The Picard crossed module of a braided tensor category

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Let $\mathcal{C}$ be a braided tensor category (e.g., the representation category of a finite dimensional Hopf algebra). One associates to $\mathcal{C}$ two groups: the group G of braided autoequivalences of $\mathcal{C}$ and the group P of invertible $\mathcal{C}$-module categories. The pair (G, P) forms a crossed module (also known as a categorical group). We discuss the structure of this crossed module and explain how it is used in the classification of fusion categories.

This is a report on joint works in progress with A. Davydov and with V. Drinfeld, S. Gelaki, and V. Ostrik.

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Coautors

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Let k be an algebraically closed field, $\mathrm{char}(k) = 0$.

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Let $\mathcal{C}$ be a *finite braided tensor category*

This means that $\mathcal{C}$ is an Abelian k-linear equipped with tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, the unit object $\mathbf{1}$, the associativity and unit constraints

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z), \quad l_X : X \otimes \mathbf{1} \xrightarrow{\sim} X, \quad r_x : \mathbf{1} \otimes X \xrightarrow{\sim} X,$$

and the braiding

$$c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X, \qquad X, Y \in Obj(\mathcal{C}).$$

satisfying natural compatibility axioms. We also require existence of duals (rigidity) and finiteness of the number of simple objects and existence of projective covers.

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Example

$\mathcal{C} = \mathrm{Rep}(H)$, the category of finite dimensional representations of a finite dimensional quasi-triangular (quasi-) Hopf algebra H.

Two groups associated to a braided tensor category

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1st group: braided autoequivalences

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2nd group: the Picard group of $\mathcal{C}$

The *Picard group* of $\mathcal{C}$ is the group $\mathrm{Pic}(\mathcal{C})$ of invertible exact $\mathcal{C}$-module categories.

We need some definitions in order to introduce it.

Exact $\mathcal{C}$-module categories (V. Ostrik, P.Etingof)

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An *exact $\mathcal{C}$-module category* $\mathcal{M}$ is an Abelian k-linear category along with an exact bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ and associativity constraints

$$\mu_{XYM} : (X \otimes Y) \otimes M \xrightarrow{\sim} X \otimes (Y \otimes M), \qquad \lambda_M : \mathbf{1} \otimes M \xrightarrow{\sim} M$$

satisfying the pentagon and triangle axioms. The exactness condition means that $P \otimes M$ is projective for any projective $P \in \mathcal{C}$ and any $M \in \mathcal{M}$.

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Classical analogy

Braided tensor categories are analogues of finite dimensional algebras

Exact module categories are analogues of projective modules

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Semisimple case

If $\mathcal{C}$ is semisimple (i.e., is a fusion category) then a $\mathcal{C}$-module category $\mathcal{M}$ is exact iff $\mathcal{M}$ is semisimple.

Tensor product of $\mathcal{C}$-module categories

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Given $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{N}$ one defines their tensor product

$$\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$$

using a universal property. The category $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ is again a $\mathcal{C}$-module category.
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A $\mathcal{C}$-module category $\mathcal{M}$ is invertible if $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \cong \mathcal{C}$ for some $\mathcal{N}$.

The equivalence classes of invertible $\mathcal{C}$-module categories form a group $\text{Pic}(\mathcal{C})$ called the *Picard group of* $\mathcal{C}$.

Relation with the Brauer group of $\mathcal{C}$

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Let A be a separable Azumaya algebra in $\mathcal{C}$. Then

$$\mathcal{M} := A - \text{modules in } \mathcal{C}$$

is an invertible exact module category. Conversely, every $\mathcal{M}$ appears in this way [Ostrik].

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We have,

$$\mathrm{Pic}(\mathcal{C}) \text{ (equivalence classes of invertible } \mathcal{C}\text{-module categories)}$$

$$\cong$$

$$\mathrm{Br}(\mathcal{C}) \text{ (Morita equaivalence classes of separable Azumaya algebras).}$$

Crossed module formed by $\mathrm{Aut}^{br}(\mathcal{C})$ and $\mathrm{Pic}(\mathcal{C})$

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Action of $\mathrm{Aut}^{br}(\mathcal{C})$ on $\mathrm{Pic}(\mathcal{C})$

By functoriality, $\mathrm{Aut}^{br}(\mathcal{C})$ acts on the 2-category of $\mathcal{C}$-module categories. In particular, $\mathrm{Aut}^{br}(\mathcal{C})$ has a canonical action on the group $\mathrm{Pic}(\mathcal{C})$.

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Homomorphism $\partial : \mathrm{Pic}(\mathcal{C}) \to \mathrm{Aut}^{br}(\mathcal{C})$

For any $\mathcal{C}$-module category $\mathcal{M}$ there is a pair tensor functors:

$$\alpha^{\pm}_{\mathcal{M}} : \mathcal{C} \to \mathrm{End}_{\mathcal{C}}(\mathcal{M}) : X \mapsto X \otimes -.$$

The $\mathcal{C}$-module functor structure on $\alpha^{\pm}_{\mathcal{M}}$ are given by $c^{\pm 1}$ (i.e., by braiding and its inverse).

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When $\mathcal{M}$ is invertible the functors $\alpha^{\pm}_{\mathcal{M}}$ are equivalences. Set

$$\partial_{\mathcal{M}} := (\alpha^{+}_{\mathcal{M}})^{-1} \circ \alpha^{-}_{\mathcal{M}} : \mathcal{C} \to \mathcal{C}.$$

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We have $\partial_{\mathcal{M}} := (\alpha^+_{\mathcal{M}})^{-1} \circ \alpha^-_{\mathcal{M}} \in \mathrm{Aut}^{br}(\mathcal{C})$ and

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Theorem [Etingof, Ostrik, speaker]

When $\mathcal{C}$ is *factorizable* the map ∂ is an isomorphism, so that

$$\mathrm{Pic}(\mathcal{C}) \cong \mathrm{Aut}^{br}(\mathcal{C})$$

For example, $\mathcal{C} =$ Rep(factorizable Hopf algebra) is factorizable. Also, Drinfeld centers of tensor categories are factorizable.

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Alternative description of $\mathrm{Pic}(\mathcal{C})$ [Davydov, speaker]

Let $\mathcal{Z}(\mathcal{C})$ denote the Drinfeld center of $\mathcal{C}$. Note $\mathcal{C}, \mathcal{C}^{op} \hookrightarrow \mathcal{Z}(\mathcal{C})$. We have

$$\mathrm{Pic}(\mathcal{C}) \cong \mathrm{Aut}^{br}(\mathcal{Z}(\mathcal{C}); \mathcal{C}),$$

where $\mathrm{Aut}^{br}(\mathcal{Z}(\mathcal{C}); \mathcal{C})$ is the group of braided autoequivalences of the Drinfeld center $\mathcal{Z}(\mathcal{C})$ trivializable on $\mathcal{C} \subset \mathcal{Z}(\mathcal{C})$.

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Any $\alpha \in \mathrm{Aut}^{br}(\mathcal{Z}(\mathcal{C}); \mathcal{C})$ maps $\mathcal{C}^{op}$ to itself and, hence determines an element of $\mathrm{Aut}^{br}(\mathcal{C})$ by restriction .
The homomorphism $\partial : \mathrm{Pic}(\mathcal{C}) \to \mathrm{Aut}^{br}(\mathcal{C})$ is identified with

$$\mathrm{Pic}(\mathcal{C}) \cong \mathrm{Aut}^{br}(\mathcal{Z}(\mathcal{C}); \mathcal{C}) \xrightarrow{restriction} \mathrm{Aut}^{br}(\mathcal{C}^{op}) \cong \mathrm{Aut}^{br}(\mathcal{C}).$$

Recall definition of the crossed module

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Definition [Whitehead]

A *crossed module* (G, C) is a pair of groups G and C together with an action of G on C, denoted $(g, c) \mapsto {}^{g}c$, and a homomorphism $\partial : C \to G$ satisfying

$$\partial({}^{g}c) = g\partial(c)g^{-1}, \quad (1)$$

$$^{\partial(c)}c' = cc'c^{-1} \quad c, c' \in C, g \in G. \quad (2)$$

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Crossed module is the same thing as a group object in the category of groupoids. They are also called *categorical groups*.

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Example

For any group G and a normal subgroup $H \subset G$ there is a crossed module $CM(G, H)$, where G acts on H by conjugation.

Theorem [Davydov, speaker]

For a braided category $\mathcal{C}$ the pair $(\mathrm{Aut}^{br}(\mathcal{C}), \mathrm{Pic}(\mathcal{C}))$ is a crossed module. We call it the *Picard crossed module* of $\mathcal{C}$.

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Let $\mathcal{C} := \mathcal{C}(A, q)$ be the pointed fusion category associated to a quadratic form $q : A \to k^\times$ on an Abelian group A [Joyal, Street]. In this case

$$
\begin{aligned}
\mathrm{Aut}^{br}(\mathcal{C}) &= O(A, q), \text{ the orthogonal group of } (A, q), \\
\mathrm{Pic}(\mathcal{C}) &= \{ \ \text{pairs } (B, \beta) \mid \text{ where } B \subset A \text{ is a subgroup and} \\
&\qquad \beta : B \times B \to k^\times \text{ is a non-degenerate bilinear form} \\
&\qquad \text{such that } \beta(x, x) = q(x),\ x \in B \ \}.
\end{aligned}
$$

One can describe the Picard crossed module of $\mathcal{C}$ explicitly.

Dyslectic modules

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The Picard crossed modules can be used to describe an important invariant of a braided tensor category: the core.

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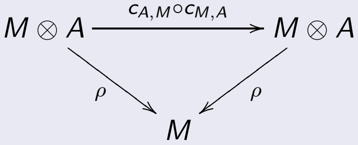
Suppose G is a group and $\mathcal{E} = \mathrm{Rep}(G) \subset \mathcal{C}$ is a Tannakian subcategory. Let $A = \mathrm{Fun}(G)$ and $\mathcal{C}_A^0 :=$ *dyslectic* $A-$modules $M \in \mathcal{C}$, i.e., such that



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Dyslectic modules

The Picard crossed modules can be used to describe an important invariant of a braided tensor category: the core.

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$$\begin{array}{ccc} M \otimes A & \xrightarrow{c_{A,M} \circ c_{M,A}} & M \otimes A \\ & {\scriptstyle \rho} \searrow \quad \swarrow {\scriptstyle \rho} & \\ & M & \end{array}$$

commutes

- The category $\mathcal{C}_A^0$ is a braided tensor category [Pareigis].
- There is a normal subgroup $H \subset G$ and a homomorphism $CM(G, H) \to \mathrm{Pic}(\mathcal{C}_A^0)$.

The core of a braided tensor category

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Theorem[Drinfeld, Gelaki, Ostrik, speaker]

Suppose $\mathcal{E} = \mathrm{Rep}(G) \subset \mathcal{C}$ is a *maximal* Tannakian subcategory. Then $\mathcal{C}_A^0$ and $\mathrm{Image}(CM(G, H) \to \mathrm{Pic}(\mathcal{C}_A^0))$ do not depend on the choice of $\mathcal{E}$.

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- The pair $(\mathcal{C}_A^0, \text{Image}(CM(G, H) \to \text{Pic}(\mathcal{C}_A^0))$ is called the *core* of $\mathcal{C}$.
- The core allows to separate the part of $\mathcal{C}$ that doesn't come from finite groups.
- One can reconstruct $\mathcal{C}$ from its core (in terms of finite groups and their cohomology)

Examples of classification and open problem

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Theorem[Drinfeld, Gelaki, Ostrik, speaker]

Let $\mathcal{C}$ be a braided fusion category. Then

- Core($\mathcal{C}$) is trivial $\Longleftrightarrow$ $\mathcal{C}$ is the relative Drinfeld center of a pointed fusion category,
- Core($\mathcal{C}$) is pointed $\Longleftrightarrow$ $\mathcal{C}$ is weakly group-theoretical (can be explicitly constructed from finite groups).

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Open problem:

Let H be a non-commutative semisimple quasitriangular (quasi-) Hopf algebra.

- Does Rep(H) contain a non-trivial Tannakian subcategory?
- In other words, does H have a non-trivial triangular quotient Hopf algebra?
- Equivalently, is Core(Rep(H)) pointed?

Thanks for listening!