Monoidal structures on the category of relative Hopf modules

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This talk is based on a joint work with Daniel Bulacu.

Let B be a bialgebra, and A a left B-comodule algebra in a braided monoidal category $\mathcal{C}$, and assume that A is also a coalgebra, with a not-necessarily associative or unital left B-action. Then we can define a right A-action on the tensor product of two relative Hopf modules, and this defines a monoidal structure on the category of relative Hopf modules if and only if A is a bialgebra in the category of left Yetter-Drinfeld modules over B.

Vrije Universiteit Brussel

Braidings on the category of bimodules, separable functors, Azumaya algebras and descent data

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Let A be an algebra over a commutative ring k.

- Classify - if any - braidings on the category ${}_A\mathcal{M}_A$

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- Classify - if any - braidings on the category ${}_A\mathcal{M}_A$
- Compute the center of ${}_A\mathcal{M}_A$ (find the local braidinga)

Part I: braidings on ${}_A\mathcal{M}_A$

Theorem 1

There is a bijective correspondence between braidings c on ${}_A\mathcal{M}_A$ and invertible elements $R = R^1 \otimes R^2 \otimes R^3 \in A^{(3)}$ (summation implicitly understood) satisfying the following conditions, for all $a \in A$:

$$\begin{aligned} R^1 \otimes R^2 \otimes aR^3 &= R^1 a \otimes R^2 \otimes R^3 \\ aR^1 \otimes R^2 \otimes R^3 &= R^1 \otimes R^2 a \otimes R^3 \\ R^1 \otimes aR^2 \otimes R^3 &= R^1 \otimes R^2 \otimes R^3 a \end{aligned}$$

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We call R the R-matrix corresponding to the braiding c, and we say that (A, R) is a quasitriangular algebra.The braiding c corresponding to R is given by the formula

$$c_{M,N}(m \otimes_A n) = R^1 n R^2 \otimes_A m R^3.$$

Easy applications

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For A commutative: game over!

If $S = S^1 \otimes S^2 \otimes S^3$ is the R-matrix corresponding to the inverse braiding, then $R^{-1} = S^2 \otimes S^1 \otimes S^3$.

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imply all the other axioms.
One first shows that R is invariant under cyclic permutation of the tensor factors:

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It then also follows that

$$R^{-1} = R^2 \otimes R^1 \otimes R^3$$

That is, $R = S$, and the braiding is symmetric.

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Azumaya algebras

we have an adjoint pair $(F = A \otimes -, G = (-)^A)$ between $\mathcal{M}_k$ and ${}_A\mathcal{M}_A$.

$$\begin{array}{rlcr} \eta_N : & N \to (A \otimes N)^A & ; & \eta_N(n) = n \otimes 1; \\ \varepsilon_M : & A \otimes M^A \to M & ; & \varepsilon_M(a \otimes m) = am. \end{array}$$

A is an Azumaya algebra if and only if (F, G) is a pair of inverse equivalences.

Separable functors (Năstăsescu, Van den Bergh, Van Oystaeyen)

$F: \mathcal{C} \to \mathcal{D}$ is called separable if the natural transformation

$$\mathcal{F}: \ \mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) \to \mathrm{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet)) \ ; \ \mathcal{F}_{C,C'}(f) = F(f)$$

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splits,that is, there is a natural transformation

$$\mathcal{P}:\ \mathrm{Hom}_{\mathcal{D}}(F(\bullet),F(\bullet)) \to \mathrm{Hom}_{\mathcal{C}}(\bullet,\bullet)$$

such that $\mathcal{P} \circ \mathcal{F}$ is the identity natural transformation.

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G is separable $\iff \varepsilon$ cosplits.

Theorem 3

Let A be a k-algebra A.
The functor $G = (-)^A : {}_A\mathcal{M}_A \to \mathcal{M}_k$ is separable

$$\Longleftrightarrow$$

there exists $R = R^1 \otimes R^2 \otimes R^3 \in A \otimes (A \otimes A)^A$ such that $R^1 R^2 \otimes R^3 = 1 \otimes 1 \otimes 1$.

Corollary

Let A be k-algebra such that the functor $G = (-)^A$ is separable. Then (A, R) is a quasitriangular algebra, and the corresponding braiding is a symmetry.
In this case the functor $F : \mathcal{M}_k \to {}_A\mathcal{M}_A$ preserves the symmetry.

Example

If A is an Azumaya algebra, then (F, G) is an equivalence of categories, and, a fortiori, G is separable.
In the case where $A = M_n(k)$ is a matrix algebra, we have an explicit formula for R, namely

$$R = \sum_{i,j,k=1}^{n} e_{ij} \otimes e_{ki} \otimes e_{jk}.$$

Part II: the center of ${}_A\mathcal{M}_A$

Descent data (Grothendieck/Knus & Ojanguren)

Let A be a commutative k-algebra. A descent datum (V, g) is a right A-module V together with a right $A^{(2)}$-module map

$$g: \ A \otimes V \longrightarrow V \otimes A$$

such that $g_2 = g_3 \circ g_1$ and

$$(m \circ g)(1 \otimes v) = v$$

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which is equivalent to g being invertible.
$\underline{\mathrm{Desc}}(A/k)$ is the category of descent data.

Sweedler canonical coring

$A \otimes A$ is an A-coring.

$$\Delta(a \otimes b) = a \otimes 1 \otimes b \in A^{(3)} \cong A^{(2)} \otimes_A A^{(2)}$$

$$\varepsilon(a \otimes b) = ab$$

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We describe right $A \otimes A$-comodules. These are right A-modules V with a right A-linear $\rho : V \to V \otimes_A A^{(2)} \cong V \otimes A$. These have to satisfy the appropriate coassociativity and counit conditions. If we write $\rho(v) = v_{[0]} \otimes v_{[1]} \in V \otimes A$, then these come down to

$$\rho(v_{[0]}) \otimes v_{[1]} = v_{[0]} \otimes 1 \otimes v_{[1]}$$

$$v_{[0]} v_{[1]} = v$$

Right A-linearity of ρ means

$$\rho(va) = v_{[0]} \otimes v_{[1]} a$$

The weak right center of ${}_A\mathcal{M}_A$

Take $(V, c_{-,V}) \in {}_A\mathcal{M}_A$. Consider

$$g = c_{A\otimes A,V} : \; A^{(2)} \otimes_A V \cong A \otimes V \to V \otimes_A A^{(2)} \cong V \otimes A$$

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Then ρ determines c completely:

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$$c_{M,V}(m \otimes_A v) = v_{[0]} \otimes_A m v_{[1]} \tag{1}$$

Furthermore, $(V, \rho) \in \mathcal{M}^{A\otimes A}$, and

$$\rho(av) = v_{[0]} \otimes a v_{[1]}$$

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We call (V, ρ) a Yetter-Drinfeld A-module. $\mathcal{YD}^A$ is the category of Yetter-Drinfeld A-modules.

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Conversely, given a Yetter-Drinfeld A-module (V, ρ), we obtain a local braiding $c_{-,V}$ using (1).

Theorem

$\mathcal{W}_r({}_A\mathcal{M}_A)$ and $\mathcal{YD}^A$ are isomorphic

Let $(V, \rho) \in \mathcal{YD}^A$. Then

$$a v_{[0]} \otimes v_{[1]} = v_{[0]} a \otimes v_{[1]}$$

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hence

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so the left A-action on V is determined by the right one.
This is the clue to the following result.

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Proof: On $(V, \rho) \in \mathcal{M}^{A\otimes A}$, define a left A-action using the formula we just obtained:

$$av = v_{[0]}av_{[1]}$$

Then show that $(V, \rho) \in \mathcal{YD}^A$.

Theorem

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is equivalent tothe invertibility of

$$g: \ A \otimes V \to V \otimes A, \quad g(a \otimes v) = av_{[0]} \otimes v_{[1]}$$

a related result is the following.

Theorem

The (right) center of the category of A-bimodules coincides with its (right) weak center: $\mathcal{Z}_r({}_A\mathcal{M}_A) = \mathcal{W}_r({}_A\mathcal{M}_A)$.

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Proof: take $(V, c_{-,V})$ in the weak center, and take the associated map $\rho: \ V \to V \otimes A$. The inverse of $c_{M,V}$ is given by

$$c^{-1}_{M,V}(v \otimes_A m) = v_{[1]} m \otimes_A v_{[0]}.$$

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If V and W are A-bimodules, then $V \otimes W$ is an $A^{(2)}$-bimodule. Let A be a k-algebra. A descent datum consists of an A-bimodule V together with an $A^{(2)}$-bimodule map $g : A \otimes V \to V \otimes A$ such that $g_2 = g_3 \circ g_1$ and $(m \circ g)(a \otimes v) = v$, for all $v \in V$.

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The last condition can be replaced by invertibility of g.
$\underline{\mathrm{Desc}}(A/k)$ is the category of descent data.
If A is commutative, then these descent data coincide with the Knus-Ojanguren descent data.

Theorem

The categories $\underline{\mathrm{Desc}}(A/k)$ and $\mathcal{YD}^A$ are isomorphic.

Conclusion

The categories $\underline{\mathrm{Desc}}(A/k)$, $\mathcal{YD}^A$, $\mathcal{M}^{A\otimes A}$, $\mathcal{W}_r({}_A\mathcal{M}_A)$ and $\mathcal{Z}_r({}_A\mathcal{M}_A)$ are isomorphic.

Conclusion

The categories $\underline{\mathrm{Desc}}(A/k)$, $\mathcal{YD}^A$, $\mathcal{M}^{A\otimes A}$, $\mathcal{W}_r({}_A\mathcal{M}_A)$ and $\mathcal{Z}_r({}_A\mathcal{M}_A)$ are isomorphic.
We have a pair of adjoint functors $(K = - \otimes A, R = (-)^{\mathrm{co}A\otimes A})$ between $\mathcal{M}_k$ and $\mathcal{M}^{A\otimes A}$.

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The categories $\underline{\mathrm{Desc}}(A/k)$, $\mathcal{YD}^A$, $\mathcal{M}^{A\otimes A}$, $\mathcal{W}_r({}_A\mathcal{M}_A)$ and $\mathcal{Z}_r({}_A\mathcal{M}_A)$ are isomorphic.
We have a pair of adjoint functors $(K = - \otimes A, R = (-)^{\mathrm{co}A\otimes A})$ between $\mathcal{M}_k$ and $\mathcal{M}^{A\otimes A}$.
(K, R) is a pair of inverse equivalences if A/k is faithfully flat.