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Absolute Continuity from a Topological Point of View

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1. Preliminaries

We study some topologies on a given Riesz space which can be generated by a solid seminorm or by a linear and positive functional. The connection with absolute continuity is studied too, more precisely absolute continuity is expressed as continuity with respect to one of these topologies.

Throughout this paper $\mathbb{R}$ will be the reals, $\mathbb{R}_+$, the positive reals (both with the natural topology), $\mathbb{N}$ the natural numbers. The empty set will be denoted by $\emptyset$.

We shall be concerned with non null Riesz spaces (or vector lattices) Y. In such spaces Y the notations $|y|$, $y \wedge z$, $y \vee z$ (for y, $z \in Y$) are the usual ones. A linear map $I : Y \to \mathbb{R}$ will be called a positive linear functional if $I(y) \geq 0$ for $y \in Y_+$, the positive elements of Y. Put $Y_+^* = Y_+ - \{0\}$.

A *Loomis system* is a triple $(X, \mathcal{A}, I)$ where $\emptyset \neq X$ is an abstract set, $\mathcal{A}$ is a lattice (a Riesz space) of functions with usual pointwise order and $I : \mathcal{A} \to \mathbb{R}$ is a positive linear functional.

In case I has the supplementary property that $I(u_n) \searrow 0$, whenever $(u_n)_{n \in \mathbb{N}}$ is a monotone decreasing sequence with $u_n \searrow 0$ pointwise, we say that I is a *Daniell integral*.

2. Three Topologies on a Riesz Space

Assume Y is a Riesz space and let $p : Y \to \mathbb{R}_+$ be a *solid seminorm* (i.e., is a seminorm having the property that

$$|u| \leq |v| \Rightarrow p(u) \leq p(v)$$

for u, v in Y). We shall define three topologies on Y (in increasing order), the second (intermediate) one being the most important for our further study.

2.1. The Topology $\tau(p)$.

We denote by $\tau(p)$ the locally convex topology generated on Y by p. Let us write, for every number $\varepsilon > 0$

$$V(\varepsilon) = \{y \in Y : p(y) < \varepsilon\}.$$

Then a basic system of open neighbourhoods of 0 for $\tau(p)$ will be

$$\mathcal{V}(0) = \{V(\varepsilon) : \varepsilon > 0\}.$$

For every $y \in Y$ we shall also write (for $\varepsilon > 0$)

$$V(y,\varepsilon) = \{z \in Y : p(z-y) < \varepsilon\}. \text{ Then}$$

$$\mathcal{V}(y) = \{V(y,\varepsilon) : \varepsilon > 0\}$$

constitutes a basic system of neigbourhoods of y for $\tau(p)$.

2.2. The Topology $\Upsilon(p)$. For every $a \in Y_+$ let us write

$$S(a) = \{y \in Y : |y| \leq a\}$$

(of course $S(0) = \{0\}$) and τ_a, the induced topology on $S(a)$ by $\tau(p)$.

It is clear that for every $y \in S(a)$, a basic system of open neighbourhoods of y in τ_a will be

$$\{V(y,\varepsilon) \cap S(a) : \varepsilon > 0\} = \mathcal{V}_a(y).$$

We get the family of topological spaces $\{(S(a),\tau_a) : a \in Y_+\}$, all of them being subspaces of the topological space $(Y,\tau(p))$. Writing for every $a \in Y_+$

$$id_a : S(a) \to Y;\ id_a(y) = y,$$

it is clear that all the mappings

$$id_a : (S(a),\tau_a) \to (Y,\tau(p))$$

are continuous.

According to general theory (see e.g. [3]), there exists an unique topology $\Upsilon(p)$ on Y having the following three properties:

(i) For every $a \in Y_+$ the map

$$id_a : (S(a),\tau_a) \to (Y,\Upsilon(p))$$

is continuous.

(ii) $\Upsilon(p)$ is the strongest topology Υ on Y having the property that for all $a \in Y_+$ the map

$$id_a : (S(a),\tau_a) \to (Y,\Upsilon)$$

is continuous (we say that $\Upsilon(p)$ is the *final topology* induced on Y by the family $(id_a)_{a\in Y_+}$ of mappings and the family $(S(a),\tau_a)_{a\in Y_+}$ of topological spaces).

(iii) The topology induced by $\Upsilon(p)$ on every $S(a)$, $a \in Y_+$ is exactly τ_a.

For a set $G \subset Y$ one has $G \in \Upsilon(p)$ if and only if, for every $a \in Y_+$, one has $G \cap S(a) = id_a^{-1}(G) \in \tau_a$.

Of course

$$\tau(p) \subset \Upsilon(p) \qquad (1)$$

and, generally speaking, inclusion (1) is strict (we shall exhibit an example at the end of this paper). For technical details one can see [3, Prop. V 5.5, pg 431].

We finish this subsection with some considerations concerning a basic system of neigbourhoods of 0 in $\Upsilon(p)$. Let $E(Y)$ be the set of functions from $Y_+^\bullet$ to $]0, +\infty[$.

Practically,

$$E(Y) = \left\{\varepsilon = (\varepsilon_a)_{a \in Y_+^\bullet} : \varepsilon_a > 0,\ \forall a \in Y_+^\bullet\right\}.$$

For every $\varepsilon \in E(Y)$, define

$$W(\varepsilon) = \bigcup_{a \in Y_+^\bullet} [V(\varepsilon_a) \cap S(a)].$$

Theorem 1. *For every neighbourhood W of 0 in $\Upsilon(p)$, there exists $\varepsilon \in E(Y)$ such that $W(\varepsilon) \subset W$*

Proof. Let W be a neighbourhood of 0 in $\Upsilon(p)$ and let $G \in \Upsilon(p)$ such that $0 \in G \subset W$. Then, for every $a \in Y_+^\bullet$ we have $0 \in G \cap S(a) \in \tau_a$, so there exists a number $\varepsilon_a > 0$ such that

$$V(\varepsilon_a) \cap S(a) \subset G \cap S(a) \subset W \cap S(a).$$

Then one forms $\varepsilon = (\varepsilon_a)_{a \in Y_+^\bullet}$ and one has

$$W(\varepsilon) = \bigcup_{a \in Y_+^\bullet} [V(\varepsilon_a) \cap S(a)] \subset \bigcup_{a \in Y_+^\bullet} [G \cap S(a)] = G \subset W. \blacksquare$$

We recall attention we are concerned with non null Riesz spaces.

Corollary 1. *The singleton $\{0\}$ is not a neighbourhood of 0 in $\Upsilon(p)$.*

Proof. Assume by absurd $\{0\}$ is a neighbourhood of 0 in $\Upsilon(p)$. Then, according to Theorem 1, we find $\varepsilon \in E(Y)$ such that $W(\varepsilon) \subset \{0\}$. But this is not true: take $a \in Y_+^\bullet$ and $n \in \mathbb{N}$ great enough. Then $\frac{1}{n}a \in V(\varepsilon_a) \cap S(a) \subset W(\varepsilon)$ and $\frac{1}{n}a \neq 0$. $\blacksquare$

2.3. The Topology $\mathcal{U}(p)$.

Theorem 2. *There exists an unique topology $\mathcal{U}(p)$ on Y such that for every $y \in Y$*

$$\mathcal{B}(y) = \{W \cap S(|y|) : W \in \mathcal{V}(y)\}$$

is a basic system of neighbourhoods of y in $\mathcal{U}(p)$.

Proof. We shall check the properties of a basic system of neighbourhoods of a fix $y \in Y$ for $\mathcal{B}(y)$, see [3, Prop 1.2.13, pg 82].

I. Of course $\mathcal{B}(y)$ is non empty.

II. Of course $y \in V$ for all $V \in \mathcal{B}(y)$.

III. For all V_1, V_2 in $\mathcal{B}(y)$, one can find $V \in \mathcal{B}(y)$ such that $V \subset V_1 \cap V_2$.

Indeed, if $V_i = W_i \cap S(|y|)$, $i = 1, 2$, one takes some $W \in \mathcal{V}(y)$ such that $W \subset W_1 \cap W_2$ and $V = W \cap S(|y|)$ will do the job.

IV. Finally, we show that for every $V \in \mathcal{B}(y)$ one can find $W \in \mathcal{B}(y)$ such that for all $z \in W$ we can find $V_z \in \mathcal{B}(z)$ with $V_z \subset V$.

Indeed, let $V = V(y, \varepsilon) \cap S(|y|) \in \mathcal{B}(y)$. Take

$$W = V\left(y, \frac{\varepsilon}{2}\right) \cap S(|y|).$$

For an arbitrary $z \in W$, take

$$V_z = V\left(z, \frac{\varepsilon}{2}\right) \cap S(|z|) \in \mathcal{B}(z).$$

Let $t \in V_z$. Then

$$p(t-y) \leq p(t-z) + p(z-y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

so $t \in V(y, \varepsilon)$. On the other hand, $z \in W$ implies $|z| \leq |y|$, so $t \in S(|z|)$ implies $t \in S(|y|)$, hence $t \in V$. ■

Theorem 3. *One has the strict inclusion*

$$\Upsilon(p) \subset \mathcal{U}(p).$$

Proof. Let $\emptyset \neq G \in \Upsilon(p)$. This means that for every $a \in Y_+^\bullet$ one has $G \cap S(a) \in \tau_a$. We shall show that $G \in \mathcal{U}(p)$.

To this aim, take an arbitrary $y \in G$. We shall show the existence of some $W \in \mathcal{V}(y)$ such that $W \cap S(|y|) \subset G$ (which means that G is a neighbourhood of y in $\mathcal{U}(p)$), thus finishing the proof.

In any case $G \cap S(|y|) \in \tau_{|y|}$, so there exists $G_1 \in \tau(p)$ such that $G \cap S(|y|) = G_1 \cap S(|y|)$. Because $y \in G_1 \in \tau(p)$, we can find $W \in \mathcal{V}(y)$ such that $W \subset G_1$, consequently $W \cap S(|y|) \subset G_1 \cap S(|y|) \subset G$.

The already proved inclusion $\Upsilon(p) \subset \mathcal{U}(p)$ is strict, because:

(a) $\mathcal{B}(0)$ contains only the set $\{0\}$, therefore $\{0\} \in \mathcal{U}(p)$.

(b) In Corollary 1, we have seen that $\{0\}$ is not a neighbourhood of 0 in $\Upsilon(p)$, hence $\{0\} \notin \Upsilon(p)$. ■

Conclusion 1. *One has*

$$\tau(p) \subset \Upsilon(p) \subset \mathcal{U}(p).$$

The last inclusion is always strict because $\{0\} \in \mathcal{U}(p) - \Upsilon(p)$. The first inclusion can be strict (see the example in the last section).

2.4. Study of the Trivial Case $p = 0$. General Non-Linearity of $\mathcal{U}(p)$. We shall divide this subsection into two parts.

First part: Study of the case $p = 0$. In this case we shall see that:

(i) One has $\tau(p) = \Upsilon(p) = \{\emptyset, Y\}$ (and, of course, $\Upsilon(p)$ is a vector topology).

Indeed, $V(y, \varepsilon) = Y$ for all $y \in Y$ and strictly positive $\varepsilon > 0$, so $\tau(p) = \{\emptyset, Y\}$. Now, let us take $\emptyset \neq G \in \Upsilon(p)$. Then

$$G = G \cap Y = \bigcup_{a \in Y_+^\bullet} [G \cap S(a)].$$

We have $G \cap S(a) \in \tau_a$ for all $a \in Y_+^\bullet$ and $\tau_a = \{\emptyset, S(a)\}$, because some $H \in \tau_a$ must be of the form $H = M \cap S(a)$, $M \in \tau(p)$. So, for every $a \in Y_+^\bullet$, one has either $G \cap S(a) = S(a)$ or $G \cap S(a) = \emptyset$.

It is impossible to have $G \cap S(a) = \emptyset$ for all $a \in Y_+^\bullet$ (this would imply $G = \emptyset$). So, let $a \in Y_+^\bullet$ such that $G \cap S(a) = S(a)$. Then $0 \in G$, hence $0 \in G \cap S(b)$ for all $b \in Y_+^\bullet$, which means that $G \cap S(b) = S(b)$ for all $b \in Y_+^\bullet$. This proves that $G = Y$. ■

(ii) The topology $\mathcal{U}(p)$ admits the class of sets

$$\{S(a);\ a \in Y_+\}$$

as a basis.

Let $a \in Y_+^{\bullet}$. We claim that $S(a) \in \mathcal{U}(p)$. To prove this, take $y \in S(a)$. Then $|y| \le a$ and $V(y,\varepsilon) = Y$ for all $\varepsilon > 0$. Then $V(y,\varepsilon) \cap S(|y|) = S(|y|) \subset S(a)$ and $y \in V(y,\varepsilon) \cap S(|y|) \in \mathcal{B}(y)$, hence $S(a)$ is a neighbourhood of y.

Now, the open sets $S(a)$, $a \in Y_+$, form a basis for $\mathcal{U}(p)$. Indeed, let an arbitrary $y \in Y$ and an arbitrary $G \in \mathcal{U}(p)$ such that $y \in G$. Then, there exists $\varepsilon > 0$ such that

$$y \in V(y,\varepsilon) \cap S(|y|) = S(|y|) \subset G.$$

Noticing also the fact that for all $a,\ b \in Y_+$ one has $S(a) \cap S(b) = S(a \wedge b)$, it is seen immediately that the class of sets $\{S(a);\ a \in Y_+\}$ is a subbase of $\mathcal{U}(p)$, i.e., $\mathcal{U}(p)$ is generated by this class. ■

Second part: In all cases $\mathcal{U}(p)$ is not a vector topology. Assume, by absurd, that $\mathcal{U}(p)$ is a vector topology. We have already seen that $\{0\} \in \mathcal{U}(p)$. Translation being a homeomorphism, it follows that for all $y \in Y$ one must have $\{y\} \in \mathcal{U}(p)$.

Take $0 \ne y \in Y$. One must find a number $\varepsilon > 0$ such that

$$\mathcal{B}(y) \ni V(y,\varepsilon) \cap S(|y|) \subset \{y\},$$

i.e. $V(y,\varepsilon) \cap S(|y|) = \{y\}$. So, there exists a number $\varepsilon > 0$ such that for every $z \ne y$ one must have either $p(y-z) \ge \varepsilon$ or $|z| > |y|$.

Take an arbitrary natural $n \ge 2$ and define $z = \left(1 - \frac{1}{n}\right) y \ne y$. Because $|z| = \left(1 - \frac{1}{n}\right)|y| < |y|$, one must have

$$\frac{1}{n} p(y) = p(z-y) \ge \varepsilon$$

for all $n \ge 2$, which is imposible. ■

3. Absolute continuity and the topology $\Upsilon(p)$

We shall generalize a little the notion of absolute continuity (see [2]).

Let Y be a Riesz space and let $I : Y \to \mathbb{R}$ be a linear and positive functional. Then I generates the solid seminorm

$$p : Y \to \mathbb{R}_+;\ p(y) = I(|y|),\ \forall y \in Y.$$

Let $J : Y \to \mathbb{R}$ be another linear and positive fuctional.

Definition. We shall say that J is *absolutely continuous* with respect to I (and we shall write $J \ll I$) if for every number $\varepsilon > 0$ and for every $a \in Y_+$, there exists a number $\delta(\varepsilon, a) = \delta > 0$ such that for every $y \in Y$ with $|y| \le a$ and $p(y) = I(|y|) < \delta$ one has $|J(y)| < \varepsilon$.

In the definition one can work only with $y \in Y_+$, because $|J(y)| \leq J(|y|)$. Rewriting what this means, we obtain:

$$J \ll I \Leftrightarrow \forall \varepsilon > 0, \forall a \in Y_+, \ \exists \delta > 0 : \ \forall y \in S(a) \cap V(\delta), |J(y)| < \varepsilon.$$

Here $V(\delta) = \{z \in Y : \ p(z) < \delta\}$ as previously used.

The last assertion means:

$$J \ll I \Leftrightarrow \forall a \in Y_+, \ J \circ id_a : (S(a), \tau_a) \rightarrow \mathbb{R} \text{is continuous at } 0.$$

Let us notice the next lemma.

Lemma 1. *The following assertions are equivalent:*

1. $\forall a \in Y_+, \ J \circ id_a : (S(a), \tau_a) \rightarrow \mathbb{R}$ *is continuous at 0.*
2. $\forall a \in Y_+, \ J \circ id_a : (S(a), \tau_a) \rightarrow \mathbb{R}$ *is continuous.*

Proof. We prove 1.⇒ 2. Take an arbitrary $a \in Y_+$ and an arbitrary $y \in S(a)$. We shall prove that $J \circ id_a : (S(a), \tau_a) \rightarrow \mathbb{R}$is continuous at y. Let $\varepsilon > 0$ be arbitrary.

In any case, there exists $\delta > 0$ such that for every $z \in S(2a)$ with $I(|z|) = p(z) < \delta$ one has $|J(z)| < \varepsilon$.

Now, take an arbitrary $y' \in S(a)$ with $y' \in V(y, \delta)$, i.e., $p(y' - y) < \delta$. This implies (because $|y' - y| \leq 2a$):

$$|J(y') - J(y)| = |J(y' - y)| < \varepsilon,$$

and 2. is true. ■

Now, we use the following property of the final topology $\Upsilon(p)$ (see [3]): for every topological space $(X, \mathcal{A})$ and for every function $f : (Y, \Upsilon(p)) \rightarrow (X, \mathcal{A})$, the following assertions are equivalent:

1. f *is continuous.*
2. *for every* $a \in Y_+$, $f \circ id_a : (S(a), \tau_a) \rightarrow (X, \mathcal{A})$ *is continuous.*

Applying this to $J : (Y, \Upsilon(p)) \rightarrow \mathbb{R}$instead of f we obtain

Theorem 4. *Consider a Riesz space Y and two linear and positive functionals $I, J : Y \rightarrow \mathbb{R}$. Define the solid seminorm $p : Y \rightarrow \mathbb{R}_+$ given by $p(y) = I(|y|)$, for every $y \in Y$. Then, the following assertions are equivalent:*

1. $J \ll I$.
2. $J : (Y, \Upsilon(p)) \to I\!R$ *is continuous.*

We are now able to show, via an example, that in case $p \neq 0$ the inclusion $\tau(p) \subset \Upsilon(p)$ is generally strict. The example is taken from [1] but, in order to make the paper self-contained, we shall give complete proofs.

Example (when the inclusion $\tau(p) \subset \Upsilon(p)$ is strict).

Define the Loomis system $(X, \mathcal{A}, I)$ as follows: take $X := [0, 1]$,

$$\mathcal{A} := \{f : [0,1] \to I\!R : f \text{ continuous}\}$$

and $I : \mathcal{A} \to I\!R$given by $I(f) := \int_0^1 f d\lambda$ (here λ is Lebesgue measure on $[0,1]$). Actually, I is Daniell.

Let us consider the λ-integrable function $h : [0,1] \to I\!R$given by $h(0) = 0$ and $h(t) = \frac{1}{\sqrt{t}}$ for $t \neq 0$. We can define the Daniell integral $J : \mathcal{A} \to I\!R$given by $J(f) = \int_0^1 f h d\lambda$.

(a) *We prove that $J \ll I$.*

(This is true in virtue of [2], but we give here a direct proof.)

Assume that $J \ll I$ is not true. So, there exists $u_0 \in \mathcal{A}_+$ and $\varepsilon_0 > 0$, such that for every $n \in I\!N$there exists $0 \leq f_n \leq u_0$, with $I(f_n) < \frac{1}{n}$, but $J(f_n) \geq \varepsilon_0$.

So, $\int f_n d\lambda < \frac{1}{n}$ for all n, which implies $f_n \to 0$ in $\mathcal{L}^1(\lambda)$. Let us find a subsequence $(f_{n_k})_k$ such that $f_{n_k} \underset{k}{\to} 0$ λ–a.e. One has, for all k, $f_{n_k} h \leq u_0 h$ and $f_{n_k} h \underset{k}{\to} 0$ λ-a.e. According to Lebesgue's dominated convergence theorem, one has

$$J(f_{n_k}) = \int f_{n_k} h \, d\lambda \underset{k}{\to} 0$$

which is a contradiction. ■

(b) $J : (\mathcal{A}, \Upsilon(p)) \to I\!R$ *is continuous.*

Writing $p(f) = I(|f|)$ for all $f \in \mathcal{A}$, we obtain the solid seminorm p on $\mathcal{A}$ and point (a) says that $J : (\mathcal{A}, \Upsilon(p)) \to I\!R$is continuous. ■

(c) $J : (\mathcal{A}, \tau(p)) \to I\!R$ *is not continuous. This will show that $\tau(p) \subset \Upsilon(p)$ is a strict inclusion.*

Indeed, $f \longrightarrow p(f)$ is a norm on $\mathcal{A}$. Assuming that J would be continuous on the normed space $(\mathcal{A}, p)$, we shall derive a contradiction. Namely, let us define the sequence $(u_n)_n \subset \mathcal{A}$ as follows:

$$u_n(x) = \begin{cases} 2\sqrt{2n}\sqrt{n}x, & x \in \left[0, \frac{1}{2n}\right[\\ \frac{1}{\sqrt{x}}, & x \in \left[\frac{1}{2n}, \frac{1}{n}\right] \\ -n\sqrt{n}(n-1)\left(x - \frac{1}{n-1}\right), & x \in \left]\frac{1}{n}, \frac{1}{n-1}\right] \\ 0, & x \in \left]\frac{1}{n-1}, 1\right] \end{cases}$$

One has, for all $n \in I\!N$:

$$p(u_n) = \int u_n d\lambda = \frac{1}{2}\frac{1}{2n}\sqrt{2n} + \int_{\frac{1}{2n}}^{\frac{1}{n}} \frac{1}{\sqrt{x}} dx + \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n}\right)\sqrt{n} \longrightarrow 0.$$

On the other hand

$$J(u_n) = \int u_n h d\lambda \geq \int_{\frac{1}{2n}}^{\frac{1}{n}} u_n(x) h(x) dx = \ln 2,$$

for all n, so $J(u_n) \longrightarrow 0$ is false. ■

Open Problem: is the topology $\Upsilon(p)$ linear?

Concerning this problem, we add that in case one could prove that for all ε in $E(Y)$ the set $W(\varepsilon)$ is a neighbourhood of 0 in $\Upsilon(p)$, then it is possible to give an example when $\Upsilon(p)$ is not linear.

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