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FUBINI–TONELLI THEOREMS WITH LOCAL INTEGRALS

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In [3] an integral extension of Lebesgue power is stated. There the extended function class $R_1(B,I)$ is defined, using a suitable "local convergence in measure", which can be traced back to Loomis [10] and Schäfke [12]. It works for arbitrary Loomis systems (X,B,I), where B is a vector lattice of real-valued functions on a set X and I is a nonnegative linear functional on B, which need not be continuous in any sense. Riemann-μ [7], Loomis's abstract Riemann [10], Dunford–Schwartz [5] integrals and essential integration of [1] with respect to an upper integral, are subsumed.

Since Fubini theorem for finitely additive integration is, in general, false (the existence of the abstract Riemann integral does not always imply the existence of the repeated integrals in the sense of Riemann), it seems therefore natural to ask for conditions under which the repeated integrals will exist and be equal. Conditions of this type were given by several authors in [6], [9] and [2]. The object of this paper is to give a more general sufficient condition concerning the boundedness of the section functions, in order that Fubini theorem still hold. Indeed, using a notion of measurability in the sense of Stone, we establish new integrability criteria of the integration in the abstract Riemann sense; also a proof for an analogue of the Fubini–Tonelli theorem is given.

It is interesting to note that the assumptions here considered may be viewed as a natural generalization of the corresponding results of Hoffmann [9] and Elsner [6].

In Section 1 product systems and relations between certain upper functionals, which will be used later, are discussed; we also summarize the abstract Riemann integration theory. In Sections 2 and 3, Fubini and Tonelli theorems for the abstract Riemann integrable functions are obtained.

The results are specialized and discussed in the $\lambda\times\mu$-finitely additive situation, and no continuity conditions (e.g. of Daniell type [11] §4, or "starke integralnorm" of Schäfke [12]) are needed.

1. In what follows we adhere to the notation and terminology of [11], [2] and [3], we will explain it whenever be necessary in order to make the paper self contained.

We extended the usual $+$ to $\mathbf{R}\times\mathbf{R}$ by $r+s:=0$ if $r=-s\in(\infty,-\infty)$, $r-s:=r+(-s)$. $\overline{\mathbf{R}}_+:=[0,\infty]$, $\overline{\mathbf{R}}:=\{-\infty\}\cup\mathbf{R}\cup\{\infty\}$.

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(1) For $j = 1, 2$, X_j is an arbitrary nonempty set, B_j is a function vector lattice $\subset \mathbf{R}^{X_j}$ and $I_j\colon B_j \to \mathbf{R}$ is a nonnegative linear functional, i.e. B_j is closed under the operations and relations $+$, $\alpha.$, $=$, $\wedge$, $\vee$, $|\,|$ ($\alpha \in \mathbf{R}$) and $I_j(h) \geqq 0$ if $0 \leqq h \in B_j$. Here $(f \wedge g)(x) := \min\bigl(f(x), g(x)\bigr)$ (similarly for $\vee$ with max) and $|f|(x) := \bigl|f(x)\bigr|$ for all $x \in X_j$.

Let $X_3 := X_1 \times X_2$ and $B_3 \subset \mathbf{R}^{X_3}$ be a vector lattice.

If $f \in \overline{\mathbf{R}}^{X_3}$ and $x \in X_1$, we define the function $f_x(y) := f(x, y)$ for each $y \in X_2$.

A system (X_3, B_3) is called a *product system* with respect to (X_1, B_1) and (X_2, B_2) whenever for all $f \in B_3$ the following conditions are satisfied:

(i) $f_x \in B_2$ for each $x \in X_1$,

(ii) $I_2 f \in B_1$, where $(I_2 f)(x) := I_2(f_x)$.

In all the following (X_3, B_3) will be a product system.

We define a nonnegative linear functional on B_3 by the rule $I_3(f) := I_1(I_2 f)$ for each $f \in B_3$.

For $f \in \overline{\mathbf{R}}^{X_j}$, $j = 1, 2, 3$, the *Riemann upper and lower integrals* are defined by $I_j^-(f) := \inf\bigl\{I_j(h);\ f \leqq h \in B_j\bigr\}$ with $\inf\emptyset := \infty$ and $I_j^+(f) := -I_j^-(-f)$.

The corresponding *local integral* is defined by

$$I_{j,\ell}^-(f) := \sup\bigl\{I_j^-(f \wedge h);\ 0 \leqq h \in B_j\bigr\}.$$

I_j^- and $I_{j,\ell}^-$ are positively homogeneous, monotonous and subadditive on $\overline{\mathbf{R}}_+^{X_j}$.

For any $f \in \overline{\mathbf{R}}^{X_3}$, we define as usual $(I_2^- f(x) := I_2^-(f_x)$ and $(I_{2,\ell}^- f)(x) := I_{2,\ell}^-(f_x)$ for all $x \in X_1$.

The following relations are easy consequences of the definitions.

For all $f \in \overline{\mathbf{R}}^{X_3}$, $I_1^-(I_2^- f) \leqq I_3^-(f)$. For all $f \in \overline{\mathbf{R}}^{X_j}$, $I_{j,\ell}^-(f) \leqq I_j^-(f)$ and $I_{j,\ell}^-(f) = I_j^-(f)$ if there exists $h \in B_j$ with $f \leqq h$.

We shall next give a brief introduction to the *abstract Riemann integral* (or R_1-integration). A general study of the integration used here is developed in [3], and in a more general setting (with local integral metrics) in [4].

(2) For $j = 1, 2, 3$, the extended function class $R_1 = R_1(B_j, I_j)$ of I_j-*integrable functions* is defined as the set of those functions $f \in \overline{\mathbf{R}}^{X_j}$ for which there exists a sequence $(h_n) \subset B_j$ with $I_{j,\ell}^-\bigl(|f - h_n|\bigr) \to 0$ as $n \to \infty$, i.e. f belongs to the closure of B_j in $\overline{\mathbf{R}}^{X_j}$ with respect to the integral seminorm $I_{j,\ell}^-(|\cdot|)$.

R_1 is closed with respect to $\pm$, $\alpha.$, $|\,|$, $\wedge$, $\vee$, and there exists a unique $I_{j,\ell}^-$-continuous extension of $I_j|B_j$ to $R_1(B_j, I_j)$, which we denote also by I_j.

(3) If $f \in R_1(B_j, I_j)$ and there exists $g \in B_j$ such that $|f| \leqq g$, then $I_j^+(f) = I_j^-(f) = I_{j,\ell}^-(f) \in \mathbf{R}$ (in this case f is called a *proper Riemann integrable function* of Loomis $R_{\text{prop}}(B_j, I_j)$, see for example [3], [10], p. 170).

A subset A of X_j is said to be $I_{j,\ell}^-$*-null* if $I_{j,\ell}^-(\chi_A) = 0$.

Looking for applications we consider the finitely additive space situation:

(4) Let Ω be a semiring of sets from X and let $\mu: \Omega \to \mathbf{R}_+$ be a finitely additive measure on Ω. With $B = B_\Omega :=$ real-valued μ-step functions over Ω and $I = I_\mu := \int . du$ on B_Ω, (B_Ω, I_μ) satisfies the assumptions in (1). Then, by using the above methods, we obtain that the space of abstract Riemann-μ-integrable functions $R_1(\mu, \mathbf{R})$ of [7] pp. 70, 199, which generalizes the space $L(X, \Omega, \mu, \mathbf{R})$ of μ-integrable functions of Dunford–Schwartz of [5] p. 112, is a special case of $R_1(B, I)$ (see [8] and [4], 3.B).

In general, the functional $I: B \to \mathbf{R}$ is not monotonely continuous, and therefore it is not an abstract Lebesgue integral; however, in the measure space situation, i.e. if Ω is a σ-ring and μ is σ-additive, with $I_\mu^-(f) = \inf\{ I_\mu(h);\ f \leqq h \in B_\Omega \}$, one has $R_1(\mu, \overline{\mathbf{R}}) = L^1(\mu, \overline{\mathbf{R}})$ (=Lebesgue-μ-integrable functions modulo nulfunctions) by [8] p. 265.

If Ω_1 and Ω_2 are semirings of sets from X_1 and X_2, and λ and μ are finitely additive measures on Ω_1 and Ω_2, respectively, it is possible to build a product additive measure $\lambda \times \mu$ in the set $X_1 \times X_2$ and the induced integral $I_{\lambda\times\mu}$ as follows: $I_{\lambda\times\mu}(\chi_{A_1\times A_2}) = (\lambda \times \mu)(A_1 \times A_2)$ for all $A_1 \times A_2 \in \Omega_1 \times \Omega_2$.

2. In [2] an analogue to the classical Fubini theorem for the proper abstract Riemann integrable functions $R_{\text{prop}}(B_3, I_3)$ is obtained, which generalizes the corresponding results of Elsner [6] and Hoffmann [9]. In this paper we extend those results of [2] to the abstract Riemann integrable functions $R_1(B_3, I_3)$, with a weaker boundedness for the section function.

In the references which we know, in order to obtain Fubini's theorems for abstract Riemann integrable functions (or for corresponding to the analogues of the Daniell extension process, but without monotone continuity assumption on the elementary integral I), it is necessary to assume certain conditions of boundedness for the section functions. For example, Elsner in [6], p. 270, gives a function $f \in R_1(B_3, I_3)$ such that $I_{1,\ell}^-(I_{2,\ell}^- f) = \infty$ and $I_{3,l}^-(f) = 0$ (see also [9], p. 141). In this case, Fubini theorem in [9] and Theorem 1 below are false, here $\sup_x |f_x|$ is not majorized by a function of B_2, neither by an integrable function, respectively.

In the sequel we will assume a product system (X_3, B_3) with the following conditions:

(5) *Given any $0 \leqq h \in B_1$ and $0 \leqq \varphi \in B_2$ there exists $k \in B_3$ such that $\varphi(y) \leqq k(x, y)$ if $h(x) > 0$, $(x, y) \in X_3$.*

Stone's condition on B_1 (if $0 \leqq h \in B_1$ then $h \wedge 1 \in B_1$) and $I_1(h \wedge \varepsilon) \to 0$ as $\varepsilon \to 0$.

Observe that the above assumptions are fulfilled in most applications, for example for step functions or continuous functions with compact support.

Now, when the usual B_2-boundedness of f_x is replaced by the more general $R_1(B_2, I_2)$-boundedness the following key lemma is proved.

LEMMA 1. *Let* (X_3, B_3) *be a product system satisfying* (5) *and let* $f \in \overline{\mathbf{R}}_+^{X_3}$ *be such that the following condition holds:*

$(*)$ There exists $g \in R_1(B_2, I_2)$ such that $f_x \leqq g$ for each $x \in X_1$.

Then $I_{1,\ell}^-(I_{2,\ell}^- f) \leqq I_{3,\ell}^-(f)$.

PROOF. For $g \in R_1(B_2, I_2)$ and $\varepsilon > 0$, there exists $\varphi_\varepsilon \in B_2$ such that $I_{2,\ell}^-(|g - \varphi_\varepsilon|) < \varepsilon$.

Now, with $0 \leqq g \leqq |g - \varphi_\varepsilon| + |\varphi_\varepsilon|$ and $(*)$, we have

$$f_x = f_x \wedge g \leqq |g - \varphi_\varepsilon| \wedge f_x + |\varphi_\varepsilon| \wedge f_x \leqq |g - \varphi_\varepsilon| + |\varphi_\varepsilon| \wedge f_x \text{ for each } x \in X_1.$$

By definition, $I_{1,\ell}^-(I_{2,\ell}^- f) := \sup\left\{ I_1^-\left((I_{2,\ell}^- f) \wedge h\right);\ 0 \leqq h \in B_1 \right\}$.

For $0 \leqq h \in B_1$ and $|\varphi_\varepsilon| \in B_2$, by (5), there exists $k \in B_3$ such that $|\varphi_\varepsilon| \leqq k_x$ on X_2 if $h(x) > 0$.

Thus, one gets

$$(I_{2,\ell}^- f)(x) := I_{2,\ell}^-(f_x)$$
$$\leqq I_{2,\ell}^-\left(|g - \varphi_\varepsilon|\right) + I_{2,\ell}^-\left(|\varphi_\varepsilon| \wedge f_x\right) < \varepsilon + I_{2,\ell}^-\left(|\varphi_\varepsilon| \wedge f_x\right).$$

Therefore

$$I_1^-\left((I_{2,\ell}^- f) \wedge h\right) \leqq I_1^-(\varepsilon \wedge h) + I_1^-(I_2^- f \wedge k) \leqq I_1(\varepsilon \wedge h) + I_3^-(f \wedge k),$$

and with $\varepsilon \to 0$ we conclude the proof. □

Without $(*)$, Lemma 1 becomes false by the above comments.

To obtain the main result, applying the inequality established in Lemma 1, we need to impose the following assumption.

(6) *For all* $g \in B_3$ *there exists* $t \in R_1(B_2, I_2)$ *such that* $|g_x| \leqq t$ *for each* $x \in X_1$.

THEOREM 1 (Fubini). *Let* (X_3, B_3) *be a product system satisfying* (5) *and* (6), *and let* $f \in R_1(B_3, I_3)$ *be such that* $|f_x| \leqq \varphi \in R_1(B_2, I_2)$, *for each* $x \in X_1$. *Then the following assertions hold:*

i) *There exist* $I_{1,\ell}^-$*-null sets* $A_n \subset X_1$, $n \in \mathbf{N}$ *such that* $f_x \in R_1(B_2, I_2)$ *for each* $x \in X_1 - \bigcup_1^\infty A_n$.

ii) *There exists* $F \in R_1(B_1, I_1)$ *defined by* $F(x) = I^-_{2,\ell}(f_x)$ *if* $f_x \in R_1(B_2, I_2)$, *and* $I^-_{1,\ell}(F) = I^-_{3,\ell}(f)$, *i.e.* $I^-_{3,\ell}(f) = I^-_{1,\ell}(I^-_{2,\ell}f)$.

The proof is obtained in a similar way as the one of Satz in [9], p. 141, by application of Lemma 1. The scheme of the proof is:

i) By (2), for $f \in R_1(B_3, I_3)$, given $\varepsilon > 0$, there exists $g \in B_3$ such that $I^-_{3,\ell}(|f-g|) < \varepsilon$. For $x \in X_1$, set $\phi(x) := \inf\left\{I^-_{2,\ell}(|f_x - h|);\ 0 \leqq h \in B_2\right\}$ and $A_n := \{x \in X_1;\ \phi(x) \geqq 1/n\}$, $n \in \mathbf{N}$. By Lemma 1, $0 \leqq I^-_{1,\ell}(\chi_{A_n}) \leqq nI^-_{3,\ell}(|f-g|) < n\varepsilon$, and the result follows.

ii) Set $F(x) := I^-_{2,\ell}(x)$ if $f_x \in R_1(B_2, I_2)$. By (2), for $f \in R_1(B_3, I_3)$ there exists $(g_n) \subset B_3$ such that $I^-_{3,\ell}(|g_n - f|) \to 0$, as $n \to \infty$. One checks easily that $I^-_{1,\ell}(|I^-_{2,\ell}g_n - k| \leqq 3I^-_{3,\ell}(|g_n - f|)$ and the result follows. (Note that, by (6), for all $g \in B_3$, $|g_x - f_x| \leqq |t| + |\varphi| \in R_1(B_2, I_2)$ for each $x \in X_1$.)

Let us remark that part of Hoffmann's results [9] (see also [6]) could be improved by using the corresponding ones in this paper, which are obtained for the localized seminorm integrals $I^-_{j,\ell}(|\cdot|)$.

3. In the context of the above R_1-integration theory, a notion of measurability in the sense of Stone is obtained in [4]. The results needed here to prove a partial converse of Fubini's theorem can be summarized briefly as follows.

Let us suppose arbitrary X, B and I are given as before (1).

A function $f \in \overline{\mathbf{R}}^X$ is said to be *I-measurable* (with respect to (X, B)) if its "truncation" by any nonnegative $h \in B$ is I-integrable, i.e. $f \cap h \in R_1(B, I)$ for all $0 \leqq h \in B$, where $f \cap h := (f \wedge h) \vee (-h)$. We denote by M the set of all I-measurable functions.

The next assertions are immediate consequences of the definitions:

$$(7)\qquad \begin{cases} f \in M \Leftrightarrow f \wedge h \in M \text{ for all } 0 \leqq h \in B \Leftrightarrow f^+, f^- \in M. \\ I^-_\ell(f \wedge h) = I^-(f \wedge h) \text{ for all } f \in \overline{\mathbf{R}}^X_+ \text{ and } 0 \leqq h \in B. \end{cases}$$

Notice that for any $f \in M$, we have $|f \cap h| = |f| \wedge h \in R_1(B, I)$ for each $0 \leqq h \in B$, hence, by (2), $f \cap h$ is a proper Riemann integrable function with $I^+(|f| \wedge h) = I^-(|f| \wedge h) \leqq I^+(|f|)$, so $I^-_\ell(|f|) \leqq I^+(|f|)$.

We also recall that always $I^+ \leqq I^-_\ell \leqq I^-$ on $\overline{\mathbf{R}}^X$.

Now we obtain the following lemma, which characterizes the R_1-integration.

LEMMA 2. *Let f be a I-measurable function such that $I^+(|f|) < +\infty$. Then f is an I-integrable function, and $I^+(|f|) = R_1$-integral $I(|f|)$.*

PROOF. By (7) we can assume $f \geqq 0$. If $I^+(f) < \infty$ then there is an I-Cauchy sequence $(h_n) \subset B$ such that $0 \leqq h_n \leqq h_{n+1} \leqq f$ and $I(h_n) \to I^+(f)$ as $n \to \infty$.

For any $0 \leqq h \in B$, we have $|h_n - f| \wedge h \leqq f \wedge (h_n + h) - h_n$. Set $t_n := f \wedge (h_n + h)$, with $t_n \in R_1(B, I)$ and $I^+(t_n) = I^-(t_n) \in \mathbf{R}$.

Now, given $\varepsilon > 0$, there exists $k \in B$ such that $t_n \leqq k$ and $I(k) \leqq I^+(f) + \varepsilon$. So $I(|h_n - f| \wedge h) \leqq I(k - h_n) \to 0$, and by (2), the result follows. □

THEOREM 2 (Tonelli). *If f is an I_3-measurable function with respect to the product system (X_3, B_3), such that there exists the iterated integral $I_1(I_2|f|)$, then $f \in R_1(B_3, I_3)$.*

PROOF. By assumption $I_2|f| \in R_1(B_1, I_1)$. Now, we have

$$I_3^+(|f|) := \sup\{I_1(I_2, k); |f| \geqq k \in B_3\}$$

$$\leqq \sup\{I_1(I_2 k);\ I_2|f| \geqq I_2 k,\ k \in B_3\} \leqq \sup\{I_1(h);\ I_2|f| \geqq h \in B_1\}$$

$$=: I_1^+(I_2|f|) = I_1(I_2|f|) < \infty,$$

and by Lemma 2 the result follows. □

Observe that if in Theorem 2 the assumptions of Theorem 1 are satisfied, then there also exist the two integrals in the equation $I_3(f) = I_1(I_2, f)$, and the equation holds.

REMARKS. a) The Example below shows that a Fubini theorem is in general false in the form $I_1(I_2 f) = I_2(I_1 f)$.

Also, this example (with B_1 and B_2 interchanged) shows that, in general, the B_2-boundedness for B_3 (i.e. if $f \in B_3$ then $|f_x| \leqq g \in B_2$ for each $x \in X_1$), does not imply B_2-boundedness for $R_1(B_3, I_3)$.

b) With (4), for the $\lambda \times \mu$-finitely additive situation, we have that (5) holds and the B_2-boundedness for f_x means that $|f|$ is bounded and there exists P in the generated ring $\mathcal{R}(\Omega_2)$ such that $\operatorname{supp}(f) \subset X_1 \times P$. Thus, with (6) satisfied all our results are applicable; for example, Theorem 1 contains Satz 10 of [6].

c) Using Lemma 2 it can be proved that $f \in R_1(B, I)$, if and only if, for all $0 \leqq h \in B$, $f \cap h$ is proper Riemann integrable and $\lim_{0 \leqq h \in B} I^+(f \cap h)$ exists. In particular, given μ and Ω as in (4), the Riemann-μ-integrable functions are exactly the improper integrable functions with respect to the proper integrable functions $R_{\text{prop}}(B, I)$ (see for example [7] p. 216 or [6] p. 275). For $\Omega = \{\text{intervals} \subset \mathbf{R}^n\}$, μ-Lebesgue measure μ_L in Ω, $R_{\text{prop}}(B_\Omega, I_\mu)$ is the classical Riemann integrable function space.

Finally, the upper Riemann integral I^- is an upper functional in the sense of Anger and Portenier [1]; in this paper, with the notations of [1], $\nu = I^-$ is admissible and $\nu^\bullet = I_l^-$ is the essential upper functional associated with I^-, so that, $R_1(B, I) = \mathcal{I}^\bullet(I^-) =$ the set of all essentially integrable functions (with respect to I^-). Consequently, a new integrability criteria of R_1 is obtained, by applying Corollary 3.7 in [1], and the results of the above sections apply to $\mathcal{I}^\bullet(I^-)$.

EXAMPLE ([6], p. 270). Let $X_1 = \mathbf{R}$, $\Omega = \{]a,b];\ a,b \in \mathbf{R},\ a \leqq b\}$, $\mu_L(]a,b]) = b-a$, $X_2 = \mathbf{N}$, $\Omega_2 = \{A \subset \mathbf{N};\ A \text{ finite or } \mathbf{N}-A \text{ finite}\}$, $\lambda(A) = 0$ if A finite, $= 0$ elsewhere. $X_3 = X_1 \times X_2$.

Let $I_1 = I_{\mu_L}$, $I_2 = I_\lambda$, $B_1 = B_{\Omega_1}$, $B_2 = B_{\Omega_2}$, $B_3 = B_{\Omega_1 \times \Omega_2}$, $I_3 = I_1 \circ I_2$ $= \int . d(\mu_L \times \lambda)$.

Let $f := \sum_{n=1}^{\infty} \chi_{]n,n+1] \times \{n\}}$. For all $x \in \mathbf{R}$, $f_x \in B_2$, $I_2(f_x) = 0$ and $I_1(I_2 f)$ $= 0$.

For all $n \in \mathbf{N} = X_2$, $f^n \in B_1$, $I_1(f^n) = 1$ and $I_2(I_1 f) = 1$.

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