

## ON ABSTRACT FUBINI THEOREMS FOR FINITELY ADDITIVE INTEGRATION

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**ABSTRACT.** A Fubini theorem for positive linear functionals on the vector lattice of the real-valued functions is given. This result properly contains that of the Riemann- $\mu$ -abstract integral.

### INTRODUCTION

In [3] one starts with a functional  $I: B \rightarrow \mathbb{R}$ , defined on the vector lattice  $B$  of real-valued functions on a set  $X$  and assumed to be positive linear. One defines the extended function class  $R_1(B, I)$  and extends  $I$  to  $R_1(B, I)$  via one or the other of three classical methods (certain limits of elementary functions, equality of the upper and lower integrals, closure of  $B$  with respect to an  $R_1$  type seminorm), and one gets the convergence theorems using a suitable "local convergence in measure". Riemann- $\mu$ , abstract Riemann-Loomis and Bourbaki integrals are subsumed.

In [5] Elsner has given a Fubini type theorem for the abstract Riemann- $\mu$ -integral. In this note, the integral extension of Lebesgue power introduced in [2] and [3] is used to develop a Fubini type theorem in quite general settings.

Let  $I_1$  and  $I_2$  be positive linear functionals on vector lattices over  $X_1$  and  $X_2$ , respectively. A methodological simplification is obtained by constructing the iterated integrals, via a suitable extension of the linear functionals. Conditions are determined for an integrable function  $f: X_1 \times X_2 \rightarrow \overline{\mathbb{R}}$ , without assuming continuity, so that the iterated integrals exist and are equal. So, our results are a reasonable substitute for Fubini's theorem for finitely additive integration (or for corresponding to the analogues to the Daniell extension process, but without continuity assumptions on the elementary integral  $I$ ).

### 1. PRODUCT SYSTEMS

On the extended real line  $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$  we adopt the usual conventions  $0(\pm\infty) := 0$  and  $\infty + (-\infty) := 0$ . We denote  $a \vee b := \max(a, b)$ ,  $a \wedge b := \min(a, b)$ ,  $a, b \in \overline{\mathbb{R}}$ .

Terminology and notation used are similar to that of [2], [3] and [10].

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(1) Throughout this note we shall assume that for  $j = 1, 2$ ,  $X_j$  is an arbitrary set,  $B_j \subset R^{X_j}$  a vector lattice (with respect to pointwise operations) and  $I_j: B_j \rightarrow \mathbb{R}$  a linear functional which is positive, i.e.,  $I_j(f) \geq 0$  for all  $f \geq 0$  in  $B_j$ .

Let  $X_3 := X_1 \times X_2$  and  $B_3 \subset R^{X_3}$  a vector lattice. For  $f \in \overline{\mathbb{R}}^{X_3}$  and for  $x \in X_1$  we define the function  $f_x$  on  $X_2$  by  $f_x(y) = f(x, y)$  for each  $y \in X_2$ . Let  $f$  be a function on  $X_3$  such that  $f_x \in B_2$  for each  $x \in X_1$ . Then setting  $(I_2f)(x) := I_2(f_x)$  for each  $x \in X_1$ , we have defined the function  $I_2f$  on  $X_1$ .

(2) A system  $(X_3, B_3)$  is called a *product system* with respect to  $(X_1, B_1)$  and  $(X_2, B_2)$ , whenever for each  $f \in B_3$  the following conditions are satisfied:

- (i)  $f_x \in B_2$  for each  $x \in X_1$ .
- (ii)  $I_2f \in B_1$ .

In all that follows  $(X_3, B_3)$  will be a product system. We define a positive linear functional on  $B_3$  by the rule  $I_3(f) := I_1(I_2f)$  for each  $f \in B_3$ .

## 2. THE ABSTRACT FUBINI THEOREM

**2.1. Proper Riemann integration.** (3) For  $f \in \overline{\mathbb{R}}^{X_j}$ ,  $j = 1, 2, 3$ , we define *Riemann upper and lower integrals* by

$$I_j^-(f) := \inf\{I_j(h); f \leq h \in B_j\}, \text{ with } \inf \emptyset := \infty \text{ and } I_j^+(f) := -I_j^-(-f).$$

$I_j^-$  is positively homogeneous and subadditive on  $\overline{\mathbb{R}}^{X_j}$ .

For  $f \in \overline{\mathbb{R}}^{X_3}$  we define the function  $I_2^-f: X_1 \rightarrow \overline{\mathbb{R}}$  by  $(I_2^-f)(x) := I_2^-(f_x)$  for each  $x \in X_1$ . Similarly,  $(I_2^+f)(x) := -I_2^-(-f_x)$ .

**Lemma 1.** *If  $f \in \overline{\mathbb{R}}^{X_3}$ , then  $I_1^-(I_2^-f) \leq I_3^-(f)$  and  $I_3^+(f) \leq I_1^+(I_2^+f)$ .*

*Proof.* By (3) and (ii) of (2), one has  $I_3^-(f) := \inf\{I_1(I_2h); f \leq h \in B_3\} \geq \inf\{I_1(I_2h); I_2^-f \leq I_2^-h = I_2h, h \in B_3\} \geq \inf\{I_1(g); I_2^-f \leq g \in B_1\} =: I_1^-(I_2^-f)$ .

The rest of the proof is similar.  $\square$

(4) The set  $R_{\text{prop}}(B_j, I_j)$  of *proper Riemann integrable functions* is defined as the set of those functions  $f \in \overline{\mathbb{R}}^{X_j}$  such that any one of the following conditions, which are equivalent, is satisfied.

- (i) Given any  $\varepsilon \in \mathbb{R}^+$ , there exist  $h, g \in B_j$  such that  $I_j(h - g) < \varepsilon$ , with  $g \leq f \leq h$ .
- (ii)  $I_j^+(f) = I_j^-(f) \in \mathbb{R}$ .

We have that  $R_{\text{prop}}(B_j, I_j)$  is the closure of  $B_j$  with respect to the integral seminorm  $I_j^-(|\cdot|)$ . If  $f \in R_{\text{prop}}(B_j, I_j)$ ,  $I_j(f) := I_j^+(f) = I_j^-(f)$  (see, for example, [1], [2]).

$A \subset X_j$  is called an  $I_j^-$ -null set iff  $I_j^-(\chi_A) = 0$ .

**Theorem 1.** *If  $f \in R_{\text{prop}}(B_3, I_3)$ , then:*

- (i)  $I_2^-f, I_2^+f \in R_{\text{prop}}(B_1, I_1)$ .
- (ii) There exist  $A_k \subset X_1$ ,  $k \in \mathbb{N}$ ,  $I_1^-$ -null sets, such that  $f_x \in R_{\text{prop}}(B_2, I_2)$  for all  $x \in X_1 - \bigcup_1^\infty A_k$ .
- (iii) There exists  $g \in R_{\text{prop}}(B_1, I_1)$  defined by  $I_2^-(f_x)$  if  $f_x \in R_{\text{prop}}(B_2, I_2)$ , and such that  $I_3(f) = I_1(g)$ .

*Proof.* (i) For  $f \in R_{\text{prop}}(B_3, I_3)$ , by (4), (3) and Lemma 1, we have

$$I_3(f) := I_3^-(f) \geq I_1^-(I_2^- f) \geq \left\{ \begin{array}{l} I_1^-(I_2^+ f) \\ I_1^+(I_2^- f) \end{array} \right\} \geq I_1^+(I_2^+ f) \geq I_3^+(f) := I_3(f).$$

Then,  $I_3(f) = I_1^-(I_2^- f) = I_1^+(I_2^- f) \in \mathbb{R}$ , and by (4)  $I_2^- f \in R_{\text{prop}}(B_1, I_1)$ . Similarly,  $I_2^+ f \in R_{\text{prop}}(B_1, I_1)$ .

(ii) For  $x \in X_1$ , set  $h(x) := I_2^-(f_x) - I_2^+(f_x)$ . One has  $0 \leq h \in R_{\text{prop}}(B_1, I_1)$  and  $I_1(h) = 0$ .

Now, let  $A_k := \{x \in X_1; h(x) \geq \frac{1}{k}\}$ ,  $k \in \mathbb{N}$ . Since  $I_1^-(\chi_{A_k}) \leq kI_1(h) = 0$ ,  $A_k$  are  $I_1^-$ -null sets, and by (4),  $f_x \in R_{\text{prop}}(B_2, I_2)$  for all  $x \in X_1 - \bigcup_1^\infty A_k$ .

(iii) Finally, if  $g \in \overline{\mathbb{R}}^{X_1}$  such that  $I_2^-(f_x) \leq g(x) \leq I_2^+(f_x)$  for all  $x \in X_1 - \bigcup_1^\infty A_k$ , then, by (4), we obtain  $g \in R_{\text{prop}}(B_1, I_1)$  and  $I_1(g) = I_1^-(I_2^- f) = I_3(f)$ .  $\square$

Observe that if  $l \in R_{\text{prop}}(B_1, I_1)$  such that  $l(x) = I_2^-(f_x)$  whenever  $f_x \in R_{\text{prop}}(B_2, I_2)$ , then  $I_1(l) = I_3(f)$  and  $I_1^-(|g - l|) = 0$ .

*Remarks 1.* 1. In general  $I_3^+(f) \geq I_1^+(I_2^+ f)$  is false for all  $f \in \overline{\mathbb{R}}^{X_3}$ , by 3.4 and Example 2 below. Therefore, an analogue to Theorem 1 for summable functions of [2] is in general not true.

Nevertheless, if  $I$  is monotone-net-continuous = Bourbaki's continuity condition, then Daniell  $L^1(B, I) \subset$  Bourbaki extension  $L^\tau$  and  $I^+ =$  upper Bourbaki extension  $I^\tau$ . In this case,  $I_3^\tau \geq I_1^\tau(I_2^\tau)$  and there is an analogue to Theorem 1 (see [2], [6] and [13], p. 186).

2. For arbitrary  $I/B$  it is easy to show that if  $(h_n) \subset B$ ,  $0 \leq h_{n+1} \leq h_n$ ,  $n \in \mathbb{N}$ ,  $I(h_n) \rightarrow 0$ , as  $n \rightarrow \infty$ , then there exist  $A_k \subset X$  such that  $I^-(\chi_{A_k}) = 0$ , and that if  $x \notin \bigcup_1^\infty A_k$ , then  $h_n(x) \rightarrow 0$ , as  $n \rightarrow \infty$ .

In general,  $\bigcup_1^\infty A_k$  is not an  $I^-$ -null set by 3.4 and Example 3 below. If  $I$  is  $\sigma$ -continuous, then  $\bigcup_1^\infty A_k$  is an  $I^\sigma$ -null set (see [9], p. 265).

**2.2. Abstract Riemann integration.** (5) For any  $f \in \overline{\mathbb{R}}^{X_j}$ ,  $j = 1, 2, 3$ , the corresponding localized functionals in the sense of Schäfer [14] are defined by

$$I_{j,l}^-(f) := \sup\{I_j^-(f \wedge h); 0 \leq h \in B_j\},$$

and for  $f \in \overline{\mathbb{R}}^{X_3}$ ,  $I_{2,l}^- f: X_1 \rightarrow \overline{\mathbb{R}}$  is defined by  $(I_{2,l}^- f)(x) := I_{2,l}^-(f_x)$  for each  $x \in X_1$ .  $I_{j,l}^-$  is monotone and subadditive on  $\overline{\mathbb{R}}^{X_j}$ .

In view of the definitions involved, we have

(6)  $(I_{j,l}^-)_{j,l} = I_{j,l}^- \leq I_j$ , and  $I_{j,l}^-(f) = I_j(f)$  if  $f \in \overline{\mathbb{R}}^{X_j}$  and  $f \leq$  some  $h \in B_j$ .

(7) For  $j = 1, 2, 3$ , the set  $R_1(B_j, I_j)$  of  $I_j$ -integrable functions is defined as the closure of  $B_j$  in  $\overline{\mathbb{R}}^{X_j}$  with respect to the integral seminorm  $I_{j,l}^- (|\cdot|)$ .

$R_1(B_j, I_j)$  is closed with respect to  $\pm, \alpha \cdot (\alpha \in \mathbb{R}); |\cdot|, \wedge, \vee$ .  $I_{j,l}^- / R_1(B_j, I_j)$  is a positive linear functional (= unique  $I_{j,l}^-$ -continuous extension of  $I_j / B_j$ ).

By [3],  $R_1(B_j, I_j)$  is the set of all  $f \in \overline{\mathbb{R}}^{X_j}$  to which there exists a sequence  $(h_n) \subset B_j$ , which is a Cauchy sequence with respect to  $I_j(|\cdot|)$  and with  $h_n \rightarrow f(I_j^-)$ ; then  $I_j(f) := \lim I(h_n)$ ,  $n \rightarrow \infty$ . In general,  $R_{\text{prop}} \subset R_1$  with

coinciding integrals, and  $\subset$  generally strict. For further properties of  $R_1$  see [3] and [10].

In all that follows we assume the following condition

- (\*)  
 To  $h \in B_1, g \in B_2$  there exists  $l \in B_3$  such that  $g(y) \leq l(x, y)$  if  $h(x) > 0$ .

**Lemma 2.** Let  $f \in \overline{\mathbb{R}}^{X_3}$  such that the following condition holds:

(\*\*)  
 $|f_x| \leq g \in B_2$  for each  $x \in X_1$ .

Then,  $I_{1,l}^-(I_{2,l}^- f) \leq I_{3,l}^-(f)$ .

*Proof.* For  $f_x \in \overline{\mathbb{R}}_+^{X_2}, x \in X_1$ , we have with (\*\*) and (5)  $(I_{2,l}^- f)(x) = (I_2^- f)(x)$ , so that  $I_{1,l}^-(I_{2,l}^- f) \leq I_{1,l}^-(I_2^- f) := \sup\{I_1^-((I_2^- f) \wedge h); 0 \leq h \in B_1\}$ .

Now, with (\*),  $(I_2^- f) \wedge h \leq I_2^- f \leq I_2^- f \wedge l$ , where  $l \in B_3$  and  $f_x \leq g \leq l_x$  for each  $x \in X_1, h(x) > 0$ . Hence, with Lemma 1 and (5), we have  $I_1^-((I_2^- f) \wedge h) \leq I_1^-(I_2^- f \wedge l) \leq I_3^-(f \wedge l) \leq I_{3,l}^-(f)$ , for all  $0 \leq h \in B_1$ , and we conclude the result.  $\square$

Without (\*\*) Lemma 2 becomes false by p. 270 of [5]. Here there exists  $f \in \overline{\mathbb{R}}_+^{X_3}$  such that  $I_{1,l}^-(I_{2,l}^- f) = \infty$  and  $I_{3,l}^-(f) = 0$ .

Theorem 2 is obtained now in a similar way as Satz p. 141 of Hoffman [11] (see also Elsner [5]).

**Theorem 2.** Let  $(X_3, B_3)$  be a product system, and let  $f \in R_1(B_3, I_3)$  satisfying (\*\*). Then the following assertions hold:

- (i) There exist  $A_k \subset X_1, k \in \mathbb{N}, I_{1,l}^-$ -null sets, such that  $f_x \in R_1(B_2, I_2)$  for each  $x \in X_1 - \bigcup_1^\infty A_k$ .
- (ii) There exists  $g \in R_1(B_1, I_1)$  defined by  $I_{2,l}^- f(x)$  if  $f_x \in R_1(B_2, I_2)$ , and such that  $I_{1,l}^-(g) = I_{3,l}^-(f)$ , i.e.  $I_{3,l}^-(f) = I_{1,l}^-(I_{2,l}^- f)$ .

*Proof.* (i) By (7), for  $f \in R_1(B_3, I_3)$ , given  $\varepsilon > 0$  there exists  $g \in B_3$  such that  $I_{3,l}^- (|f - g|) < \varepsilon$ .

For each  $x \in X_1$  set  $\varphi(x) := \inf\{I_{2,l}^- (|f_x - h|), \text{ for all } h \in B_2\}$  and set  $A_k := \{x \in X_1; \varphi(x) \geq \frac{1}{k}\}, k \in \mathbb{N}$ . By virtue of Lemma 2, it can be easily proved that the sets  $A_k, k \in \mathbb{N}$ , are  $I_{1,l}^-$ -null, and (i) follows immediately.

To prove (ii) it suffices to see that there is  $(I_{2,l}^- g_n) \subset B_1$  such that  $I_{2,l}^- g_n \rightarrow g(I_{1,l}^-)$ , where  $(g_n) \subset B_3$  and  $I_{3,l}^- (|g_n - f|) \rightarrow 0$ , as  $n \rightarrow \infty$ .

In fact, a calculation analogous to the proof given in [11] (Hauptsatz, p. 139, with ‘‘Fubini-integral norms’’), and having in mind the properties stated in (6) and (7), permits to show the inequality

$$I_{1,l}^- (|(I_{2,l}^- g_n) - g|) \leq 3I_{1,l}^- (I_{2,l}^- |g_n - f|) \leq 3I_{3,l}^- (|g_n - f|).$$

Besides,  $I_{1,l}^- (I_{2,l}^- g_n) \rightarrow I_{1,l}^- (g)$ , as  $n \rightarrow \infty$ , and  $I_{1,l}^- (g) = I_{3,l}^- (f)$ .  $\square$

*Remarks 2. 1.* In the above statement usually all assumptions are essential. There exist counterexamples for the  $\lambda \times \mu$ -case (additive measure space, see 3.1. below) in [5], Bem.4.b, and 4.c.p.270. Similar examples show that one cannot substitute  $|f_x| \leq g$  for  $f_x \leq g$  in (\*\*).

2. There are simple examples of  $f \in R_1(\lambda \times \mu, \mathbb{R})$  with (\*\*), but  $f \notin R_{\text{prop}}(\lambda \times \mu, \mathbb{R})$ :

$$f = \chi_{X_1 \times M}, M \in \text{ring } \Omega_2, \text{ with } \mu(X_1) = \infty.$$

3. Let us finally remark that our results can be reformulated for Banach space-valued functions, using  $f \cap g := \|f\|^{-1}(\|f\| \wedge g)f$ , with  $f: X \rightarrow E = \text{Banach space}$ ,  $g \in \mathbb{R}_+^X$ , of [9], p. 327.

### 3. APPLICATIONS AND EXAMPLES

1. If  $\Omega$  is a semiring of sets  $\subset X$  and  $\mu: \Omega \rightarrow [0, \infty[$  is additive, then  $B = B_\Omega := \text{real-valued step functions over } \Omega$  and  $I = I_\mu := \int \cdot d_\mu$  satisfying (1).

Then the *proper Riemann- $\mu$ -integrable functions*  $R_{\text{prop}}(\mu, \mathbb{R}) = I_\mu^-$ -closure of  $B_\Omega$  in  $\mathbb{R}^X$ , in the sense of Aumann [1], p. 448.

The space of *abstract Riemann- $\mu$ -integrable functions*  $R_1(\mu, \mathbb{R})$  was presented essentially by Loomis [12]. For Banach space-valued functions it has been introduced by Dunford-Schwartz [4], and in more general form by Günzler [8], [9].  $R_{\text{prop}}(\mu, \mathbb{R}) \subset \text{Dunford-Schwartz integral } L(X, \Omega, \mu, \mathbb{R}) \subset R_1(\mu, \mathbb{R})$ , with coinciding integrals; all  $\subset$  are in general strict (see Lemma 9 of [10] and [9], pp. 199, 70).

In Gould [7], Stone's axiom  $B \wedge 1 \subset B$  is assumed, so by [8] his results are already subsumed by the abstract Riemann integral (see, for example, [9], pp. 57, 268).

If  $\Omega_1$  and  $\Omega_2$  are semirings of sets from  $X_1$  and  $X_2$ , and  $\mu_1$  and  $\mu_2$  are additive measures on  $\Omega_1$  and  $\Omega_2$ , respectively, one can construct a product additive measure  $\mu_3$  in the set  $X_3 := X_1 \times X_2$  and the induced integral  $I_{\mu_3}$ .

If we set  $\Omega_3 := \{A_1 \times A_2; A_j \in \Omega_j, j = 1, 2\}$ , then,  $\mu_3(A_1 \times A_2) := \mu_1(A_1) \cdot \mu_2(A_2) = I_{\mu_3}(\chi_{A_1 \times A_2})$ . See [13], §16; [11], p. 125.

2. If  $B = B_\Omega$  with  $\Omega = \sigma$ -ring and  $I = I_\mu$  with  $\mu$   $\sigma$ -additive, then  $R_1(\mu, \mathbb{R}) = L^1(\mu, \mathbb{R})$  ( $:= \text{Lebesgue-}\mu\text{-integrable functions}$ ), and  $f_n \rightarrow f$   $\mu$ -almost everywhere implies  $f_n \rightarrow f(I_\mu^-)$  for  $\mu$ -measurable  $f_n$ , by [9], p. 265; and we get the usual Lebesgue convergence theorems.

In [5] Elsner has given a very thorough and interesting treatment of the Fubini theorem for the abstract Riemann- $\mu$ -integral. Our results contain properly that of [5], for which we obtain simplified proofs. Indeed, Example 1 below shows that there exist functions for which Theorem 1 is applicable, but not the corresponding result of [5] or even [11], p. 129.

Observe that for the  $\lambda \times \mu$ -case, (\*) holds and (\*\*) means that  $|f|$  is bounded and there exists  $P \in \text{ring generated } \Omega_2$  such that  $\text{supp}(f) \subset X_1 \times P$ .

In [14] integration with local Loomis-Schäpfke integral seminorms is obtained. With (7), we have  $R_1(B, I) \cap \mathbb{R}^X = \text{Schäpfke local } I_1^-$ -closure of  $B$ , and  $R_1(B, I) \cap \mathbb{R}^X = \text{"one-sided completion" of Loomis [12], p. 170}$ .

3. We denote by  $B_1 \otimes B_2$  the vector space of functions on  $X_3$  generated by the family  $\{g \otimes h; g \in B_1, h \in B_2\}$ , where  $(f \otimes k)(x, y) := f(x) \cdot k(y)$  for arbitrary  $f$  and  $k$  on  $X_1$  and  $X_2$ , respectively. If  $|f| \in B_1 \otimes B_2$  whenever  $f \in B_1 \otimes B_2$ , then  $B_1 \otimes B_2$  is a product system with respect to  $(B_1, I_1)$  and  $(B_2, I_2)$  (see [13], §15; [6], p. 187), and Sections 1 and 2 are applicable.

4. Using Examples 1-3 below it is not difficult to check that there are finitely additive  $\lambda, \mu$  on rings and  $f, g, h_n$  with  $0 \leq f = I_{\lambda \times \mu}^-$ -null functions, but

$f_x \in R_{\text{prop}}(\mu, \mathbb{R})$  for no  $x \in X_1$  (Example 1);  $g \in R_1(\lambda \times \mu, \mathbb{R})$ , all  $g^y \in B_1$ ,  $I_1(g^y) \in B_2$ , but  $\int g d(\lambda \times \mu) \neq \int (\int g d\lambda) d\mu$  (Example 2);  $0 \leq h_{n+1} \leq h_n$ ,  $\int h_n d\mu \rightarrow 0$ , but  $h_n(x) \rightarrow 0$  for no  $x$  (Example 3).  $g$  can be found in Elsner [5], p. 270.

**Example 1.** Let  $X_1 = X_2 = \mathbb{N}$ ,  $\Omega_1 = \Omega_2 = \{\mathbb{N} - E, E; \text{finite set } \subset \mathbb{N}\}$ ,  $\mu_1 = \mu_2 = \mu$  finitely additive measure, such that  $\mu(E) := 0$ ,  $\mu(\mathbb{N}) := 1$ .

Let  $X_3 := \mathbb{N} \times \mathbb{N}$ ,  $\Omega := \{X_3 - E, E; E \text{ finite set } \subset X_3\}$ , and  $\nu: \Omega \rightarrow \mathbb{R}$ ,  $\nu(E) := 0$ ,  $\nu(X_3) := 1$ .

**Example 2.** Let  $X_1 = \mathbb{R}$ ,  $\Omega_1 := \{]a, b]; a, b \in \mathbb{R}, a \leq b\}$ ,  $\lambda(]a, b]) := b - a$ ;  $X_2 = \mathbb{N}$ ,  $\Omega_2 := \{\mathbb{N} - E, E; E \text{ finite set } \subset \mathbb{N}\}$ ,  $\mu(\mathbb{N}) = 1$ ,  $\mu(E) = 0$ , and  $X_3 := X_1 \times X_2 = \mathbb{R} \times \mathbb{N}$ . Let  $I_1 = I_\lambda$ ,  $I_2 = I_\mu$ ,  $B_1 = B_{\Omega_1}$ ,  $B_2 = B_{\Omega_2}$ ,  $B_3 := B_{\Omega_1 \times \Omega_2}$ ,  $I_3 := I_1 \circ I_2 = \int \cdot d(\lambda \times \mu)$ .

**Example 3.** Let  $X = \mathbb{N}$ ,  $\Omega = \{\mathbb{N} - E, E; E \text{ finite set } \subset \mathbb{N}\}$ ,  $\mu(\mathbb{N}) = 1$ ,  $\mu(E) = 0$ ,  $B = B_\Omega$ , and  $I = I_\mu$ .

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