_2) = \sum_{s \in S} e_s \nu(E_s) \leqslant \beta\nu\Big(\bigcup_{s \in S} E_s\Big) < \frac{\varepsilon}{2}.$$

Therefore $I_\nu(h) = I_\nu(h_1) + I_\nu(h_2) < \varepsilon$, which completes the proof. □

(ii) Assume that $J \ll I$ and let $\varepsilon > 0$. Given $f \in +R_{\mathrm{prop}}(B, I)$ we can take $k_\varepsilon = k_\varepsilon(f) \in +B$ such that $f \leqslant k_\varepsilon$. I-continuity of J gives $\delta = \delta(\varepsilon, f) > 0$ such that

$$\forall\, h \in +B,\ h \leqslant k_\varepsilon,\ I(h) < \delta \Rightarrow J(h) < \varepsilon.$$

Let $\varrho := \frac{1}{2}\delta$. Given $g \in +R_{\mathrm{prop}}(B, I)$, $g \leqslant f$ with $I(g) < \varrho$ there are $h_\delta, k_\delta \in +B$ with $h_\delta \leqslant g \leqslant k_\delta$ and $I(k_\delta - h_\delta) < \varrho$.

Taking $h_\varepsilon := k_\varepsilon \wedge k_\delta \in +B$, we have $g \leqslant k_\varepsilon \wedge k_\delta = h_\varepsilon$ and

$$I(h_\varepsilon) \leqslant I(k_\delta) \leqslant I(k_\delta - h_\delta) + I(h_\delta) < \varrho + I(g) < \varrho + \varrho = \delta.$$

Therefore, we deduce that $J(h_\varepsilon) < \varepsilon$ and hence

$$J(g) = J^-(g) = \inf\{J(h)\colon\ h \in +B,\ g \leqslant h\} \leqslant J(h_\varepsilon) < \varepsilon,$$

that is, $J \ll I$ on $R_{\mathrm{prop}}(B, I)$. □

We are able to give a first sufficient condition for finitely additive measures induced by absolutely continuous integrals to be absolutely continuous.

Given two positive functionals I and J, let $(X, \Omega(I), \mu_I)$ and $(X, \Omega(J), \nu_J)$ be their respective finitely additive measure induced spaces, that is,

$$\Omega(I) = \{A \subseteq X\colon\ \chi_A \in R_1(B, I)\},\quad \mu_I(A) = I(\chi_A),\ \forall A \in \Omega,$$
$$\Omega(J) = \{A \subseteq X\colon\ \chi_A \in R_1(B, J)\},\quad \nu_J(A) = J(\chi_A),\ \forall A \in \Omega.$$

Proposition 5.5. *If $1 \in R_{\mathrm{prop}}(B, I)$ and $J \ll I$, then $\nu_J \ll \mu_I$ (on $\Omega(I) \cap \Omega(J)$).*

Proof. Since $J \ll I$, Theorem 5.4 says that $J \ll I$ on $R_{\mathrm{prop}}(B, I) \subseteq R_{\mathrm{prop}}(B, J)$. Thus, for $\varepsilon > 0$ and $1 \in R_{\mathrm{prop}}(B, I)$ there exists $\delta > 0$ such that

$$\forall h \in R_{\mathrm{prop}}(B, I) \text{ with } h \leqslant 1 \text{ and } I(h) < \delta \Rightarrow J(h) < \varepsilon.$$

Given $A \in \Omega(I) \cap \Omega(J)$ with $\mu_I(A) < \delta$ we have $\chi_A \wedge h \in R_{\mathrm{prop}}(B, I), \forall h \in +B$, $\chi_A \wedge h \leqslant 1$ and $I(\chi_A \wedge h) \leqslant I(\chi_A) = \mu_I(A) < \delta$.

Therefore, it follows that $J(\chi_A \wedge h) < \varepsilon$, $\forall h \in +B$ and, keeping in mind that $\chi_A \in +R_1(B, J)$, we conclude that

$$\nu_J(A) = J(\chi_A) = J_l^-(\chi_A) = \sup\{J^-(\chi_A \wedge h)\colon\ h \in +B\} < \varepsilon.$$

□

In the following section, the condition $1 \in R_{\mathrm{prop}}(B, I)$ will be relaxed to $1 \in R_1(B, I)$ and $\Omega(I) \cap \Omega(J)$ will be, in fact, $\Omega(I)$ (see Corollary 6.9).

At this point, since absolute continuity has a good behaviour with respect to local convergence, one can expect that if J is absolutely I-continuous then $R_1(B,I) \subseteq R_1(B,J)$, but this is not, in general, true.

Example 6.4. Let $X :=]0,1]$, let Ω be the ring generated by the semi-ring $\{]a,b]\colon\ 0 < a < b < 1\}$, let $B := B_\Omega$ be the vector lattice of all Ω-simple functions and I its canonical elemental integral.

Consider the function f defined by

$$f(x) := \sum_{n=1}^{+\infty} n\chi_{]1/(n+1)^2, 1/n^2]}, \forall x \in]0,1],$$

and the linear functional $J\colon\ B \longrightarrow \mathbb{R}$ given by $J(h) := I(fh)$, $\forall h \in B$.

Let us see that $f \in R_1(B,I)$. Since $f \wedge h \in B \subseteq R_{\text{prop}}(B,I)$ for all $h \in B$, we only have to check that $I^+(f) < +\infty$. To see this, let $I_k =]1/(k+1)^2, 1/k^2]$ for each $k \in \mathbb{N}$ and consider the functions $h_n := \sum\limits_{k=1}^{n} k\chi_{I_k}$. It is easy to check that for each $h \in +B$ with $h \leqslant f$, there exists $m \in \mathbb{N}$ such that $h \leqslant h_m + m\chi_{]0,1/(m+1)^2]}$.

Therefore, $I^+(f) = \sup\{I(h)\colon\ h \leqslant f, h \in +B\}$ can be bounded in the following way:

$$I^+(f) \leqslant \lim_{m\to+\infty} I(h_m) + \lim_{m\to+\infty} m\, I(\chi_{]0,1/(m+1)^2]}) \leqslant \sum_{k=1}^{+\infty} \frac{2k^2+k}{k^2(k+1)^2} < +\infty.$$

Since $f \in R_1(B,I)$, the functions in B are bounded and $BB \subseteq B$, Lemma 3.3 guaranties that J is well-defined.

Moreover, if λ is the Lebesgue measure on X and ν is the measure given by $\nu(A) := \int_A f\,\mathrm{d}\lambda$, both defined on the σ-algebra $\sigma(\Omega)$ generated by Ω, then it is clear that $\nu \ll \lambda$ on $\sigma(\Omega)$ and so, in particular, on Ω. Thus, Theorem 4.4 says that $J \ll I$ on B (since I and J on B are induced by λ and ν on Ω, respectively).

However, $f \notin R_1(B,J)$, since $J^+(f) = I^+(f^2) = +\infty$.

To find the condition under which $R_1(B,I) \subseteq R_1(B,J)$ holds, we have to consider the measurable functions. The characterization (1) of $R_1(B,I)$, given in [9], suggests the following definition of measurability (in the sense of Stone, [16]).

Definition 6.5. The class of *measurable functions* with respect to a Loomis system (X,B,I) (I-measurable functions) is defined by

$$M_1(B,I) := \{f \in \overline{\mathbb{R}}^X\colon\ f^\pm \wedge h \in R_{\text{prop}}(B,I),\ \forall h \in +B\}.$$

Thus, we have that every integrable function is measurable and that every measurable function with $I^+(|f|) < +\infty$ is, in fact, integrable. Moreover, note that we can

Let $g \in +R_1(B, I)$ with $g \leqslant f$ and $I^+(g) < \delta$, and let us prove that $J^+(g) < \varepsilon$. Given $u \in +B$ with $u \leqslant g$, it is clear that $u \wedge h \in +B$, $u \wedge h \leqslant h$ and $I(u \wedge h) \leqslant I(u) \leqslant I^+(g) < \delta$ and hence $J(u \wedge h) < \frac{1}{4}\varepsilon$.

Thus, $J(u)+J(h) = J(u\wedge h)+J(u\vee h) \leqslant J(u\wedge h)+J^+(f)$ implies that $J(u) < \frac{1}{2}\varepsilon$.

Therefore, $J(u) < \frac{1}{2}\varepsilon$ for all $u \in +B$, $u \leqslant g$; that is, $J^+(f) \leqslant \frac{1}{2}\varepsilon < \varepsilon$, which gives $J \ll I$ on $R_1(B, I)$. □

We are now in a position to give the announced sufficient conditions for finitely additive measures induced by absolute continuous functionals to be absolutely continuous.

Consider the finitely additive measure space induced by I, that is, (X, Ω, μ_I) with $\Omega = \{A \subseteq X\colon \chi_A \in R_1(B, I)\}$ and $\mu_I(A) = I(\chi_A)$ $\forall A \in \Omega$. Under the assumptions of Corollary 6.7, $R_1(B, I) \subseteq R_1(B, J)$ and, therefore, we can also define the finitely additive measure μ_J on Ω as $\mu_J(A) := J(\chi_A)$, $\forall A \in \Omega$.

Corollary 6.9. *If $1 \in R_1(B, I)$, $J \ll I$ and $J^+(f) < +\infty$ for all $f \in +\overline{\mathbb{R}}^X$ with $I^+(f) < +\infty$, then $\nu_J \ll \mu_I$.*

P r o o f. Theorem 6.8 and Corollary 6.7 say that $J \ll I$ on $R_1(B, I) \subseteq R_1(B, J)$. Thus, for $\varepsilon > 0$ and $1 \in R_1(B, I)$ there exists $\delta > 0$ such that

$$\forall g \in R_1(B, I) \text{ with } g \leqslant 1 \text{ and } I(g) < \delta \Rightarrow J(g) < \varepsilon.$$

Given $A \in \Omega(I) \cap \Omega(J)$ with $\mu_I(A) < \delta$ we have $\chi_A \wedge h \in R_{\text{prop}}(B, I), \forall h \in +B$, $\chi_A \wedge h \leqslant 1$ and $I(\chi_A \wedge h) \leqslant I(\chi_A) = \mu_I(A) < \delta$.

Therefore, it follows that $J(\chi_A \wedge h) < \varepsilon$, $\forall h \in +B$ and, since $\chi_A \in R_1(B, I) \subseteq R_1(B, J)$, we conclude that

$$\nu_J(A) = J(\chi_A) = J_l^-(\chi_A) = \sup\{J^-(\chi_A \wedge h)\colon\ h \in +B\} < \varepsilon.$$

□

Furthermore, assuming that the Loomis system (X, B, I) is $C_{+\infty}$, we are able to obtain the absolute continuity of certain induced finitely additive measures μ_f with respect to the finitely additive measure μ induced by I.

To be more specific, given $f \in +R_1(B, I)$, Lemma 3.4 allows us to define the finitely additive measure μ_f by $\mu_f(A) := I(f\chi_A), \forall A \in \Omega$, where $\Omega = \{A \subseteq X\colon\ \chi_A \in R_1(B, I)\}$. Setting $\mu(A) := I(\chi_A), \forall A \in \Omega$, we can prove that $\mu_f \ll \mu$.

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