Congruence property and Galois Symmetry of modular categories

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The natural representation of $SL(2,\mathbb{Z})$ associated to a Rational Conformal Field Theory (RCFT) has been conjectured, by Eholzer, to be t-rational and have a congruence kernel. It is further conjectured by Coste and Gannon a Galois symmetry of this representation. Some of these conjectures have been proved mathematically in the context of modular categories via the machinery called generalized Frobenius-Schur indicators. In this talk, I will report recent progress of these conjectures for modular categories.

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Modular invariance

- Recall that the modular group $\mathrm{SL}(2, \mathbb{Z})$ is a group generated by

$$\mathfrak{s} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathfrak{t} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$ subject to the relations

$$(\mathfrak{s}\mathfrak{t})^3 = \mathfrak{s}^2 \text{ and } \mathfrak{s}^4 = 1.$$

- Associated to a RCFT is a representation $\rho : SL(2, \mathbb{Z}) \to \mathrm{GL}(V)$, where V is spanned by the characters $\chi_a(\tau)$ of the primary fields $a \in \Pi$.
- If $\rho\left(\begin{bmatrix} c & d \\ e & f \end{bmatrix}\right) = [M_{ab}]_\Pi$, then

$$\chi_a\left(\frac{c\tau + d}{e\tau + f}\right) = \sum_{a \in \Pi} M_{ab}\chi_b(\tau).$$

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Fusion Rules and the Verlinde Formula

- The free $\mathbb{Z}$-module $\mathbb{Z}[\Pi]$ is a $\mathbb{Z}$-algebra given by the fusion rules:

  a family of non-negative integers N_{ab}^c $(a, b, c \in \Pi)$ such that

  $$a \otimes b = \sum_{c \in \Pi} N_{ab}^c c \, .$$

  The identity element 1 is a member of Π
- $S = \rho(\mathfrak{s})$ and $T = \rho(\mathfrak{t})$ are simply called the S and T-matrices.
- Verlinde formula:

  $$N_{ab}^c = \sum_{r \in \Pi} \frac{S_{ar} S_{br} \overline{S_{cr}}}{S_{r1}} \, .$$

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Questions on such representations of $\mathrm{SL}(2, \mathbb{Z})$

- Question: What can we say about such representations of $\mathrm{SL}(2, \mathbb{Z})$?
- Conjecture [Eholzer]:
  (i) A representations of $\mathrm{SL}(2, \mathbb{Z})$ associated to a RCFT has a congruence kernel of level N.
  (ii) The representation is $\mathbb{Q}_N$-rational, i.e. both $S, T \in \mathrm{GL}(\Pi, \mathbb{Q}_N)$.
- Conjecture [Coste-Gannon]:
  Let $\rho : \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{GL}(\Pi, \mathbb{C})$ be a representation associated to a RCFT. For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then $\sigma^2\rho$ is equivalent to ρ.
- Partial results on these conjectures were proved in the contexts of vertex operator algebras, conformal nets, and modular categories.

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Modular categories

- Another formalism of RCFT
- Integral ingredient of quantum invariants of knots and 3-manifolds. (cf. Reshetikhin-Turaev)
- In short, a modular category is a $\mathbb{C}$-linear semisimple tensor category $\mathcal{C}$ with
  - (i) $\Pi = \mathrm{Irr}(\mathcal{C})$ finite containing the unit object $\mathbf{1}$,
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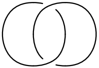

Representations of $SL(2,\mathbb{Z})$ associated to a modular category

- [Vafa] If $\mathcal{C}$ is modular, the matrix

$$T = [\delta_{ab}\theta_a]_{a,b\in\Pi}$$

has finite order. Note that θ_a is a scalar for $a \in \Pi$.

- The assignment $\overline{\rho}_{\mathcal{C}} : SL(2,\mathbb{Z}) \to PGL(\Pi, \mathbb{C})$ given by

$$\mathfrak{s} \mapsto S \quad \text{and} \quad \mathfrak{t} \mapsto T,$$

defines a projective representation of $SL(2,\mathbb{Z})$.

- $\overline{\rho}_{\mathcal{C}}$ can be lifted to a linear representation ρ of $SL(2,\mathbb{Z})$:

$$\begin{array}{ccc} SL(2,\mathbb{Z}) & \xrightarrow{\rho} & GL(\Pi,\mathbb{C}) \\ & \searrow^{\overline{\rho}_{\mathcal{C}}} & \downarrow \eta \\ & & PGL(\Pi,\mathbb{C}) \end{array}$$

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- Let $\mathcal{C}$ be a modular category,
- Π the set of isomorphism classes of simple objects of $\mathcal{C}$.
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An example: Modular category and Weil Representation

- Let G be a finite abelian group of odd order with a non-degenerate quadratic form $q : G \to \mathbb{Q}/\mathbb{Z}$.
- $q(g) = q(g^{-1})$ and $b_q(g, h) = q(gh) - q(g) - q(h)$ is a non-degenerate bilinear map.
- $\mathcal{C} = \mathrm{Rep}(\mathbb{C}[G]^*)$=category finite dimensional G-graded $\mathbb{C}$-linear spaces.
- Note that $\mathbb{C}[G]^*$ is a commutative semisimple Hopf algebra over $\mathbb{C}$.
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Modular category and Weil Representation

- The dual of $e(g)$ is $e(g^{-1})$.
- braiding $c_{e(g),e(h)} : e(g) \otimes e(h) \to e(h) \otimes e(g)$ is the scalar $\exp(2\pi i b_q(g,h)/2)$
- ribbon structure (or twist) $\theta_{e(g)} = \exp(2\pi i q(g))$.
- Thus $T = [\delta_{gh} \exp(2\pi i q(g))]$ and

$$S_{hg} = \exp(-2\pi i b_q(h,g))$$

- The projective representation of $\mathrm{SL}(2,\mathbb{Z})$ associated to $\mathcal{C}$ is equivalent to the Weil representation of associated with the pair (G,q).

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- [Müger] Given any spherical fusion category $\mathcal{C}$, the center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ is modular.
- If H is a semisimple Hopf algebra, then $\mathrm{Rep}(D(H)) = \mathcal{Z}(\mathrm{Rep}(H))$ where $D(H)$ is called the Drinfeld double of H.
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Let $\mathcal{C}$ be a modular category, and $\rho : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{GL}(\Pi, \mathbb{Q}_m)$ a lifting of the projective representation $\overline{\rho}_{\mathcal{C}} : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{PGL}(\Pi, \mathbb{C})$, where $m = \mathrm{ord}\,\rho(\mathfrak{t})$.

1. *Then ρ has a congruence kernel of level m.*
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Some applications of Galois symmetry

- Let $\mathcal{C}$ be an integral modular category.
- The quotient $\frac{p^+}{p^-}$ is a 4-th root of unity, where

$$p^{\pm} = \sum_{a \in \Pi} d(a)^2 \theta_a^{\pm 1}$$

are the Gauss sums of $\mathcal{C}$.
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The end

Thanks for your attention!