On functors which fail to be monadic

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A relevant result concerning monads is the so called Beck's monadicity (tripleability) theorem which characterizes right adjoint functors R which are monadic, i.e., such that the Eilenberg-Moore category of algebras (over the canonical monad associated to the adjunction) is equivalent, through the so-called comparison functor, to the domain category of R. In this talk we investigate those right adjoint functors R which fail to be monadic and measure how far they are to fulfil monadicity. To this aim we propose the definition of comparable functor. The obtained results are tested on a series of examples which also involve (braided) Lie theory and Module theory. This is part of a joint research with A. Ardizzoni (University of Ferrara) and J. Gómez-Torrecillas (University of Granada).

Hopf algebras and tensor categories

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Joint work with
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THANKS TO THE ORGANIZERS!!!

A monad on a category $\mathscr{A}$ is a triple $\mathbb{T} = (T, m, u)$, where

- $T : \mathscr{A} \to \mathscr{A}$ is a functor,
- $m : TT \to T$ and
- $u : \mathrm{Id}_{\mathscr{A}} \to T$ are functorial morphisms

satisfying the associativity and the unitality conditions:

$$m \circ mT = m \circ Tm \qquad \text{and} \qquad m \circ Tu = \mathrm{Id}_T = m \circ uT.$$

A module for a monad $\mathbb{T} = (T, m, u)$ over $\mathscr{A}$ is a pair (X, μ_X) where

- $X \in \mathscr{A}$ and
- $\mu_X : TX \to X$ is a morphism in $\mathscr{A}$ such that

$$\mu_X \circ T\mu_X = \mu_X \circ mX \quad \text{and} \quad \mathrm{Id}_X = \mu_X \circ uX.$$

A morphism between two $\mathbb{T}$-modules (X, μ_X) and $(X', \mu_{X'})$ is a morphism $f : X \to X'$ in $\mathscr{A}$ such that

$$\mu_{X'} \circ Tf = f \circ \mu_X.$$

We will denote by

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This is the so-called Eilenberg-Moore category.

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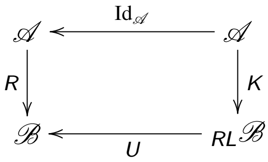

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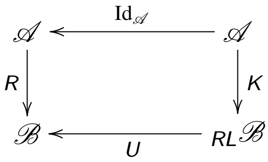

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A functor R is called

monadic $=$ R has a left adjoint L such that the adjunction (L,R) is monadic.

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Λ is full and faithful $\Leftrightarrow$ R preserves coequalizers of elements in S.

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Compare with the construction performed in Manes [1.5.5, page 49] .

E. G. Manes, *A TRIPLE MISCELLANY: SOME ASPECTS OF THE THEORY OF ALGEBRAS OVER A TRIPLE*. Thesis (Ph.D.)–Wesleyan University. 1967.

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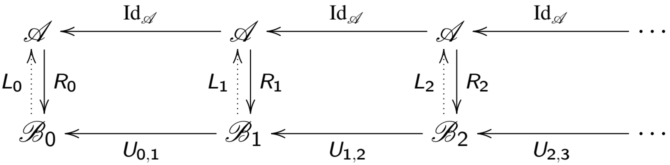

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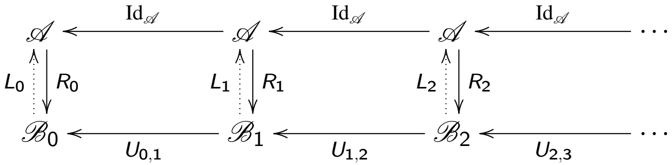

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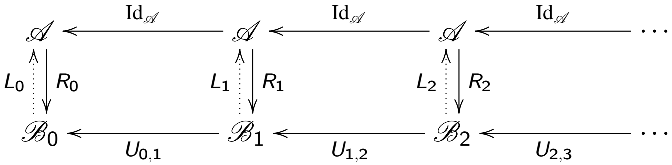

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- R_0 has a left adjoint L_0;

and, for $n > 0$,

- $\mathscr{B}_n$ is the category of $(R_{n-1}L_{n-1})$-modules ${}_{R_{n-1}L_{n-1}}\mathscr{B}_{n-1}$;
- R_n is the comparison functor of the adjunction (L_{n-1}, R_{n-1});
- R_n has a left adjoint L_n.

Note that for a comparable functor $R : \mathscr{A} \to \mathscr{B}$, we have a diagram

$$\begin{array}{ccccccc}
\mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \cdots \\
L_0 \uparrow \downarrow R_0 & & L_1 \uparrow \downarrow R_1 & & L_2 \uparrow \downarrow R_2 & & \\
\mathscr{B}_0 & \xleftarrow[U_{0,1}]{} & \mathscr{B}_1 & \xleftarrow[U_{1,2}]{} & \mathscr{B}_2 & \xleftarrow[U_{2,3}]{} & \cdots
\end{array}$$

where, for $n = 0$,

- $\mathscr{B}_0 = \mathscr{B}$;
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and, for $n > 0$,

- $\mathscr{B}_n$ is the category of $(R_{n-1}L_{n-1})$-modules ${}_{R_{n-1}L_{n-1}}\mathscr{B}_{n-1}$;
- R_n is the comparison functor of the adjunction (L_{n-1}, R_{n-1});
- R_n has a left adjoint L_n.
- $U_{n-1,n}$ is the forgetful functor ${}_{R_{n-1}L_{n-1}}U$.

Define a category $\mathscr{B}_{\infty}$ as follows

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- An object in $\mathscr{B}_\infty$ is a sequence $\mathbb{B}_\infty := (\mathbb{B}_n)_{n\in\mathbb{N}}$ where

$$\mathbb{B}_n \in \mathscr{B}_n \qquad \text{and} \qquad U_{n,n+1}(\mathbb{B}_{n+1}) = \mathbb{B}_n \quad \text{for all} \quad n \in \mathbb{N}.$$

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- A morphism $f_\infty : \mathbb{B}_\infty \to \mathbb{B}'_\infty$ is a sequence $f_\infty := (f_n)_{n\in\mathbb{N}}$ where

$$f_n : \mathbb{B}_n \to \mathbb{B}'_n \quad \text{is in } \mathscr{B}_n \qquad \text{and} \qquad U_{n,n+1}(f_{n+1}) = f_n \quad \text{for all} \quad n \in \mathbb{N}.$$

For all $n \in \mathbb{N}$, consider the functors

$$U_n = U_{n,\infty} : \mathscr{B}_\infty \to \mathscr{B}_n$$

$$U_{n,\infty}(\mathbb{B}_\infty) := \mathbb{B}_n \qquad \text{and} \qquad U_{n,\infty}(f_\infty) := f_n.$$

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Note that, for all $n \in \mathbb{N}$ the following diagram commutes.

$$\begin{array}{ccc} \mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} \\ {\scriptstyle R_n}\downarrow & & \downarrow{\scriptstyle R_\infty} \\ \mathbb{B}_n & \xleftarrow[U_{n,\infty}]{} & \mathbb{B}_\infty \end{array}$$

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$$L_n R_n L_n \mathbb{B}_n \underset{\varepsilon_n L_n \mathbb{B}_n}{\overset{L_n \mu_n}{\rightrightarrows}} L_n \mathbb{B}_n \overset{\pi_{n+1,n}}{\longrightarrow} L_{n+1} \mathbb{B}_{n+1}.$$

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(1) R_∞ has a left adjoint, say L_∞.

(2) For each $\mathbb{B}_\infty \in \mathscr{B}_\infty$, we can choose a specific direct limit in $\mathscr{A}$ for the direct system $(L_n \mathbb{B}_n, \pi_{n+1,n})_{n \in \mathbb{N}}$.

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Assume (2) holds. Then

$$(L_\infty \mathbb{B}_\infty, \pi_n : L_n \mathbb{B}_n \to L_\infty \mathbb{B}_\infty) = \varinjlim (L_n \mathbb{B}_n, \pi_{n+1,n})_{n \in \mathbb{N}}$$

Moreover L_∞ is full and faithful if and only if R preserves

$$\varinjlim (L_n \mathbb{B}_n, \pi_{n+1,n})_{n \in \mathbb{N}}$$

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is stationary after n steps
whenever $U_{t,t+1}$ is an isomorphism of categories, for all $t \geq n$.

$$
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(2) $U_{n,n+1}$ is an isomorphism of categories.

$$\begin{array}{ccccccc}
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$$
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If one of these conditions holds, then $U_n : \mathscr{B}_\infty \to \mathscr{B}_n$ is an isomorphism of categories such that

$$
\begin{array}{ccccccc}
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L_0 \uparrow \downarrow R_0 & & L_1 \uparrow \downarrow R_1 & & L_2 \uparrow \downarrow R_2 & & \\
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If one of these conditions holds, then $U_n : \mathscr{B}_\infty \to \mathscr{B}_n$ is an isomorphism of categories such that $U_n \circ R_\infty = R_n$ and $L_\infty := L_n \circ U_n$ is a left adjoint of R_∞.

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$$
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\mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} \\
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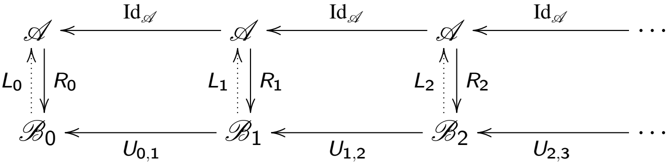
$$



Let

$$U_{0,n} = U_{0,1} U_{1,2} \cdots U_{n-1,n} : \mathscr{B}_n \to \mathscr{B}_0 \qquad \text{for every } n \in \mathbb{N}$$

and

$$U_{0,\infty} : \mathscr{B}_\infty \to \mathscr{B}_0, U_0(\mathbb{B}_\infty) := \mathbb{B}_0 \qquad \text{and} \qquad U_{0,\infty}(f_\infty) := f_0.$$

Let $R_0 : \mathscr{A} \to \mathscr{B}_0$ be a comparable functor and let $n \in \mathbb{N} \cup \{\infty\}$.
Since $U_{0,n} R_n = R_0$, we have

$$\mathrm{Im} R_0 \subseteq \mathrm{Im} U_{0,n}.$$

Moreover whenever R_n is surjective on objects up to isomorphism we also have

$$\mathrm{Im} U_{0,n} \subseteq \mathrm{Im} R_0.$$

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This simple statement can be considered as a "general descent theory" result.
In fact we deduce that the objects of $\mathscr{B}_0$ which are isomorphic to objects of the form $R_0 A$, for some $A \in \mathscr{A}$,
are exactly those of the form $U_{0,n} B_n$ where $B_n \in \mathscr{B}_n = {}_{R_{n-1} L_{n-1}} \mathscr{B}_{n-1}$.

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If (c) holds, then

1) R is comparable,
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satisfying the usual properties of unit and counit of an adjunction.

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In fact we can prove much more, namely that L_2 is full and faithful.

[Ag] A. L. Agore, *Categorical Constructions for Hopf Algebras*. Comm. Algebra, 1532-4125, Vol. **39**(4), (2011), 1476-1481.

[Ar2] A. Ardizzoni, *A Milnor-Moore Type Theorem for Primitively Generated Braided Bialgebras*, J. Algebra, Vol. **327**(1) (2011), 337-365.

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- $\operatorname{char}\mathbb{k} = 0$. Then, for all $x, y \in V_0$ we have that $xy - yx \in R_0 L_0 V_0$. Define a map $[-,-] : V_0 \otimes V_0 \to V_0$ by setting $[x, y] := \mu_0(xy - yx)$. Then $(V_0, [-,-])$ is an ordinary Lie algebra and $L_2 V_2$ is the universal enveloping algebra

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Now, by [Ar2, Example 6.10], if $\mathrm{char}(\Bbbk) = 0$, and [Ar3, Example 3.13], if $\mathrm{char}(\Bbbk) \neq 0$, (V_0, c) is a braided vector spaces of combinatorial rank at most one.

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[Ar1] A. Ardizzoni, *On Primitively Generated Braided Bialgebras*, Algebr. Represent. Theory, to appear.

[Ar3] A. Ardizzoni, *Universal Enveloping Algebras of PBW Type*, Glasg. Math. J., to appear. (arXiv:1008.4523)