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Absolute Continuity and Abstract Riemann Integration

Abstract. *Absolute continuity for functionals is studied in the context of abstract Riemann integration with a twofold purpose. On the one hand, we look for relations between the integrable functions of absolutely continuous integrals and we deal with the possibility of preserving absolute continuity when extending the integrals. On the other hand, we examine the relation with absolute continuity for finitely additive measure giving results in both directions: integrals coming from measures and measure induced by integrals.*

Key Words: *Finitely additive integration, Abstract Riemann integration, Absolute continuity.*

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1. Introduction

It is well known that, if you want to make a Measure Theory, then you have, at least, two classical ways.

On the one hand, there is the set theoretic starting point, which we will denote by (μ/Ω): X is a non empty set, Ω is a σ-algebra of the power set of X and μ is a measure on Ω. Standard and classical methods leads to the class $L_1(\mu, \Omega)$ of the Lebesgue integrable functions, in this context.

When the continuity of the measure is dropped (and we work without or with weaker continuity conditions) arises a new paradigm: Dundford-Schwartz treatise, [10], introduces vector finitely additive set functions

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3. Proper and abstract Riemann integration

Let (X, B, I) be a Loomis system. For $f \in \overline{\mathbb{R}}^X$, we define

$$I^-(f) := \inf\{I(h) : h \in B, h \geq f\}$$
$$I^+(f) := \sup\{I(h) : h \in B, h \leq f\}$$

the corresponding upper and lower integrals of f (with convention $\sup \emptyset = -\inf \emptyset = -\infty$) which verify $-\infty \leq I^+(f) \leq I^-(f) \leq +\infty$, $\forall f \in \overline{\mathbb{R}}^X$, I^- is sub-additive, I^+ is super-additive, and both are positively homogeneous.

The class of the *properly Riemann integrable functions* is defined by

$$R_{\text{prop}}(B, I) := \{f \in \mathbb{R}^X : I^+(f) = I^-(f) \in \mathbb{R}\},$$

or, equivalently, by

$$R_{\text{prop}}(B, I) := \{f \in \mathbb{R}^X : \forall \varepsilon > 0,\ \exists h, g \in B,\ h \leq f \leq g \text{ and } I(g-h) < \varepsilon\}$$

and it is a vector lattice where the functional $I := I^+ = I^-$ is linear and increasing, i.e., it is an integral which extends the original I.

For this class, there are not satisfactory Lebesgue convergence type theorems to make a consistent Measure Theory. Therefore, it is necessary to introduce a "local convergence" to ensure this kind of results.

The *local I-convergence* for sequences of functions $\{f_n\}$ in $\overline{\mathbb{R}}^X$ to a function f in $\overline{\mathbb{R}}^X$, denoted by $\{f_n\} \longrightarrow f(I^-)$, means that $\{I^-(|f_n - f| \wedge h)\} \to 0$, $\forall h \in +B$ and it can be used to define the class $R_1(B, I)$ of the *abstract Riemann integrable functions* in this way:

$$R_1(B, I) := \{f \in \overline{\mathbb{R}}^X : \exists \{h_n\} \text{ in } B,\ I\text{-Cauchy};\ \{h_n\} \longrightarrow f(I^-)\}.$$

Moreover, for $f \in R_1(B, I)$, we set $I(f) := \lim_{n\to\infty} I(h_n)$ for any sequence $\{h_n\}$ in B such that $\{h_n\} \longrightarrow f(I^-)$.

The definition does not depend on the particular sequence $\{h_n\}$ and no confusion arises with this notation since $R_{\text{prop}}(B, I) \subseteq R_1(B, I)$ with coinciding integrals I.

Further relations between the classes $R_{\text{prop}}(B, I)$ and $R_1(B, I)$ are given by the following characterizations:

(i) $f \in R_1(B, I) \ \Leftrightarrow\ f^{\pm} \wedge h \in R_{\text{prop}},\ \forall h \in +B$ and $I^+(|f|) < \infty$.
(ii) $f \in R_{\text{prop}}(B, I) \ \Leftrightarrow\ f \in R_1(B, I)$ and $\exists h \in +B : |f| \leq h$.

Finally, we introduce two classes related to $R_1(B, I)$: The class of the *null-functions*,

$$N_1(B, I) := \{f \in R_1(B, I) : I(|f|) = 0\},$$

Unfortunately, this definition does not work well since I-continuity is, in fact, a kind of boundness condition:

PROPOSITION 4.2. *Let (X, B, I) be a Loomis system and J a positive functional on B. The following are equivalent:*

(i) *J is I-continuous*
(ii) $\exists M > 0 \; : \; J(h) \leq MI(h), \; \forall h \in +B$ *($J \leq MI$, for abbreviation).*

This equivalence shows us that integrals induced by absolute measures needn't be continuous in this sense, that is, there exist measures μ and ν such that $\nu \ll \mu$ but I_ν is not I_μ-continuous (see [1]).

Therefore we have to weaken I-continuity in order to define a satisfactory notion of absolute continuity for functionals.

DEFINITION 4.3. Let (X, B, I) be a Loomis system and J a positive functional on B. J is said to be *absolute I-continuous* (absolute continuous with respect to I), and it is denoted by $J \ll I$, if

$$\forall \varepsilon > 0, \forall h \in +B, \; \exists \delta > 0 \; : \; \forall k \in +B, \; k \leq h, \; I(k) < \delta \Rightarrow J(k) < \varepsilon \, .$$

Given (X, Ω, μ) with μ a finite finitely additive measure and Ω a ring, we set (X, B_Ω, I_μ) to be the induced Loomis system, where

$$B_\Omega := \left\{ h \in \mathbb{R}^X : h = \sum_{i=1}^{n} a_i \chi_{A_i}, \;\; a_i \neq 0, A_i \in \Omega \text{ pairwise disjoint} \right\},$$

and

$$I_\mu(h) := \sum_{i=1}^{n} a_i \mu(A_i), \quad \forall h \in B_\Omega \, .$$

Next proposition makes evident that absolutely continuos finitely additive measures yield absolutely continuous elementary integrals:

PROPOSITION 4.4. *Let μ and ν be finite finitely additive measures. If $\nu \ll \mu$ then $I_\nu \ll I_\mu$.*

PROOF. Assume that $\nu \ll \mu$ let be $\varepsilon > 0$ and $f \in +B_\Omega$. There are $a_i > 0$ and $A_i \in \Omega$ pairwise disjoint such that $f = \sum_{i=1}^{n} a_i \chi_{A_i}$. Set $A := \bigcup_{i=1}^{n} A_i \in \Omega$ and $\beta := \sup\{a_i : i = 1, \ldots, n\} > 0$.

If $\nu(A) = 0$, then

$$I_\nu(f) = \sum_{i=1}^{n} a_i \nu(A_i) \leq \beta \sum_{i=1}^{n} \nu(A_i) = \beta \nu(A) = 0$$

and therefore $I_\nu(h) \leq I_\nu(f) = 0 < \varepsilon$, $\forall h \in +B_\Omega$ with $h \leq f$.

5. Absolute continuity and proper Riemann integration

In this section we will study the good behaviour of absolute continuity with respect to proper Riemann integration.

First result says that absolute continuity of J with respect to I transfers convergence to 0 for B-bounded sequences from the integral I to the integral J:

LEMMA 5.1. *Assume that* $J \ll I$ *and let* $\{h_n\}$ *be a sequence in* $+B$ *such that* $\exists\, h \in +B$ *with* $h_n \le h, \forall\, n \in \mathbb{N} n$ *and* $I(h_n) \to 0$. *Then* $J(h_n) \to 0$.

PROOF. Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall\, k \in +B,\ k \le h,\ I(k) < \delta \Rightarrow |J(k)| < \varepsilon\,.$$

Since $I(h_n) \to 0$ there exists $m \in \mathbb{N}$ such that $\forall\, n \ge m,\ I(h_n) < \delta$. Therefore, $\forall\, n \ge m,\ |J(h_n)| < \varepsilon$, that is, $J(h_n) \to 0$. ■

In particular, we have the following results:

COROLLARY 5.2. *If* $J \ll I$ *and* I *is Daniell then* J *is Daniell too.*

COROLLARY 5.3. *If* $J \ll I$ *and* $\{h_n\}$ *is a* I*-Cauchy sequence in* $+B$ *such that* $\exists\, h \in +B$ *with* $|h_n - h_m| \le h\, \forall\, n \in \mathbb{N} n, m$, *then* $\{h_n\}$ *is* J*-Cauchy.*

THEOREM 5.4. *If* $J \ll I$ *then*

(i) $R_{\text{prop}}(B, I) \subseteq R_{\text{prop}}(B, J)$

(ii) $\tilde{J} \ll \tilde{I}$ *(where* $\tilde{J}$ *and* $\tilde{I}$ *are the extensions of* I *and* J *to* $R_{\text{prop}}(B, I)$*).*

PROOF. (i) Let $f \in R_{\text{prop}}(B, I)$ and $\varepsilon > 0$. There are $k_\varepsilon, h_\varepsilon \in B$ such that

$$k_\varepsilon \le f \le h_\varepsilon \quad \text{and} \quad I(h_\varepsilon - k_\varepsilon) < \varepsilon\,.$$

For $\varepsilon > 0$ and $h_\varepsilon - k_\varepsilon \in +B$, since $J \ll I$, there exists $\delta > 0$ such that

$$\forall\, g \in +B, \quad g \le h_\varepsilon - k_\varepsilon, \quad I(g) < \delta \Rightarrow J(g) < \varepsilon\,. \tag{2}$$

We can also take $k_\delta, h_\delta \in B$ such that

$$k_\delta \le f \le h_\delta \quad \text{and} \quad I(h_\delta - k_\delta) < \delta\,.$$

Since δ only depends on ε, we can consider the following functions:

$$k'_\varepsilon := k_\varepsilon \vee k_\delta \quad \text{and} \quad h'_\varepsilon := h_\varepsilon \vee h_\delta,$$

which verify that $k'_\varepsilon, h'_\varepsilon \in B$, and $k'_\varepsilon \le f \le h'_\varepsilon$.

6. Absolute continuity and abstract Riemann integration

The definition of absolute continuity given in 4.3 allows us to prove an analog of Lemma 5.1 where I and J are replaced by I^- and J^-, respectively.

THEOREM 6.1. *Assume that* $J \ll I$ *and let* $\{g_n\}$ *be a sequence in* $\overline{\mathbb{R}}_+^X$ *sucht that* $\exists h \in +B$ *with* $g_n \leq h, \forall n \in \mathbb{N}n$ *and* $I^-(g_n) \to 0$. *Then* $J^-(g_n) \to 0$.

PROOF. Let be $\varepsilon > 0$. Since $J \ll I$, for those ε and h, there exists $\delta > 0$ such that

$$\forall k \in +B,\ k \leq h,\ I(k) < \delta \Rightarrow J(k) < \varepsilon. \tag{4}$$

Since $I^-(g_n) \to 0$, $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $I^-(g_n) < \delta$. Therefore, for each $n \geq n_0$ we can find $k_n \in B$, $g_n \leq k_n$ with $I(k_n) < \delta$.

Set $k'_n := k_n \wedge h \in +B$, $\forall n \geq n_0$. Obviously, $k'_n \leq h$ and $I(k'_n) \leq \leq I(k_n) < \delta$, and from (4) it follows that $J(k'_n) < \varepsilon$, $\forall n \geq n_0$. Since $g_n \leq k'_n$, we deduce that $J^-(g_n) \leq J(k'_n) < \varepsilon, \forall n \geq n_0$. Thus, we have proved that $J^-(g_n) \to 0$, as we wanted. ■

As an immediate consequence of Theorem 6.1 local I-convergence implies local J-convergence whenever J is absolutely continuous with respect to I.

COROLLARY 6.2. *If* $J \ll I$ *and* $f_n \longrightarrow f(I^-)$ *then* $f_n \longrightarrow f(J^-)$, $\forall f_n, f \in \overline{\mathbb{R}}^X, \forall n \in \mathbb{N}n$.

PROOF. Assume that $f_n \longrightarrow f(I^-)$ and let $h \in +B$. We have $I^-(|f_n - f| \wedge h) \to 0$, that is, $I^-(g_n) \to 0$ where $g_n := |f_n - f| \wedge h$ verify that $0 \leq g_n \leq h\ \forall n \in \mathbb{N}n$. Theorem 6.1 says that $J^-(g_n) \to 0$. Thus, $J^-(|f_n - f| \wedge h) \to 0$, $\forall h \in +B$, that is, $f_n \longrightarrow f(J^-)$. ■

OPEN QUESTION 6.3. Does $J \ll I$ imply $R_1(B, I) \subseteq R_1(B, J)$?

However, Corollary 6.2 is the key for proving that absolute continuity establish a good relation between the respective null functions and measurable functions of the involved integrals:

THEOREM 6.4. *If* $J \ll I$ *then*

(i) $N_1(B, I) \subseteq N_1(B, J)$
(ii) $M_1(B, I) \subseteq M_1(B, J)$

COROLLARY 6.6. *If* (X, B, I) *is a* C_∞ *Loomis system and* $f \in +R_1(B, I)$ *then*

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } A \in \Omega \text{ with } \mu(A) < \delta \Rightarrow \mu_f(A) < \varepsilon\,.$$

REFERENCES

[1] E. DE AMO - I. CHITESCU - M. DÍAZ-CARRILLO, *An Approximate Functional Radon-Nikodym Theore*, Rend. del Circ. Mat. de Palermo, Serie II, Tomo **48** (1999), pp. 443-450

[2] E. DE AMO - I. CHITESCU - M. DÍAZ-CARRILLO, *An Exact Functional Radon-Nikodym Theorem for Daniell Integrals*, Studia Mathematica, **148** (2) (2001), pp. 97-110.

[3] E. DE AMO - M. DÍAZ-CARRILLO, *On abstract Fubini theorems for finitely additive integration*, Proc. Amer. Math. Soc., **123** No. 9, Septiembre (1995), pp. 2739-2744.

[4] G. AUMANN, *Integralerweiterungen mittels Normen*, Arch. Math., **3** (1952), pp. 441-450.

[5] R. DEL CAMPO, *Contribuciones a la teoria de la integración finitamente aditiva*, Ph. Dr. Dissertation, under supervision.

[6] M. DÍAZ-CARRILLO - H. GÜNZLER H., *Abstract Daniell-Loomis spaces*, Bull. Austral. Math. Soc., **53** (1996), pp. 135-142.

[7] M. DÍAZ-CARRILLO - H. GÜNZLER H., *Daniell-Loomis integrals*, Rocky Mount. J. Math., **27** No. 4 (1997), pp. 1-13.

[8] M. DÍAZ-CARRILLO - P. MUÑOZ-RIVAS, *Finitely additive integration: integral extension with local-convergence*, Ann. Sci. Math. Québec, **17** (1993), pp. 145-154.

[9] M. DÍAZ-CARRILLO - P. MUÑOZ-RIVAS, *Positive linear functionals and improper integration*, Ann. Sci. Math. Québec, **18** No. 2 (1994), pp. 149-156.

[10] N. DUNFORD - J.T. SCHWARTZ, *Linear Operartors, Part I, General Theory*, Interscience, New-York, 1957.

[11] H. GÜNZLER, *Integration*, Bibliogr. Institut, Mannheim, 1985.

[12] L. H. LOOMIS, *Linear Functionals and Content*, Amer. J. Math., **76** (1954), pp. 168-182.

[13] W. F. PFEFFER, *Integrals and Measures*, Dekker, New-York, 1977.

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