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LOCAL AND IMPROPER DANIELL-LOOMIS INTEGRALS

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In this paper we start from previous results obtained in [7] on the abstract space of Daniell-Loomis integrable functions L, which is constructed like to the Daniell extension process, but without continuity assumptions on the elementary integral.

The localized integral is used to prove that L consists of those functions whose local upper and lower integrals are equal and finite, or that L is closed with respect to improper integration.

Our results are also holded in integration with respect to finitely additive measures.

1. Introduction.

The Daniell-Bourbaki integral extension has been generalized with the integral $\overline{I}: \overline{B} \to I\!R$, introduced in [5], starting with any nonnegative linear functional I on a vector lattice B of real-valued functions on X. In [6] an abstract space of integrable functions L is constructed similar to the Daniell L^1, using an appropriate local convergence in measure, which is very useful to obtain convergence theorems in a form analogous to the classical ones, but contrary to that L^1 case, no continuity conditions on the starting elementary integral $I \mid B$, e.g. of Daniell type or "starke" integral norm of [13], are needed. It allows to discuss an unified functional analytic approach to integration, in an abstract Riemann spirit; which subsume previous results obtained by Aumann [4], Loomis [10], Gould [8] and Schäfke [13].

On the other hand, this also leads to treat set-theoretical aspects of integration with respect to finitely additive measures μ on semirings Ω of sets

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(abbreviated $\mu \mid \Omega$). Always, proper Riemann- [6], abstract Riemann- [9] and Dunford-Schwartz [7] μ-integration are subsumed by L. This abstract measure theory is developed by proving Fubini theorems for finitely additive measures [2] and an approximate functional Radon-Nikodym theorem [1].

An important source of information on finitely additive measures is the paper by W.A. Luxemburg [11], which gives an extensive bibliography and treatment of the subject that may be useful in applications.

Since the cornerstone of our approach to integration is the concept of a localized integral, it seem interesting to discuss new characterizations of the abstract space of integrable functions L given in [6].Thus, one obtains L via one of the three classical methods: certain limits of elementary functions, the closure of B with respect to an L-type seminorm; and, in this paper, via equality and finiteness of the localized upper and lower integrals (Theorem 8) and improper integrable functions (Theorem 13).

We recall that the set of the integrable functions L coincides with L^1 in the classical case. Always $\overline{B}$ (summable in [5]) and $R_1(B, I)$ (abstract-Riemann integrable functions in [6]) are contained in L.

For an upper functional in the sense of Anger and Portenier [3], essential integration gives new characterizations of abstract Riemann integration with respect to $I \mid B$. Then, we have in mind future applications to Riesz representation theorems (see [1] and [2]), regularity and Radon integrals. Such as we mentioned before, we already have incorporated to this abstract integration theory Fubini and Radon-Nikodym thorems, which are not treated in [6].

2. General framework. Preliminaires.

Notations and conventions used are similar to that of [5] and [6], and will be explained it whenever be necessary in order to mke the paper self-contained.

We extend the usual $+$ in $\mathbb{R}$ to $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ by $a + b := 0$, $a \dot{+} b := \infty$ if $a = -b \in \{-\infty, \infty\}$; $a - b := a + (-b)$, etc.. With $a \vee b := \max(a, b)$, $a \wedge b := \min(a, b)$, $a \cap b := (a \wedge b) \vee (-b)$ if $b \geq 0$, $a^+ := a \vee 0$, $a^- := (-a)^+$, one has for $a, b, c, d, e \in \overline{\mathbb{R}}$, $s, t \in \overline{\mathbb{R}}_+ := [0, \infty]$, the inequalities

$$|a \cap t - b \cap t| \leq 2(|a - b| \wedge t)$$

(1)
$$|(a + b) - (c + d)| \leq |a - c| + |b - d|$$

$$||a| - |b|| \leq |a - b| \leq |a - c| + |c - d|\,, \qquad a \leq b \dot{+} (a - b)$$

(Aumann [4, *b), *c)]); $+$, $\dot{+}$, $\underset{\cdot}{+}$ are conmutative, $+$ distributive with

$0(\pm\infty) := 0$, but not asociative; $\dot{+}$ is associative and the above inequalities hold for $\dot{+}$.

On the set $\overline{\mathbb{R}}^X$ of functions $f : X \to \overline{\mathbb{R}}$, we define $=, \pm, \dot{+}, \wedge, \vee, \cup, \cdot\alpha$, $|\bullet|$, $\leq$, pointwise on X. Given $M \subset \overline{\mathbb{R}}^X$, $+M := \{f \in M; \quad f \geq 0\}$ and for an arbitrary functional q on $\overline{\mathbb{R}}^X$, q_* denotes the functional defined on $\overline{\mathbb{R}}^X$ by $q_*(f) := -q(-f)$.

In all that follows, B will be a function vector lattice (or Riesz space) $\subset \mathbb{R}^X$ and $I : B \to \mathbb{R}$, a linear functional with $I(h) \geq 0$ for $h \in +B$.

For such $I \mid B$ context, we need the following results of [5] and [6], in somewhat modified notation:

(2)
$$
\begin{aligned}
&B^\tau := \sup\{M; \emptyset \neq M \subset B\}\\
&I^+(f) := \sup\{I(h); h \in B, h \leq f\}, \text{ for } f \in \overline{\mathbb{R}}^X, \text{ with } \sup\emptyset := -\infty\\
&B_\tau := \{g \in B^\tau; I^+(f+g) = I^+(f) + I^+(g), \text{ for all } f \in B^\tau\}\\
&\overline{I}(f) := \inf\{I^+(g); f \leq g \in B_\tau\}, \qquad \underline{I}(f) := (\overline{I})_*(f), \text{ for } f \in \overline{\mathbb{R}}^X
\end{aligned}
$$

The elements of

$$\overline{B} := \{f \in \overline{\mathbb{R}}^X; \overline{I}(f) = \underline{I}(f) \in \mathbb{R}\}$$

are called I-*summable functions.*

B^τ and B_τ are $+$ and $\vee$-closed, B_τ is also $\wedge$-closed. $\overline{I}$ is -subadditive on $\overline{\mathbb{R}}^X$, $\overline{I}$ and I^+ are $+\mathbb{R}_0$-homogeneous and monotone on $\overline{\mathbb{R}}^X$.

$B_{(\tau)}$ denotes $\{f \in B_\tau; I^+(f) < +\infty\}$. $B_{(\tau)}$ is $\wedge$-closed and $B \subset B_{(\tau)} \cup (-B_{(\tau)}) \subset \overline{B}$. If $f \in B_\tau$, then $I^+(f) = \overline{I}(f) = \underline{I}(f)$.

(3) $\overline{B}$ is closed under $+, \dot{+}, \wedge, \vee, \cdot\alpha, |\bullet|$; $\overline{B}$ is the closure of B in $\overline{\mathbb{R}}^X$ with respect to the integral seminorm $\overline{I}$, $\overline{I} \mid \overline{B}$ is the unique $\overline{I}$-continuous extension of $I \mid B$ to $\overline{B}$ and is "linear" on $\overline{B}$ ([5], [6]).

(4) Using the corresponding definitions, the following result holds: $f \in \overline{B}$ iff for any $\varepsilon > 0$ there exist $h, g \in B_\tau$ such that $-h \leq f \leq g$ and $I^+(g) + I^+(h) < \varepsilon$.

3. Local integrals.

In [7] an abstract integration theory is developed for general integral metrics.

A functional $q : +\overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$ is called an *integral metric* if $q(0) = 0$ and $q(f) \leq q(g) + q(h)$ if $f \leq g + h$, $f, g, h \in +\overline{\mathbb{R}}^X$.

(5) For any $T: \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$ we define the *localization*

$$T_B(f) := \sup\{T(f \wedge h); h \in +B\}$$

for all $f \in \overline{\mathbb{R}}^X$.

This is a simplified version of Schäfke's definition [13, p.120]. If $T = q =$ integral metric, q_B is also an integral metric. In al the following, we assume $q = \overline{I}$ integral metric on $+\overline{\mathbb{R}}^X$. From above definitions, one gets

$$(\overline{I}_B)_*(f) := -(\overline{I}_B)(-f) = \inf\{\underline{I}(f \vee (-h)); h \in +B\}.$$

We have $(\overline{I}_B)_* \le \overline{I}_B \le \overline{I}$ on $\overline{\mathbb{R}}^X$ and $\overline{I}_B(f) < +\infty$. Moreover, if $\overline{I}(f) < +\infty$, then $\overline{I}_B(f) = \overline{I}(f)$. Simple consequences of the definitions are

$$I^+ \le \underline{I}_B \le \underline{I} =: \overline{I}_* \le (\overline{I}_B)_* \le \overline{I} \le (I^+)_*.$$

DEFINITION 1. *The set $L := L(B, I)$ of I-integrable functions is defined as the clousure of B in $\overline{\mathbb{R}}^X$ with respect to the integralmetric $\overline{I}_B(|\bullet|)$.*

(6) As in the proof of Theorem 1.5 of Schäfke [13], one shows that $L(B, I) =$ set of all those $f \in \overline{\mathbb{R}}^X$ for which there exists an $\overline{I}$-Cauchy sequence $(h_i) \subset B$ such that $h_i \to f(\overline{I})$, i.e., $\overline{I}(|f - h_i| \wedge h) \to 0$ for each fixed $h \in +B$. Then $J(f) := \lim I(h_i)$, and (h_i) is called a defining sequence for f (see [6, Sec.2]).

One gets $\overline{B} \subset \overline{L}(B, I)$ and $\overline{I}(f) = J(f)$ for any $f \in \overline{B}$. Also, L is closed with respect to $+$, $\dot{+}$, $\wedge$, $\vee$, $\cdot\alpha$, $|\bullet|$ and J is linear and monotone on L.

In [6], covergence theorems for $L(B, I)$ are given in an analogous form to the classical ones, and various descriptions of the set L have been treated.

Additionaly, we can obtain the following:

1. If $f \in L(B, I)$, $I^+(f) \le \underline{I}(f) = J(f)$
2. $L(B, I) = B_+^* - B_+^*$, where

$$B_+^* := \{f \in +\overline{\mathbb{R}}^X; f \wedge h \in B, \forall h \in B, \underline{I}(f) < +\infty\}.$$

(7) We summarize applications given in [6, Sec.5], in the situation $\mu \mid \Omega : \Omega$ is a semiring of sets from X, and $\mu : \Omega \to [0, +\infty[$ is a finititely additive measure on Ω, $B = B_\Omega :=$ real valuedstep functions over Ω and $I = I_\mu := \int \bullet d\mu$ are admissible.

Then,

$R^1_{prop}(\mu, \mathbb{R})$ (proper Riemann μ-integrable functions in [6]) $\subset$

$$L(X, \Omega, \mu, \mathbb{R}) \text{ (Dunford-Schwartz integrable functions in [8])} \subset$$

$$R_1(\mu, \overline{\mathbb{R}}) \text{ (abstract Riemann } \mu\text{-integrable functions in [9])} \subset L(B_\Omega, I_\mu),$$

with coinciding integrals; all inclusions are strict.

If Ω is a δ-ring and μ is σ-additive, then

$$R_1(\mu, \overline{\mathbb{R}}) = L^1(\mu, \overline{\mathbb{R}}) \text{ (Lebesgue integrable functions modulo nullfunctions in [7])} \subset L(B_\Omega, I_\mu);$$

and $f_n \to f$ μ-almost everywhere implies $f_n \to f$ (I_μ^-) for μ-measurable (f_n).

For X = open sets $\subset \mathbb{R}^n, \Omega$ = intervals, μ =Lebesgue measure on X. $B = C_0(\mathbb{R}^n, \mathbb{R})$ = continuous real valued functions on Ω with compact support, and $I: X \to \Omega$ =the classical Riemann integral on B, one has $\overline{B} = L = L^1$.

The following basic properties, which will be useful in our subsequent studies, are new here. The inequality needed here reads: if $a, b \in \overline{\mathbb{R}}$, $c \in \overline{\mathbb{R}}$, $a \geq b$, $a \geq 0$, then $(b + c) \wedge a = c \wedge (a - b) + b$.

LEMMA 2. *If $f, k \in \overline{\mathbb{R}}^X$, $h \in B$ such that $\overline{I}_B(k) < +\infty$ and $k \leq f + h$, then $\overline{I}_B(k) \leq \overline{I}_B(f) + I(h)$.*

Proof. For every $\varepsilon > 0$ there exists $t_\varepsilon \in +B$ such that $\overline{I}_B(k) - \varepsilon < \overline{I}(k \wedge t_\varepsilon)$. Set $t_\varepsilon \geq h$, now with $t_\varepsilon - h \in +B$ and $k \leq f + h$, we have

$$k \wedge t_\varepsilon \leq (f + h) \wedge t_\varepsilon = f \wedge (t_\varepsilon - h) + h.$$

Therefore, $\overline{I}_B(k) - \varepsilon < \overline{I}(k \wedge t_\varepsilon) \leq \overline{I}(f \wedge (t_\varepsilon - h)) + I(h) \leq \overline{I}_B(f) + I(h)$ for all $\varepsilon > 0$, and the result follows. □

(8) Note that if $f \in L(B, I)$, then $\overline{I}_B(f) < +\infty$.

In fact, one has $f \leq |h_n| + |f - h_n|$ where (h_n) is a defining sequence for f; by (3) and since $\overline{I}_B$ is $+$-subadditive on $+\overline{\mathbb{R}}^X$, the result follows.

The above lemma will be generalized in Proposition 4.

LEMMA 3. *If $f \in L(B, I)$, then $J(f) = \overline{I}(f) = (\underline{I})_*(f) \in \mathbb{R}$.*

Proof. By definition 1, given $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ and $h_n \in B$ such that $\overline{I}_B(|f - f_n|) < \varepsilon$, if $n \geq n_0$.

We have $f \leq h_n + |f - h_n| =: g$, with $g \in L(B, I)$ and $\overline{I}_B(f) < +\infty$. Now, with lemma 2, one gets

$$\overline{I}_B(f) \leq I(h_n) + \overline{I}_B(|f - h_n|) < I(h_n) + \varepsilon.$$

Furthermore, since $f - h_n \geq -|f - h_n|$, lemma 2 yields

$$\overline{I}_B(-|f - h_n|) \leq \overline{I}_B(f - h_n) \leq \overline{I}_B(f) - I(h_n).$$

Besides,

$$(\overline{I}_B)_*(|f - h_n|) := -\overline{I}_B(-|f - h_n|) \leq \overline{I}_B(|f - h_n|) < \varepsilon,$$

so that

$$-\varepsilon < \overline{I}_B(-|f - h_n|) < \overline{I}_B(f) - I(h_n), \text{ if } n \geq n_0.$$

Thus,

$$J(f) := \lim I(h_n) = \overline{I}_B(f) \in \mathbb{R}.$$

Finally,

$$(\overline{I}_B)_*(f) := -(\overline{I}_B)(-f) = -J(-f) = J(f). \qquad \square$$

Note that the inequality $(f + g) \wedge h \leq f \wedge h + g \wedge h$ is always valid for $f, g \in +\overline{\mathbb{R}}^X$; so, $\overline{I}_B$ is subadditive on $+\overline{\mathbb{R}}^X$, i.e., an integral metric.

For arbitrary functions $f \in \overline{\mathbb{R}}^X$, the following additional properties of $\overline{I}_B$, extending those in Lemma 2, can be given.

PROPOSITION 4. *For a given function* $f \in \overline{\mathbb{R}}^X$,

1. *If* $h \in B$, *we have* $\overline{I}_B(f + h) \leq \overline{I}_B(f) + I(h)$.
2. *If* $\overline{I}_B(f) < +\infty$ *and* $g \in L(B, I)$, *we have* $\overline{I}_B(f \dot{+} g) \leq \overline{I}_B(f) + \overline{I}_B(g)$.

Proof: 1. It is clear that if $\overline{I}_B(f + h) = +\infty$ (so, $\overline{I}_B(f) = +\infty$) or $= -\infty$, then $\overline{I}_B(f + h) \leq \overline{I}_B(f) + I(h)$.

Now, suppose that $\overline{I}_B(f + h) < +\infty$. For an arbitrary $\varepsilon > 0$, there exists $k \in +B$, $k \geq h$, such that

$$\overline{I}_B(f + h) - \varepsilon < \overline{I}\,((f + h) \wedge k) = \overline{I}\,(f \wedge (k - h) + h) \leq$$

$$\overline{I}\,(f \wedge (k - h)) + I(h) \leq \overline{I}_B(f) + I(h),$$

so that

$$\overline{I}_B(f + h) \leq \overline{I}_B(f) + I(h).$$

2. For $g \in L(B, I)$, there exists $h_n \in B$ such that $\overline{I}_B(|g - h_n|) \to 0$. Now, $I_B(g - h_n) < +\infty$, $I_B(g_n) < +\infty$, by 1. and remark below,

$$\left|\overline{I}(g) - I(h_n)\right| \leq \overline{I}(|g - h_n|) \to 0,$$

so $I(h_n) \to \overline{I}(g)$.

Since $|f + h_n| \leq |f + g| + |g - h_n|$ and $\overline{I}_B$ is subadditive on $\overline{\mathbb{R}}^X$, we have

$$\overline{I}_B(f + h_n) \leq \overline{I}_B(f + g) + \varepsilon,$$

and with $|f + g| \leq |f + h_n| + |h_n - g|$,

$$\overline{I}_B(f + g) \leq \overline{I}_B(f + h_n) + \varepsilon,$$

the result follows. □

Observe that, with $\overline{I}_B(f) < +\infty$ and the above reasoning in 1., one gets $\overline{I}_B(f) \leq \overline{I}_B(f + h) - I(h)$, so that,

$$\overline{I}_B(f + h) = \overline{I}_B(f) + I(h).$$

The proof follows the same arguments of lemma 2 and those of remark (8).

DEFINITION 5. *(Stone) A function* $f \in \overline{\mathbb{R}}^X$ *is called* I*-measurable if* $f \cap h \in L(B, I)$ *for all* $h \in +B$*. Obviously,* $\overline{B} \subset L \subset M_\cap := \{f \in \overline{\mathbb{R}}^X; \quad f$ I*-measurable*$\}$.

In [7], the following results are given:

1. f *is* I*-measurable and* $|f| \leq$ *some* I*-integrable* g*, implies* f *is* I*-integrable.*
2. $f \in M_\cap$ *iff* $f^{\pm} \in M_\cap$.

The concept of -measurability enable to give, with [6, th.3], the following **Integrability Criterion**:

(9) $f \in L(B, I)$ iff f is I-measurable and $\overline{I}_B(|f|) < +\infty$.
(Note that $\overline{I}$ is aditive on B, so B-semiadditive.)

PROPOSITION 6. *If* $f \in +\overline{\mathbb{R}}^X$ *is* I*-measurable with* $\underline{I}(f) \in \mathbb{R}$*, then there exist* $(g_n) \subset +\overline{B}$*,* $g_n \leq g_{n+1} \leq f$*,* I*-Cauchy and* $g_n \to f(\overline{I})$.

Proof. By (2), there exist g and (g_n) in $-B_{(\tau)} \subset \overline{B}$, $g_n \leq g_{n+1} \leq g \leq f$, $\overline{I}(g_n) \to \sup \overline{I}(g) = \underline{I}(f) \in \mathbb{R}$.

Then, $\overline{I}(|g_n - g_m|) = \overline{I}(g_n) - \overline{I}(g_m) < \varepsilon$, if $n \geq m \geq n_0(\varepsilon)$, so (g_n) is $\overline{I}$-Cauchy.

Now, if not $g_n \to f(\overline{I})$, by (6), there exist $h_0 \in +B$, $\delta_0 > 0$, $n_k \nearrow +\infty$, such that $\overline{I}((f - g_{n_k}) \wedge h_0) \geq \delta_0$, $k \in \mathbb{N}$.

We have $(f - g_{n_k}) \wedge h_0 \in \overline{B}$, so there exists $l_k \in \overline{B}$ such that $(f - g_{n_k}) \wedge h_0 \geq l_k \geq 0$ and $\overline{I}(l_k) \geq \dfrac{\delta_0}{2}$.

Then, $\overline{I}(g_{n_k}) + \overline{I}(l_k) \leq \underline{I}(f)$, but $\overline{I}(g_{n_k}) \to \overline{I}(f)$, which implies contradiction with $\overline{I}(l_k) \geq \dfrac{\delta_0}{2} > 0$, $k \in$ □

LEMMA 7. *If* $f \in \overline{\mathbb{R}}^X$ *is such that* $(\overline{I}_B)_*(f) = \overline{I}_B(f) \in \mathbb{R}$, *then* $f^{\pm}$ *and* f *are* I*-measurable.*

Proof. Let $h_0 \in +B$. For a given $\varepsilon > 0$, there exists $h_1 \in +B$ such that, with $h_1 \geq h_0$, $\overline{I}_B(f) - \varepsilon < \overline{I}(f \wedge h_1) \leq \overline{I}_B(f)$.

Now, for $h_1 \in +B$, there exists $h_2 \in B_{(+)}$ such that $f \wedge h_1 \leq h_2$ and $I(h_2) < \overline{I}(f \wedge h_1) + \varepsilon \leq \overline{I}_B(f) + \varepsilon$; one can assume $h_2 \leq h_1$ since $B \subset B_{(+)} \subset \overline{B}$ and $B_{(+)}$ is $\wedge$-closed. Then

$$\left|I(h_2) - \overline{I}_B(f)\right| < \varepsilon \qquad (\underline{1})$$

For h_0, h_1, there exists $-k_1 \in +B$, $k_1 \leq -(h_0 \vee h_1 \vee |h_2|)$, such that

$$\left(\overline{I}_B\right)_*(f) \leq \underline{I}(f \vee k_1) < \overline{I}_B(f) + \varepsilon.$$

Now, for k_1, there exists $k_2 \in -B_{(+)} \subset \overline{B}$, $k_2 \leq f \vee k_1$, with $k_1 \leq h_2$, such that $\underline{I}(k_2) > \underline{I}(f \vee k_1) - \varepsilon$. Then

$$\left|\underline{I}(k_2) - \left(\overline{I}_B\right)_*(f)\right| < \varepsilon \qquad (\underline{2})$$

Finally, for $h_1, k_1, k_2 \in \overline{B}$, there exists $h_3 \in \overline{B}$, $h_3 \geq h_1 \vee k_1 \vee k_2$, such that

$$\overline{I}_B(f) - \varepsilon < \overline{I}(f \wedge h_3) \leq \overline{I}_B(f),$$

and for h_3, there exists $h_4 \in B_{(+)}$ such that $f \wedge h_3 \leq h_4 \leq h_3$ and

$$\overline{I}_B(f) - \varepsilon < \overline{I}(f \wedge h_3) \leq \overline{I}(h_4) < \overline{I}(f \wedge h_3) + \varepsilon < \overline{I}_B(f) + \varepsilon,$$

so that

$$\left|\overline{I}(h_4) - \overline{I}_B(f)\right| < \varepsilon \qquad (\underline{3})$$

One gets

$$|h_4 - k_2| \leq h_4 - k_2 + 2\rho, \text{ with } \rho := h_4 \vee k_2 - h_4 \qquad (\underline{4})$$

and

$$f + \rho \leq f \vee k_1 \qquad (\underline{5})$$

By lemma 2, $\overline{I}_B$ applied to ($\underline{5}$), with $\left(\underline{2}\right)$, yields to

$$\left(\overline{I}_B\right)_*(f) + \overline{I}(\rho) \leq \left(\overline{I}_B\right)_*(f+\rho)$$
$$\leq \left(\overline{I}_B\right)_*(f \vee k_1) \leq \underline{I}(f \vee k_1) \leq \left(\overline{I}_B\right)_*(f) + \varepsilon;$$

hence,

$$0 \leq \overline{I}(\rho) < \varepsilon. \qquad (\underline{6})$$

Moreover, one verifies by checking cases,

$$\left|h_4^+ \cap h_0 - f^+ \cap h_0\right| \leq |h_4 - h_2|\,. \qquad (\underline{7})$$

Next, let $l := h_4^+ \cap h_0 \in B_{(+)} \subset \overline{B}$; and with $\left(\underline{4}\right)$, $\left(\underline{7}\right)$, $\left(\underline{6}\right)$, $\left(\underline{2}\right)$ and $\left(\underline{3}\right)$, one gets Therefore, since B is $\overline{I}$-dense in $\overline{B}$, we conclude that $f^+ \cap h_0 \in \overline{B}$, for all $h_0 \in +B$, hence f^+ is I-measurable.

For f^- it is enough to consider that $f^- := (-f)^+$, and the previous facts for positive functions. Since, for an arbitrary function f we have $f \cap h_0 = f^+ \wedge h_0 - f^- \wedge h_0 \in \overline{B}$, for all $h_0 \in +B$, the I-measurability of f follows. □

The integrability criterion (6), together Lemma 7, allows to us to show the following characterization of I-integrable functions (the upper and lower localized integrals are equal and finite).

THEOREM 8. *A function* $f \in \overline{\mathbb{R}}^X$ *is* I*-integrable iff* $(\overline{I}_B)_*(f) = \overline{I}_B(f) \in \mathbb{R}$.

Proof. Lemma 3 gives the sufficiency. To prove the necessity, with (6) and Lemma 7, we only have to prove that $\overline{I}_B(|f|) < +\infty$.

For $0 \leq h \leq l \in B$, one has, with $f^+ \wedge l$, $f^+ \wedge h \in \overline{B} \subset L$,

$$f \wedge l = f \wedge h + (f^+ \wedge l - f^+ \wedge h). \qquad (\underline{1})$$

First, we claim that $\overline{I}_B(f^\pm) < +\infty$. If $\overline{I}_B(f) < +\infty$, for a given $\varepsilon > 0$, there exists $h_\varepsilon \in +B$ such that $\overline{I}_B(f) - \overline{I}(f \wedge h_\varepsilon) < \varepsilon$.

Let $h := h_\varepsilon$, $0 \leq h_\varepsilon \leq l$, then

$$\overline{I}(f \wedge h_\varepsilon) \leq \overline{I}(f \wedge l) \leq \overline{I}_B(f).$$

Next, $\overline{I}$ applied to $\left(\underline{1}\right)$ gives

$$\overline{I}(f \wedge l) = \overline{I}(f \wedge h_\varepsilon) + \overline{I}(f^+ \wedge l) - \overline{I}(f^+ \wedge h_\varepsilon) \leq \overline{I}_B(f),$$

so that,

$$\overline{I}(f^+ \wedge l) < \overline{I}(f^+ \wedge h_\varepsilon) + \varepsilon < +\infty$$

for all $l \geq h_\varepsilon$, hence $\overline{I}_B(f^+) < +\infty$, so $f^+ \in L(B, I)$.

Because analogously $f^- \in L(B, I)$, we have

$$\overline{I}_B(|f|) \leq \overline{I}_B(f^+) + \overline{I}_B(f^-) < +\infty,$$

and therefore $f \in L(B, I)$. □

For any $f \in \overline{\mathbb{R}}^X$, the *lower and upper Darboux integrals* are defined as in [6, Def.4]:

$$J_*(f) := \sup\{J(g);\, g \leq f,\, g \in L(B, I)\}$$

and $J^*(f) := -J_*(-f)$. One check easily that J^* is an integral metric on $\overline{\mathbb{R}}^X$.

With Theorem 8 and [6, Th.4], I-integrability can be characterized in its more general form (as in the classical cases), without any measurability assumptions:

COROLLARY 9. *For any $f \in \overline{\mathbb{R}}^X$, the followig conditions are equivalent:*

1. $f \in L(B, I)$
2. $(\overline{I}_B)_*(f) = \overline{I}_B(f) \in \mathbb{R}$
3. $J^*(f) = J_*(f) \in \mathbb{R}$.

In this case, $J(f)$ coincides with all the above integrals.

We conclude this section with a more general sufficient condition for I-integrability, which is directly proved using (4).

PROPOSITION 10. *For $f \in \overline{\mathbb{R}}^X$, if $\underline{I}_B(f) = \overline{I}_B(f) \in \mathbb{R}$, then $f \in L(B, I)$ and, in this case, $J(f) = \underline{I}(f)$.*

Proof. Let $h_0 \in +B$ and $\varepsilon > 0$. By (5),there are (g_n), $(h_n) \subset +B$ such that

$$\overline{I}_B(f) - \varepsilon < \overline{I}(f \wedge g_n) < \overline{I}_B(f) + \varepsilon$$

and

$$\underline{I}_B(f) - \varepsilon < \underline{I}(f \wedge h_n) < \underline{I}_B(f) + \varepsilon.$$

One can assume $h_n = g_n$ and $h_n \geq h_0$ (take $h_n \vee g_n$, $h_n \wedge h_0$). By (2), there are $l_n, k_n \in B_\tau$ such that $-k_n \leq f \wedge h_n \leq l_n$ and

$$I^+(l_n) < \overline{I}(f \wedge h_n) + \frac{\varepsilon}{2}, \qquad \underline{I}(f \wedge h_n) - \frac{\varepsilon}{2} < -I^+(k_n).$$

Then, $I^+(l_n) + I^+(k_n) \to 0$, if $n \to +\infty$.

Furthermore, $-k_n \wedge h_0 \le f \wedge h_0 \le l_n \wedge h_0$, if $n \ge n_0(\varepsilon)$, with $-(-k_n \wedge h_0) = k_n \vee (-h_0) \in B_\tau$.

This gives, with (4), $f \cap h_0 = (f \wedge h_0) \vee (-h_0) \in \overline{B}$ for all $h_0 \in +B$, so f is I-measurable. By 2. in definition 5, $f^\pm$ are I-measurable, with $f^- := -(f \wedge 0) \in \overline{B}$. Now, by the proof of the finiteness of $\overline{I}_B(|f|)$ in theorem 8, one gets $f \in L(B, I)$; and, by lemma 3 and (5), $J(f) = \underline{I}(f)$. □

Example 13 shows that $\underline{I}_B = J$ on $L(B, I)$ is false in general. If additionaly, for $f \in L(B, I)$, there exists $h \in B$ with $f \ge h$ (or equivalently, $I^+(f) > -\infty$), the converse of Proposition 6 holds; the proof is mostly similar to those above using our earlier results.

4. Improper integrals.

In the present Section, we discuss improper I-integrability with respect to I-summable functions and give an I-integrability criterion.

When an integral T on a set $M \subset \overline{\mathbb{R}}^X$ of integrable functions is given, a function $f \in \overline{\mathbb{R}}^X$ is called *improper T-integrable* (w.r.t. M) if $f \cap h \in M$ for all $h \in +B =$ nonnegative elementary functions (e.g., step functions) and exists $\lim_{+B} T(f \cap h) \in \mathbb{R}$, with $+B$ a set directed by $\le$.

So, for $I \mid B$ as in Section 1, and with $T = \overline{I}$, the class $\overline{B}_\cap := \{f \in \overline{\mathbb{R}}^X; f$ improper integrable$\}$ and $\overline{I}_\cap :=$ this limit on $\overline{B}_\cap$, are well defined.

LEMMA 11. *We have $L(B, I) \subset \overline{B}_\cap$ and $J = \overline{I}_\cap$ on $L(B, I)$.*

Proof. With $f \in L(B, I)$, because $|f| \in L(B, I)$, for a given $\varepsilon > 0$, there exists $h \in +B$ such that $\overline{I}_B(|f| - h) < \varepsilon$.

If $h \le k \in +B$, one gets

$$|f \cap k - f| \le |f \cap h - f| = |f \cap h - f \cap |f|| \le |h - |f||,$$

where $f \cap k - f \in L(B, I)$. Therefore,

$$\left|\overline{I}_B(f \cap k) - \overline{I}_B(f)\right| \le \overline{I}_B(h - |f|) < \varepsilon;$$

since $\overline{I} = \overline{I}_B$ on $\overline{B}$, we have $f \in \overline{B}_\cap$ and $J = \overline{I}_B = \overline{I}_\cap$ on $L(B, I)$. □

LEMMA 12. *For $f \in \overline{\mathbb{R}}^X$, $f \in \overline{B}_\cap$ if and only if $f^\pm \in \overline{B}_\cap$.*

Proof. Let $f \in \overline{B}_\cap$. For $h \in +B$, $f^+ \cap h \in \overline{B}$, since $\overline{B}$ is $\wedge$-closed. Now, if there exists $\lim_{+B} \overline{I}(f \cap h) \in \mathbb{R}$, chose $h_0 \in +B$ with

$$\overline{I}(f \cap k) \leq \overline{I}(f \cap h_0) + 1$$

if $h_0 \leq k \in +B$; since $f \cap h \leq f \cap (h + h_0) + |f \cap h_0|$ for $h \in +B$, with $|f \cap h_0| \in \overline{B}$, one gets

$$\overline{I}(f \cap h) \leq \overline{I}(f \cap (h + h_0)) + \overline{I}(|f \cap h_0|) \leq \overline{I}(f \cap h_0) + 1 + \overline{I}(|f \cap h_0|) =: \alpha.$$

For the existence of $\lim_{+B} \overline{I}(f^+ \cap h)$, it is enough to show that

$$\sup\{\overline{I}(f^+ \cap h); h \in +B\} < +\infty.$$

But, if the above sup is $+\infty$, there exists $h \in +B$ such that $\overline{I}(f^+ \wedge h) > \alpha + 2$.

We have $f \cap k = f^+ \wedge k - f^- \wedge k$ and $f^- \wedge k \leq \left|(f^+ \wedge h) - k\right|$ since $f^+ = 0$ where $f^- := (-f)^+ > 0$, so that

$$\overline{I}(f^+ \wedge k) \leq \overline{I}(f \wedge k + \left|(f^+ \wedge h) - k\right|) \leq \overline{I}(f \cap k) + 1 \leq \alpha + 1.$$

We conclude

$$\alpha + 2 < \overline{I}(f^+ \wedge h) \leq \overline{I}(f^+ \wedge h - f^+ \wedge k) + \overline{I}(f^+ \wedge k) \leq$$

$$\overline{I}\left(f^+ \wedge \left(f^+ \wedge k\right) - \left(f^+ \wedge k\right)\right) + \alpha + 1 \leq \overline{I}\left(\left|(f^+ \wedge h) - k\right|\right) + \alpha + 1 < \alpha + 2,$$

a contradiction.

Because $\overline{I}$ is linear on $\overline{B}$, which is closed for addition, with $f \cap h = f^+ \cap h - f^- \cap h$, we have the "$\Longleftarrow$" implication, and this completes the proof. □

We recall that [6, Th.1] gives a substitute for the general missing completeness of $L(B, I)$:

(10) If $(f_n) \subset L(B, I)$ is a J-Cauchy sequence with $f_n \to f(\overline{I})$, for $f \in \overline{\mathbb{R}}^X$, then $f \in L(B, I)$ and $J(f_n) \to J(f)$, if $n \to +\infty$.

THEOREM 13. *For $f \in \overline{\mathbb{R}}^X$, $f \in L(B, I)$ if and only if f is improper I-integrable (w.r.t. $\overline{B}$) and, in this case, $J = \overline{I}_\cap$.*

Proof. By lemma 11 it is necessary only to prove that $\overline{B}_\cap \subset L(B, I)$.

Now, by lemma 12, if $f \in \overline{B}_\cap$, then $f^\pm \in \overline{B}_\cap$. Because $L(B, I)$ is closed for addition, we can assume $f \in +\overline{B}_\cap$. There exists $h_n \in +B$ with $h_n \leq h_{n+1}$ and $\overline{I}(f \wedge h_n) \to \overline{I}_\cap(f) =: \alpha$, where $f \wedge h_n \in \overline{B}$.

For any $k \in +B$, one gets

$$|f - f \cap h_n| \wedge k = (f - (f \cap h_n)) \wedge k = f \wedge (k + (f \wedge h_n)) - (f \wedge h_n) \in \overline{B}.$$

If $\alpha - \varepsilon < \overline{I}(f \wedge h_n)$ for all $n \geq n(\varepsilon)$, we have

$\overline{I}(|f - f \cap h_n| \wedge k) \leq \overline{I}\,(f \wedge (k + (f \wedge h_n))) - \overline{I}\,(f \wedge h_n) \leq \alpha - (\alpha - \varepsilon)\,;$

with $g_n := |f - f \cap h_n| \wedge k \in \overline{B}$, then $\overline{I}\,(g_n) \to 0$ and $f \wedge h_n \in \overline{B}$ is an $\overline{I}$-Cauchy sequence, and by (7) we obtain that $f \in L(B, I)$. □

Specially, in the situation $\mu \mid \Omega$ one can also consider improper integration with respect to Ω-unbounded domains:

$$(\overline{B}_\Omega)_\cap := \{f \in \overline{\mathbb{R}}^X;\ f\chi_A \in \overline{B}_\Omega \text{ if } A \in \Omega,\ \left(I_\mu\right)_\cap (f): = \lim_{r\Omega} \overline{I}_\mu(f\chi_A) \text{ exists } \in \mathbb{R}\},$$

where $r\Omega$ is the ring generted by Ω.

Example 15. Let $X := [0, 1]$, $\Omega := \{[a, b[\,;\, a, b \in \mathbb{R}\}$ and $\mu :=$ Lebesgue measure. If we consider

$$f(x) := \begin{cases} -\dfrac{1}{\sqrt{x}}, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}$$

we obtain that $f \in L(B, I)$, with $J(f) = \int_0^1 f = 2$ and $I^+(f) = -\infty$.

Example 14. Let $X := [0, \infty[$, $\Omega := \{M \subset X;\ M$ or $X - M$ is finite and $\subset [1, \infty[\}$, and $\mu := \delta_0 =$ Dirac measure on 0 (so, with $E \subset [1, \infty[$ and finite, we have $\mu(E) = 0$ and $\mu(X - E) = 1$).

In this case:

$$R^1_{prop}(\mu, \mathbb{R}) \subsetneqq R_1(\mu, \overline{\mathbb{R}}) \subsetneqq L^1(\mu, \overline{\mathbb{R}}) \subsetneqq \overline{B} = L^\tau(B, I) \subset L(B, I)$$

with $B = B_\Omega$, $I = I_\mu$, and $L^\tau =$ Bourbaki extension.

Remarks.

1. If $\nu : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$ is an upper functional in the sense of Anger and Portenier [3], with the notations and results in [5], the functional $q := \nu_{|+\overline{\mathbb{R}}^X}$ is an integral metric, $B := J(\nu) \cap \mathbb{R}^X$ is a function vector lattice and $I := \nu_{|B}$ is linear and monotone, where

$$J(\nu) := \{f \in \overline{\mathbb{R}}^X;\ \nu(f) = \nu_*(f) \in \mathbb{R}\}$$

and $\nu = \overline{I}$ is admissible, then $J(\nu) = \overline{B}$.

2. [3, Cor.3.7] and our Theorem 1 give that the class $J(\nu^\bullet)$ of the essential ν-integrable functions coincides with $\overline{B}_\cap$, where

$$\nu^\bullet(f) := \inf_{u \in J_-} \sup_{v \in J_-} \nu\,[(f \wedge (-v)) \vee u]$$

and

$$J_- := J(\nu) \cap\,]-\infty, +\infty]^{X.}$$

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