Galois theory for Hopf-Galois extensions

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For a comodule algebra over a Hopf algebra over arbitrary commutative ring we construct a Galois correspondence between the complete lattices of of subalgebras and the complete lattice of generalised quotients of the structure Hopf algebra. The construction involves techniques of lattice theory and of Galois connections. Such a 'Galois Theory' generalises the classical Galois Theory for field extensions, and some important results of S. Chase and M. Sweedler, F. van Oystaeyen, Y.H. Zhang and P. Schauenburg.

Galois Theory for Hopf Galois Extensions

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Hopf Galois Extensions

Definition (Hopf Galois Extension)

Let H be a Hopf algebra (over a ring R),
*let A be an H-**comodule algebra**, i.e.*

- *algebra over R,*
- *(left) H-comodule: $\delta : A \longrightarrow H \otimes A$,*
- *δ is an algebra homomorphism.*

$^{co\,H}A := \{a \in A : \delta(a) = 1_H \otimes a\}$ *the **subalgebra of coinvariants**.*

*$A/\,^{co\,H}A$ is H-**Hopf Galois** iff the canonical map:*

$$can : A \otimes_{^{co\,H}A} A \longrightarrow H \otimes A, \quad a \otimes b \mapsto \delta(a)b$$

is an isomorphism (in $^{H}Mod_A$).

Galois Connections

Let $(P, \succcurlyeq_P)$, $(Q, \succcurlyeq_Q)$ be two posets. A pair of antimonotonic maps:

$$P \underset{\psi}{\overset{\varphi}{\rightleftarrows}} Q$$

is called a **Galois connection** if

$$\varphi \circ \psi \succcurlyeq id_P \text{ and } \psi \circ \varphi \succcurlyeq id_Q$$

*An element $p \in P$ ($q \in Q$) is called **closed** if*

$$\varphi\psi(p) = p \qquad \psi\varphi(q) = q$$

We let $\overline{P}$, $\overline{Q}$ denote the subsets of *closed elements.*

Properties:

1. $\overline{P} = \psi(Q)$ and $\overline{Q} = \varphi(P)$,
2. $\varphi|_{\overline{P}}$ and $\psi|_{\overline{Q}}$ are *inverse bijections* between $\overline{P}$ and $\overline{Q}$,
3. φ is *mono* (epi) iff ψ is *epi* (mono),
4. φ (ψ) is *isomorphism* then ψ (φ) is its *inverse.*

The lattice of generalised quotients

Definition

Let H be a Hopf algebra. Then

$$\mathrm{Quot}_{gen}(H) := \{H/\mathcal{J} : \ \mathcal{J} - \textit{coideal left ideal}\}$$

*is a **poset**, with partial order:* $H/I \geq H/\mathcal{J} \Leftrightarrow I \subseteq \mathcal{J}$

The poset $\mathrm{Id}_{gen}(H)$ is a *lattice*, i.e. there exists finite infima and suprema: for $\mathcal{J}_1, \mathcal{J}_2 \in \mathrm{Id}_{gen}(H)$.

$$\mathcal{J}_1 \vee \mathcal{J}_2 = \mathcal{J}_1 + \mathcal{J}_2, \qquad \mathcal{J}_1 \wedge \mathcal{J}_2 = \sum_{\substack{\mathcal{J} \subseteq \mathcal{J}_1 \cap \mathcal{J}_2 \\ \mathcal{J} \in \mathrm{Id}_{gen}(H)}} \mathcal{J} \quad \subseteq \ \mathcal{J}_1 \cap \mathcal{J}_2$$

Proposition

The lattice $\mathrm{Quot}_{gen}(H)$ *is **complete**, i.e. there exists arbitrary suprema and infima.*

Modules with intersection property

Definition

Let M, N be an R-modules, and let $(N_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in I}$ be a family of submodules of an R-module N. If M is flat then there exists the canonical map:

$$M \otimes_R \Big(\bigcap_{\boldsymbol{\alpha}\in I} N_{\boldsymbol{\alpha}}\Big) \longrightarrow \bigcap_{\boldsymbol{\alpha}\in I} (M \otimes_R N_{\boldsymbol{\alpha}})$$

We say that a flat module M has the ***intersection property with respect to*** *N if the above homomorphism is an* ***isomorphism*** *for any family of submodules $(N_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in I}$.*

We say that M has ***intersection property*** *if the above condition holds for any R-module N.*

Modules with intersection property

Proposition

*Every **projective module** has the **intersection property**.*

Example

Let p be a prime ideal of the ring $\mathbb{Z}$. Then $\bigcap_i p^i = \{0\}$. Let $\mathbb{Z}_p$ denote the localisation of $\mathbb{Z}$ at the prime ideal p.

The $\mathbb{Z}$-module $\mathbb{Z}_p$ is flat. Then $\mathbb{Z}_p \otimes_{\mathbb{Z}} \bigcap_i p^i = \{0\}$. From the other side $\bigcap_i \mathbb{Z}_p \otimes_{\mathbb{Z}} p^i = \bigcap_i \mathbb{Z}_p = \mathbb{Z}_p$.

By similar argument $\mathbb{Q} \otimes_{\mathbb{Z}} -$ doesn't posses the intersection property.

The reason for this is that, the *intersection property* is stable under arbitrary *sums* but not under *cokernels*.

Modules with intersection property

Theorem

Every flat R-module has the ***intersection property*** *if and only if for any exact sequence:*

$$0 \to M' \to M \to M'' \to 0$$

with M', M projective, M'' flat and any family of submodules $(\mathrm{N}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in I}$ of an R-module N the sequence:

$$0 \to \bigcap_{\boldsymbol{\alpha}} \left(M' \otimes_R \mathrm{N}_{\boldsymbol{\alpha}}\right) \to \bigcap_{\boldsymbol{\alpha}} \left(M \otimes_R \mathrm{N}_{\boldsymbol{\alpha}}\right) \to \bigcap_{\boldsymbol{\alpha}} \left(M'' \otimes_R \mathrm{N}_{\boldsymbol{\alpha}}\right) \to 0$$

is ***exact****.*

Galois Theory for H-comodule algebras

Theorem

Let A/B be an H-comodule algebra over a ring R, such that one of the following conditions is satisfied:

- *every finitely generated R-submodule of A is projective and A has the intersection property with respect to H, or*
- *A is projective as an R-module.*

Then there exists a ***Galois connection****:*

$$\mathrm{Sub}_{\mathrm{Alg}}(A) \underset{{}^{co\,Q}A \longleftarrow\!\shortmid\, Q}{\overset{\psi}{\rightleftarrows}} \mathrm{Quot}_{gen}(H)$$

where

$$\psi(A') = \bigvee\{Q \in \mathrm{Quot}_{gen}(H) : A' \subseteq A^{co\,Q}\}$$

F. van Oystaeyen and Y. Zhang result

Theorem *(F. van Oystaeyen and Y. Zhang)*
Let $k \subseteq \mathbb{F}$ be a field extension and let H be a commutative and cocommutative k-Hopf algebra. Let $\mathbb{F} \subseteq \mathbb{E}$ be a field extension and an H-Hopf–Galois extension.
Then there is a ***one-to-one correspondence****:*

$$\left\{ \begin{array}{c} H\text{-subcomodule} \\ \text{subfields of } \mathbb{E} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{Hopf ideals of} \\ \mathbb{F} \otimes_k H \end{array} \right\}$$

If I – a Hopf ideal, and $\mathbb{M}$ – an intermediate field extension of $\mathbb{E}/\mathbb{F}$, which corresponds to each other then

$$\mathbb{E}/\mathbb{M} \text{ is } \mathbb{F} \otimes_k (H/I)\text{-Hopf–Galois.}$$

P. Schauenburg Galois correspondence

For a right H-Hopf Galois extensions A of the base ring R there exists a Hopf algebra $L(H, A)$ such that A becomes $L(H, A)$-H-biGalois extension.
As an algebra $L(H, A)$ is a subalgebra of $A \otimes A^{op}$ given by $(A \otimes A)^{co\,H}$.

Theorem

*Let $R \subseteq A$ be a faithfully flat H-Hopf–Galois extension of a ring R. Then there is the following **Galois connection**:*

$$\mathsf{Sub}_{\mathsf{Alg}}(A) \underset{}{\overset{\psi}{\rightleftarrows}} \mathsf{Quot}_{gen}(L(H, A))$$

$${}^{co\,Q}A \longleftarrow\!\mid Q$$

$$\psi(B) = (A \otimes_B A)^{co\,H}$$

It restricts to bijection between:

- $Q \in \mathsf{Quot}_{gen}(L(H, A))$ *such that $L(H, A)$ is (right, left) faithfully flat over Q, and Q is flat R-module;*
- $B \in \mathsf{Sub}_{\mathsf{Alg}}(A)$ *such that A is (right, left) faithfully flat B module.*

Closed elements

Proposition

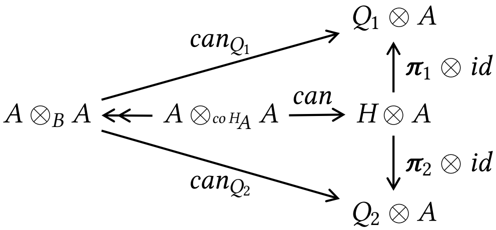
Let A be an H-comodule algebra with surjective canonical map:

$$A \otimes_{{}^{co\,H}A} A \xrightarrow{can} H \otimes A$$

*Let $Q_1, Q_2 \in \mathrm{Quot}_{gen}(H)$ such that $A/{}^{co\,Q_1}A$ and $A/{}^{co\,Q_2}A$ are Q-**Galois extensions**, then:*

$${}^{co\,Q_1}A = {}^{co\,Q_2}A \iff Q_1 = Q_2$$

Proof.



Closed elements

Corollary

*Let A be an H-comodule algebra with surjective canonical map. Let $Q \in \mathrm{Quot}_{gen}(H)$ be such that $A/ {}^{co\,Q}A$ is Q-**Galois extension**. Then Q is a **closed element** in:*

$$\mathrm{Quot}_{gen}(H) \rightleftarrows \mathrm{Sub}_{alg}(A/ {}^{co\,H}A)$$

Note: if H is *finite dimensional* then every $Q \in \mathrm{Quot}_{gen}(H)$ is *closed* and thus:

$$\varphi : \mathrm{Quot}_{gen}(H) \rightarrowtail \mathrm{Sub}_{\mathrm{Alg}}(A)$$

is a *monomorphism.*

Closed elements

Definition

Let C be a k-coalgebra and let $C \to \overline{C}$ be a quotient coalgebra.

Then $\widetilde{C}$ is called ***left (right) admissible*** *if it is k-flat and C is left (right) faithfully coflat over $\widetilde{C}$.*

Let S belong to $\mathrm{Sub}_{\mathrm{Alg}}(A/B)$ for an H-Hopf Galois extension.
We call S ***left (right)*** *H**-admissible*** *if the following conditions are satisfied:*

1. *A is left (right) faithfully flat over S,*
2. $can_S : A \otimes_S A \to A \otimes_{{}^{co\,\psi(S)}A} A \xrightarrow{can_{\psi(S)}} \psi(S) \otimes A$ *which is well defined since $S \subseteq {}^{co\,\psi(S)}A$, is a bijection, and*
3. *$\psi(S)$ is flat over R.*

An element is called ***admissible*** *if it is both left and right admissible.*

Closed elements

***Theorem** (P. Schauenburg)*

Let A/B be a faithfully flat H-Galois extension over a ring R, such that A is projective as an R-module.
Then the GALOIS CONNECTION*:*

$$\mathsf{Sub}_{\mathsf{Alg}}(A) \rightleftarrows \mathsf{Quot}_{gen}(H)$$
$$^{co\,Q}A \longleftarrow\!\!| \; Q$$

*gives rise to a **bijection** between **(left, right) admissible** objects (thus (left, right) admissible objects are closed).*

Note: *the proof of P. Schauenburg for Hopf Galois extensions of the base ring R works without a change.*

Galois coextensions

The base ring R is now assumed to be a field.

Definition

*Let C be a H-**module coalgebra**. (the H-module structure map is a coalgebra map.) Then $C^H := {}^{C}/_{CH^+}$ is a quotient coalgebra.*
*We call $C \to C^H$ a **Galois coextension** if the canonical map:*

$$can_H : C \otimes H \longrightarrow C \,\square_{C^H}\, C \quad can_H(c \otimes h) = c_{(1)} \otimes c_{(2)} h$$

is an isomorphism.

Let K be a right coideal of H. Then $C \to C^K := {}^{C}/_{CK^+}$ is a quotient coalgebra.
*We call C/C^K a **Galois coextension** if the canonical map:*

$$can_K : C \otimes K \longrightarrow C \,\square_{C^K}\, C \quad can_K(c \otimes k) = c_{(1)} \otimes c_{(2)} k$$

is an isomorphism.

Galois coextensions

Theorem

Let C be an H-module coalgebra over a field k. Then there exists a ***Galois connection****:*

$$\mathrm{Quot}\left({}^{C}\!/_{C^H}\right) \rightleftarrows \mathrm{cold}_r(H)$$

$$C/CI^+ \longleftarrow\!\shortmid H/I$$

where $\mathrm{cold}_r(H)$ *is a complete lattice of right coideals of H, and*

$$\mathrm{Quot}\left({}^{C}\!/_{C^H}\right) := \left\{\widetilde{C} \in \mathrm{Quot}(C) : C \twoheadrightarrow \widetilde{C} \twoheadrightarrow C^H\right\}$$

Closed elements

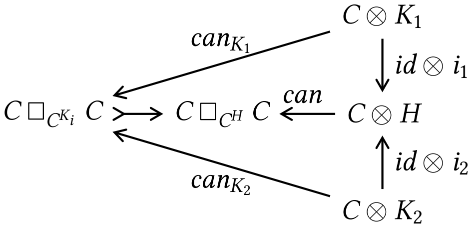
Proposition

Let C be an H-Galois coextension over a field k and $K_1, K_2 \in \mathsf{cold}_r(H)$ such that both:

$$can_{K_i} : C \otimes K_i \xrightarrow{\simeq} C \,\Box_{C^{K_i}}\, C \quad i = 1, 2$$

are bijections. Then $K_1 = K_2$, whenever $C^{K_1} = C^{K_2}$.

Proof.



□

Closed elements

Corollary

Let C be and H-Galois coextension. Then K - a right coideal of H is a ***closed element*** *of the Galois connection:*

$$\mathrm{Quot}\left({}^{C}\!/_{C^H}\right) \rightleftarrows \mathrm{cold}_r(H)$$

if C is K-Galois.

Let H be a Hopf algebra. Then there exists a *Galois connection*:

$$\{K \subseteq H:\ K\text{ - right coideal subalgebra}\} \underset{\varphi}{\overset{\psi}{\rightleftarrows}} \{H/I:\ I\text{ - left ideal coideal}\}$$
$$=: \mathsf{Sub}_{gen}(H) \qquad\qquad =: \mathsf{Quot}_{gen}(H)$$

where $(\varphi(Q) = {}^{co\,Q}H, \psi(K) = H/HK^{+})$ is the considered Galois connection. This Galois correspondence restricts to normal elements:

$$\{K \subseteq H:\ K\text{ - normal Hopf subalgebra}\} \underset{\varphi}{\overset{\psi}{\rightleftarrows}} \{H/I:\ I\text{ - normal Hopf ideal}\}$$
$$=: \mathsf{Sub}_{nHopf}(H) \qquad\qquad =: \mathsf{Quot}_{normal}(H)$$

1. $K \in \mathsf{Sub}_{gen}(H)$ such, that H is *faithfully flat* over K, is a ***closed element***,
2. $Q \in \mathsf{Quot}_{gen}(H)$ such, that H is *faithfully coflat* over Q, is a ***closed element***,
3. if H is *finite dimensional* then φ and ψ are ***inverse bijections***.

- Whenever H is finite dimensional, for every Q the extension ${}^{coQ}H \subseteq H$ is Q-Galois.
  Hence every Q is *closed* and the map
  $$\varphi : \mathsf{Quot}_{gen}(H) \longrightarrow \mathsf{Sub}_{gen}(H), \quad \varphi(Q) = {}^{co\,Q}H$$
  is a monomorphism.
- To show that we have a pair of isomorphisms it is enough to prove that
  $$\psi : \mathsf{Sub}_{gen}(H) \longrightarrow \mathsf{Quot}_{gen}(H), \quad \psi(K) = H/HK^{+}$$
  is a monomorphism.
  Let us consider H^{*}. To distinguish φ and ψ for H and H^{*} we will write φ_H and ψ_H for H and φ_{H^*} and ψ_{H^*} considering H^{*}.
  It turns out that:
  - $(\psi_H(K))^{*} = \varphi_{H^*}(K^{*})$, i.e. $(H/HK^{+})^{*} = {}^{co\,K^{*}}H^{*}$,
  - $can_{K^*} = (can_K)^{*} : H^{*} \otimes_{{}^{co\,K^*}H^*} H^{*} \to H^{*} \otimes K^{*}$.

  But can_{K^*} is an isomorphism, hence can_K is an isomorphism for every right coideal subalgebra K of H.

Theorem

Let H be a Hopf algebra Then the correspondence:

$$\{K \subseteq H:\ K - \textit{right coideal subalgebra}\} \overset{\simeq}{\rightleftarrows} \{H/I:\ I - \textit{left ideal coideal}\}$$

is a ***bijection*** *if and only if*

- *for every* $Q \in \mathsf{Quot}_{gen}(H)$, *the extension* ${}^{co\,Q}H \subseteq H$ *is Q-Galois.*
- ${}^{co\,H/K^{+}H}H \subseteq K$ *for every right coideal subalgebra K of H.*

Cleft Extensions

Theorem *(Doi and Takeuchi)*

Let $A/{}^{co\,H}A$ be an H-extension. Then

$$A/{}^{co\,H}A \text{ is } \textbf{cleft} \iff \begin{array}{l} A/{}^{co\,H}A \text{ is } H\text{-}\textbf{Galois} \text{ and} \\ \text{has the } \textbf{normal basis property} \end{array}$$

Theorem

*An element $Q \in \mathrm{Quot}_{gen}(H)$ is **closed** in the Galois connection:*

$$\mathrm{Quot}_{gen}(H) \underset{}{\overset{Q \longmapsto {}^{co\,Q}A}{\rightleftarrows}} \mathrm{Sub}_{alg}(A/{}^{co\,H}A)$$

*if and only if the extension $A/{}^{coQ}A$ is Q-**Galois**.*

Thank you.