Towards Sweedler's cohomology for weak Hopf algebras

Ana Belén Rodríguez Raposo (University of A Coruña, Spain)
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Let H be a cocommutative weak Hopf algebra and B a H-module algebra. We extend to this context some of the notions that arise in the study of Sweedler's cohomology. In particular, we define the concept of 2-cocycle and of cohomologous 2-cocycles. We use this notion to classify weak crossed products of B and H.

We also study H-extensions of B, and we obtain that a normal 2-cocycle induces a weak cleft H-extension of B. As a consequence of the classification of weak crossed products via cohomologous 2-cocycles, we finally obtain that cohomologous 2-cocycles induce isomorphic weak cleft H-extensions.

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In colaboration with J.M. Fernández Vilaboa and R. González Rodríguez

Outline

Some notation and conventions

- Our base category is a symmetric monoidal one, with base object K and split idempotents.
- (A, μ_A, η_A) is an algebra with multiplication μ_A and unit η_A.
- $(C, \delta_C, \varepsilon_C)$ is a coalgebra with comultiplication δ_C and counit ε_C.
- $A^{(n)} = A \otimes \overset{n \text{ times}}{\cdots} \otimes A$.
- If (A, μ_A, η_A) is an algebra then μ_A^n is the corresponding multiplication of $A^{(n)}$.
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Sweedler's cohomology and other related topics

Let H be a cocommutative Hopf algebra and B a commutative H-module algebra.

Semi-simplicial complex

$$K \longrightarrow Reg(H,B) \overset{\partial_3}{\underset{\partial_1}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{\partial_2}}}}} Reg(H \otimes H, B) \overset{\partial_4}{\underset{\partial_1}{\overset{\longrightarrow}{\underset{\longrightarrow}{\overset{\xrightarrow{\partial_3}}{\xrightarrow{\partial_2}}}}}} Reg(H^{(3)}, B) \ldots$$

1. $\partial_1(f) = \varphi_B(H \otimes f)$
2. $\partial_2(f) = f(\mu_H \otimes H)$
3. $\partial_3(f) = f(H \otimes \mu_H)$
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2-cocycle

$\sigma \in Reg(H \otimes H, B)$ is a 2-cocycle if $\partial_1(\sigma) \wedge \partial_3(\sigma) = \partial_4(\sigma) \wedge \partial_2(\sigma)$
The 2-cocycle τ is cohomologous to σ if $\tau \wedge \partial_2(\gamma) = \partial_1(\gamma) \wedge \partial_3(\gamma) \wedge \sigma$ for $\gamma \in Reg(H, B)$. $H^2(H, B)$ is the 2nd. group of cohomology.

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Crossed product

$B \otimes H$ is a crossed product if $\mu_{\varphi_B,\sigma}$ is associative and $\eta_B \otimes \eta_H$ is the unit.

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If H is a Hopf algebra, B an algebra and A a H-comodule algebra:

$B \hookrightarrow A$ is a cleft extension if and only if there exists $\varphi_B : B \otimes H \to B$ a weak action and a 2-cocycle $\sigma \in Reg(H \otimes H, B)$ such that $A \simeq B \otimes_{\sigma}^{\varphi_B} H$

$B \otimes_{\sigma}^{\varphi_B} H \simeq B \otimes_{\tau}^{\varphi_B} H$ if and only if σ and τ are cohomologous.

Clasifying crossed products

$H^2(H, B)$ gives a classification of crossed products with invertible cocycle and of cleft extensions.

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$B \hookrightarrow A$ is a cleft extension if and only if there exists $\varphi_B : B \otimes H \to B$ a weak action and a 2-cocycle $\sigma \in Reg(H \otimes H, B)$ such that $A \simeq B \otimes_{\sigma}^{\varphi_B} H$

$B \otimes_{\sigma}^{\varphi_B} H \simeq B \otimes_{\tau}^{\varphi_B} H$ if and only if σ and τ are cohomologous.

Clasifying crossed products

$H^2(H, B)$ gives a classification of crossed products with invertible cocycle and of cleft extensions.

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Recall that:

Weak Hopf algebras

A weak Hopf algebra H is an algebra and a coalgebra such that $\delta_H \circ \eta_H \neq \eta_H \otimes \eta_H$ (at least not necessarily).

- The behavior of $\delta_H \circ \eta_H$ is encoded by some idempotent morphisms $\Pi_{R,L} : H \to H$ and $\bar{\Pi}_{R,L} : H \to H$, that in the non weak case become $\eta_H \otimes \varepsilon_H$.

Preunits

If $\mu_A : A \otimes A \to A$ is an associative multiplication, $\nu : K \to A$ is a *preunit* if $\mu_A \circ (A \otimes \nu) = \mu_A \circ (\nu \otimes A)$ and $\mu \circ (\nu \otimes \nu) = \nu$. $\nabla_\nu = \mu_A(\nu \otimes A)$ is an idempotent morphism whose image is an algebra.

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Groupoids

Definition

A groupoid G is a (small) cateogory whose morphisms are isomorphisms. We denote by G_0 the set of objects and by G_1 the morphisms.
A group is a groupoid with one object.

Changing our point of view

Let H be a cocommutative Hopf algebra, (B, φ_B) a commutative H-module algebra and $f : H^{(n)} \to B$.

$Reg^n(H, B)$

$f \in Reg^n(H, B)$ if there exists $f^{-1} : H^{(n)} \to B$ such that

$$f \wedge f^{-1} = f^{-1} \wedge f = \varepsilon_H^n \otimes \eta_B$$

$Reg^n(H, B)$ is a group $\forall n \in \mathbb{N}$ with unit $\varepsilon_H^n \otimes \eta_B$.

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Some facts

For (B, φ_B) a H-module algebra:

1st. fact $\varphi_B \circ (H \otimes \eta_B) = \varepsilon_H \otimes \eta_B$ if H is a Hopf algebra.

2nd. fact If H is a weak Hopf algebra, then $\varphi_B \circ (H \otimes \eta_B)$ does not have to equal $\varepsilon_H \otimes \eta_B$.

3rd. fact If $B \hookrightarrow A$ is a weak cleft extension with cleaving morphism f, then $f \wedge f^{-1} = \varphi_B(H \otimes \eta_B)$.

An idea

Why do we not change the unit(s) of $Reg(H, B)$ by $\varphi_B \circ (H \otimes \eta_B)$?

Why do we not change groups by groupoids?

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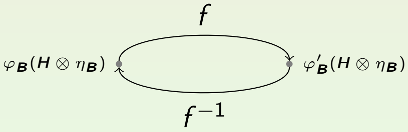
Outline

H is a cocommutative weak Hopf algebra and B is an algebra.

Changin $Reg(H, B)$ by $M(H, B)$.



Objects

$\varphi_B(H \otimes \eta_B)$, where $\varphi_B : B \otimes H \to H$ is a weak action.

Arrows

Morphisms $f : H \to B$ such that there exists $f^{-1} : H \to B$ satisfiying

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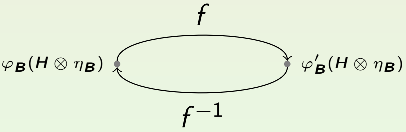

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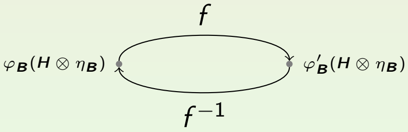

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2. $\varphi_B \circ (\mu_H \otimes \eta_B) = \varphi_B \circ (\mu_H \circ (H \otimes \Pi_L) \otimes \eta_B)$
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The objects of $M^n(H, B)$

$M^n(H, B)_0 = \{\varphi_B \circ (H^{(n)} \otimes \eta_B)\}$ for $\varphi_B : H^{(n)} \otimes B \to B$ a weak action.

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and we generalize these properties to $\varphi_B : H^{(n)} \otimes B \to B$.

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Gauge transformations (the morphisms)

If $\varphi_B, \varphi'_B : H \otimes B \to B$ are weak actions, $f : H \to B$ is a *gauge transformation between φ_B and φ'_B* if there exists $f^{-1} : H \to B$:

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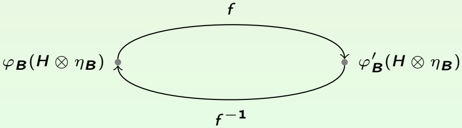

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Arrows and composition of $M^n(H, B)$

$M^n(H,B)_1 = \{f : H^{(n)} \to B\}$ where f is a gauge transformation between φ_B and φ'_B.
We define the composition by $f * g = f \wedge g$ if $t(g) = s(f)$.

The operators

Morphisms of groupoids

$$M(H,B) \underset{\partial_1}{\overset{\partial_3}{\underset{\partial_2}{\rightrightarrows}}} M(H\otimes H,B) \begin{array}{c}\xrightarrow{\partial_4}\\ \xrightarrow{\partial_3}\\ \xrightarrow{\partial_2}\\ \xrightarrow{\partial_1}\end{array} M^3(H,B)\ldots$$

- $\partial_1^{\varphi_B}(\varphi_B(H\otimes\eta_B)) = \varphi_B(H\otimes\varphi_B(H\otimes\eta_B))$
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- $\partial_3(\varphi_B(H\otimes\eta_B)) = \varphi_B(\mu_H(H\otimes\Pi_L)\otimes\eta_B)$

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The operators

Morphisms of groupoids

$$M(H,B) \overset{\partial_3}{\underset{\partial_1}{\overset{\partial_2}{\rightrightarrows}}} M(H\otimes H,B) \overset{\partial_4}{\underset{\partial_1}{\overset{\partial_3, \partial_2}{\rightrightarrows}}} M^3(H,B)\dots$$

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If $f \in M(H,B)_1$, $\varphi_B(H\otimes\eta_B) \xrightarrow{f} \varphi'_B(H\otimes\eta_B)$:

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The operators

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$$M(H,B) \underset{\partial_1}{\overset{\partial_3}{\underset{\partial_2}{\rightrightarrows}}} M(H\otimes H,B) \;\overset{\partial_4,\,\partial_3,\,\partial_2,\,\partial_1}{\rightrightarrows}\; M^3(H,B)\dots$$

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2-cocycles

Let $\sigma : H \otimes H \to B$ be a morphism in the groupoid $M(H \otimes H, B)$ such that

$$\varphi_B(\mu_H \otimes \eta_B) \underset{\sigma^{-1}}{\overset{\sigma}{\rightleftarrows}} \varphi_B(H \otimes \varphi_B(H \otimes \eta_B))$$

Twisted condition

As $\sigma \in M(H \otimes H, B)$:

$$\varphi_B \circ (H \otimes \varphi_B \circ (H \otimes B)) =$$

$$\mu_B^3 \circ (\sigma \otimes \varphi_B(\mu_H \otimes B) \otimes \sigma^{-1}) \circ (\delta_{H\otimes H} \otimes c_{H\otimes H,B}) \circ (\delta_{H\otimes H} \otimes B)$$

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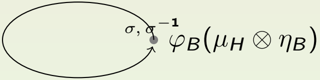

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2-cocycles

2-cocycles

σ is a 2-cocycle with respect to φ_B if

$$\partial_1^{\varphi_B}(\sigma) * \partial_3(\sigma) = \partial_4(\sigma) * \partial_2(\sigma)$$

Let σ be a 2-cocycle with respect to φ_B and let τ be a 2-cocycle with respect to φ'_B and $\gamma : H \to B$ a morphism in $M(H, B)$.

Equivalent 2-cocycles

σ and τ are equivalent if

$$\tau * \partial_2(\gamma) = (\partial_3(\gamma) \wedge \partial_1^{\varphi_B}(\gamma)) * \sigma$$

provided that $\gamma : \varphi_B(H \otimes \eta_B) \to \varphi'_B(H \otimes \eta_B)$. We denote by $H^2(H, B)$ the set of equivalence classes.

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Normal 2-cycles

If σ is a 2-cocycle with respect to φ_B

Normal 2-cocycle

σ is a *normal 2-cocycle* if:

$$\sigma \circ (H \otimes \eta_H) = \sigma \circ (\eta_H \otimes H) = \varphi_B(H \otimes \eta_B).$$

Proposition

If σ is a 2-cocycle with respect to φ_B, there exists a normal 2-cocycle with respect to φ_B equivalent to σ.

Classes of equivalence

All the equivalence classes of cohomologous 2-cocycles have a normal representative element.

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Outline

Weak crossed products

H is a weak Hopf algebra, B an algebra and $\varphi_B : H \otimes B \to B$ and $\sigma : H \otimes H \to B$ morphisms.

$\varphi_B, \sigma \rightsquigarrow \mu_{\varphi_B,\sigma}$

$\mu_{\varphi_B,\sigma} : B \otimes H \otimes B \otimes H \to B \otimes H$

Normal crossed product

$B \otimes H$ is a crossed product if $\mu_{\varphi_B,\sigma}$ is associative and $\eta_B \otimes \eta_H$ is a preunit.

- $B \otimes H$ is not an algebra.
- For $\nabla_{\eta_B \otimes \eta_H} : B \otimes H \to B \otimes H$, $Im\nabla = B \times_{\sigma}^{\varphi_B} H$ is an algebra.
- φ_B is a weak action.

σ satisfies:

- Twisted condition.
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- $\sigma M(H \otimes H, B)$.
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Equivalent weak crossed products

Definition: $B\sharp_{\sigma}^{\varphi_B}H$ and $B\sharp_{\tau}^{\varphi'_B}H$ are equivalent if and only if $B\times_{\sigma}^{\varphi_B}H \simeq B\times_{\tau}^{\varphi'_B}H$.

Theorem

Two weak crossed products $B\sharp_{\sigma}^{\varphi_B}H$ and $B\sharp_{\tau}^{\varphi'_B}H$ are equivalent if and only if σ and τ are cohomologous.

The equivalence classes of weak crossed products (with invertible cocycle) are in bijective correspondence with $H^2(H, B)$.

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Outline

H-extensions of algebras

If A is an H-comodule algebra define

$$A^{coH} \xrightarrow{\iota_A} A \underset{(A\otimes \Pi_L)\circ \rho_A}{\overset{\rho_A}{\rightrightarrows}} A\otimes H$$

H-extension

$B \hookrightarrow A$ is a H-extension if

1. $B \simeq A^{coH}$ with inclusion $\iota_A : B \to A$.
2. $\iota_A \circ \varphi_B \circ (H \otimes \eta_B) = (A \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes H) = \bar{e}_{LL}$.

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Another Reg

Let $f : H \to A$ be a morphism in $\mathcal{C}$. $f \in Reg(H, A)$ if there exists $f^{-1} : H \to A$ such that

1. $f^{-1} \wedge f = e_{RR}$, $f \wedge f^{-1} = \bar{e}_{LL}$.
2. $f \wedge f^{-1} \wedge f = f$, $f^{-1} \wedge f \wedge f^{-1} = f^{-1}$.

$$\begin{aligned} e_{RR} &= (A \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,A} \otimes H) \circ (H \otimes (\rho_A \circ \eta_A)) \\ \bar{e}_{LL} &= (A \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes H) \end{aligned}$$

A new groupoid!

$Reg(H, A)$ is a groupoid with two objects e_{RR} and $\bar{e}_{LL}$.

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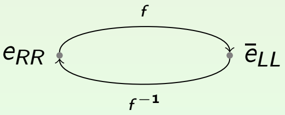

Cleft H-extensions

H-cleft extensions

The H-extension $B \hookrightarrow A$ is *cleft* if there exits $f \in Reg(H, A)$ of H-comodules and such that f^{-1} satisfies

1. $(A \otimes \Pi_R) \circ \rho_A \circ f^{-1} = (A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes (\rho_A \circ f^{-1})) \circ \delta$
2. $f \circ \eta_H = \eta_A, \qquad f^{-1} \circ \eta_H = \eta_A.$

- As $B \hookrightarrow A$ is a H-extension, $\bar{e}_{LL} = \iota_A \circ \varphi_B(H \otimes \eta_B)$ so

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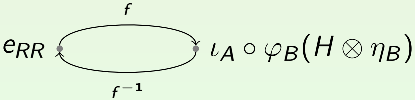

Equivalent H-extensions

If $B \hookrightarrow A_1$ and $B \hookrightarrow A_2$ are H-extensions then $T : A_1 \to A_2$ is a *morphism of extensions* if

1. T is of H-comodule algebras.
2. $\iota_{A_2} = T \circ \iota_{A_1}$.

T is an *isomorphism of extensions* if $T : A_1 \to A_2$ is an isomorphism.

Proposition

If $T : (B \hookrightarrow A_1) \to (B \hookrightarrow A_2)$ is of H-extensions and $B \hookrightarrow A_1$ is cleft then:

- $B \hookrightarrow A_2$ is also cleft.
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We can classify (cleft) H-extensions by isomorphisms of extensions.

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Cleft extensions and weak crossed products

Induced weak crossed product

If $B \hookrightarrow A$ is a cleft H-extension then $B \otimes H$ can be endowed with a normal weak crossed product with invertible cocycle structure such that $A \simeq B \times^{\sigma}_{\varphi_B} H$

A cleft extension from a weak crossed product

If $B \sharp^{\sigma}_{\varphi_B} H$ is a normal weak crossed product with invertible cocycle, then $B \hookrightarrow B \times H$ is a cleft extension.

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Classifying weak crossed products

Theorem

Let H be a cocommutative weak Hopf algebra and B an algebra. Then the classes of equivalence of cleft H-extensions are in bijective correspondence with $H^2(H, B)$ and with the classes of equivalence of weak crossed products with invertible cocycle.

Muchas gracias