Galois-Grothendieck duality, Tannaka duality and Hopf (co)monads

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In SGA 1, Alexandre Grothendieck defines the algebraic fundamental group of a scheme S. As is his wont, Grothendieck adopts a very general setting: given an abstract category C (think of it as the category of étale coverings of S) endowed with a functor ω to finite sets, satisfying certain conditions, Grothendieck constructs a profinite group G, the group of automorphisms of ω, and shows that C is equivalent to the category of continuous finite G-sets.

Similarly, Tannaka duality (in the larger sense) associates with a tensor category C endowed with a fiber functor ω a sort of group G (affine group, gerbe in the commutative setting, Hopf algebra, Hopf algebroid in the non-commutative setting) in such a way that C is equivalent to the category of finite dimensional representations of G.

We will propose a general setting which encompasses these two analogous situations; given a monoidal functor $\omega : C \to B$, we will show that, under general conditions on C, B and ω, there exists a Hopf monad T on B such that the category of ind-objects of C is equivalent to the category of 'representations' of T, and C itself, to the category of representations 'of finite type' of T.

Hopf monads generalize groups and Hopf algebras in a non-braided setting.

We will also explain how this result yields Galois-Grothendieck duality as well as Tannaka duality, and other results on tensor functors.

Galois-Grothendieck Duality, Tannaka Duality, Hopf (co)monads and tensor functors

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2. Hopf Monads - a sketchy survey
3. Main result
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In SGA1 (1960) Grothendieck defined the *étale fundamental group* of a scheme S as follows.

Let p be a geometric point of S. If $E \to S$ is a finite étale covering of S denote by E_p the fibre of E above p. The assignment $E \mapsto E_p$ defines the *fibre functor*

$$\omega_p : \mathcal{C} \to \mathrm{set}$$

where $\mathcal{C}$ is the category of finite étale coverings of S and set the category of finite sets.

The étale fundamental group of S is the 'profinite group of automorphisms' of ω_p. It is the limit of the groups of automorphisms of 'Galois coverings'.

Grothendieck shows that

The category of finite étale coverings of S is equivalent to the category of continuous finite G-sets.

However Grothendieck's result is more general, pertaining not to algebraic geometry...

... but to category theory :

Theorem

Let C be a category and $\omega : C \to \mathrm{set}$ be a functor, and assume :

1. *C has finite limits and colimits, and ω preserves them;*
2. *in C any morphism factorizes as an epi followed by a mono;*
3. *in C epis are strict;*
4. *in C monos are summands;*
5. *ω is conservative.*

Then one constructs a profinite group G such that C is equivalent to the category G – set of finite continuous G-sets.

Tannaka theory in its algebraic form goes back to the thesis of Grothendieck's student Saavedra Rivano's in 1972.

Theorem

Let $\mathcal{C}$ be a symmetric tensor category over a field $\Bbbk$. Let B be a non-zero commutative $\Bbbk$ algebra and let

$$\omega : \mathcal{C} \to \mathrm{Mod}\, B$$

be a symmetric fibre functor, that is, a strong monoidal symmetric $\Bbbk$-linear exact functor. Then one constructs an affine algebraic groupoid G with base $\mathrm{Spec} B$ such that $\mathcal{C}$ is equivalent to the category $\mathrm{rep} G$ of representations of G of finite type.

This theorem has been given non-commutative (non-symmetric) generalizations.

We would like to provide a general setting unifying Galois-Grothendieck duality and Tannaka duality (and non-commutative generalizations).

Question

Let $F : \mathcal{C} \to \mathcal{B}$ be a strong monoidal functor. Can one describe $\mathcal{C}$ as the category of 'representations' of an algebraic structure, a sort of 'group', living at the level of $\mathcal{B}$?

An incomplete answer to this problem has been given by Xavier Rochard in his unpublished thesis (1998).
Our first step to address this question will be to introduce (co) monads, and more precisely Hopf (co) monads, which serve exactly that kind of purpose.

1 Galois-Grothendieck duality and Tannaka duality

2 Hopf Monads - a sketchy survey
- Definition
- Examples
- Some aspects of the general theory

3 Main result

4 Applications

5 Structure of Hopf modules on Hopf monads and applications

Monads

Let $\mathcal{C}$ be a category. The category $\mathrm{EndoFun}(\mathcal{C})$ is strict monoidal ($\otimes$=composition, $\mathbb{1} = 1_{\mathcal{C}}$)

A monad on $\mathcal{C}$ is an algebra (=monoid) in $\mathrm{EndoFun}(\mathcal{C})$:

$$T\colon \mathcal{C} \to \mathcal{C}, \quad \mu\colon T^2 \to T \text{ (product)}, \quad \eta\colon 1_{\mathcal{C}} \to T \text{ (unit)}$$

A T-module is a pair (M, r), $M \in \mathrm{Ob}(\mathcal{C})$, $r\colon T(M) \to M$ s. t.

$$r\mu_M = rT(r) \quad \text{and} \quad r\eta_M = \mathrm{id}_M.$$

$\rightsquigarrow$ $\mathcal{C}^T$ category of T-modules.

Example

A algebra in a monoidal category $\mathcal{C}$

$\rightsquigarrow$ $T = ? \otimes A : X \mapsto X \otimes A$ is a monad on $\mathcal{C}$ and $\mathcal{C}^T = \mathrm{Mod}\text{-}A$

$T' = A \otimes ?$ is a monad on $\mathcal{C}$ and $\mathcal{C}^{T'} = A\text{-}\mathrm{Mod}$

Monads and adjunctions

A monad T on a category $\mathcal{C}$ $\rightsquigarrow$ an adjunction $\begin{tikzcd} \mathcal{C}^T \arrow[d, bend left, "U^T"] \\ \mathcal{C} \arrow[u, bend left, "F^T"] \end{tikzcd}$

where $U^T(M, r) = M$ and $F^T(X) = (T(X), \mu_X)$.

An adjunction $\begin{tikzcd} \mathcal{D} \arrow[d, bend left, "U"] \\ \mathcal{C} \arrow[u, bend left, "F"] \end{tikzcd}$ $\rightsquigarrow$ a monad $T = (UF, \mu := U(\varepsilon_F), \eta)$ on $\mathcal{C}$

where $\eta : 1_\mathcal{C} \to UF$ and $\varepsilon : FU \to 1_\mathcal{D}$ are the adjunction morphisms

$\rightsquigarrow$ $\begin{tikzcd} \mathcal{D} \arrow[dr, bend left, "U"] & & \mathcal{C}^T \arrow[dl, bend left, "U^T"] \\ & \mathcal{C} \arrow[ul, bend left, "F"] \arrow[ur, bend left, "F^T"] & \end{tikzcd}$ $\begin{tikzcd} \mathcal{D} \arrow[rr, bend left, "K"] \arrow[dr, bend left, "U"] & & \mathcal{C}^T \arrow[dl, bend left, "U^T"] \\ & \mathcal{C} \arrow[ul, bend left, "F"] \arrow[ur, bend left, "F^T"] & \end{tikzcd}$

$K : D \mapsto (U(D), U(\varepsilon_D))$
(the *comparison functor*)

The adjunction (F, U) is *monadic* if K equivalence.

Bimonads [Moerdijk]

$\mathcal{C}$ monoidal category, (T, μ, η) monad on $\mathcal{C} \rightsquigarrow \mathcal{C}^T$, $U^T : \mathcal{C}^T \to \mathcal{C}$

T is a *bimonad* if and only if $\mathcal{C}^T$ is monoidal and U^T is strict monoidal. This is equivalent to:

- T is comonoidal endofunctor
  (with $\Delta_{X,Y} : T(X \otimes Y) \to TX \otimes TY$ and $\varepsilon : T\mathbb{1} \to \mathbb{1}$)
- μ and η are comonoidal natural transformations.

Axioms similar to those of a bialgebra except the compatibility between μ and Δ:

$$\begin{tikzcd}
T^2(X \otimes Y) \arrow[r, "T\Delta_{X,Y}"] \arrow[d, "\mu_{X\otimes Y}"'] & T(TX \otimes TY) \arrow[r, "\Delta_{TX,TY}"] & T^2X \otimes T^2Y \arrow[d, "\mu_X \otimes \mu_Y"] \\
T(X \otimes Y) \arrow[rr, "\Delta_{X,Y}"'] & & TX \otimes TY
\end{tikzcd}$$

No braiding involved!

Hopf monads

For a bimonad T define the (left and right) *fusion morphisms*

- $H^l(X,Y) = (\mathrm{id}_{TX} \otimes \mu_Y)\Delta_{X,TY} \colon T(X \otimes TY) \to TX \otimes TY$,
- $H^r(X,Y) = (\mu_X \otimes \mathrm{id}_{TY})\Delta_{TX,Y} \colon T(TX \otimes Y) \to TX \otimes TY$.

A bimonad T is a *Hopf monad* if the fusion morphisms are isomorphisms.

Proposition

For T bimonad on $\mathcal{C}$ rigid, equivalence:

(i) $\mathcal{C}^T$ is rigid;

(ii) T is a Hopf monad;

(iii) (older definition) T admits a left and a right (unary) antipode $s^l_X \colon T({}^\vee TX) \to {}^\vee X$ and $s^r \colon T(TX^\vee) \to X^\vee$.

There is a similar result for closed categories (monoidal categories with internal Homs).

Hopf comonads

The notion of a Hopf monad is not self-dual, unlike that of a Hopf algebra: if you reverse the arrows in the definition, you obtain the notion of a *Hopf comonad.* A Hopf comonad is a monoidal comonad such that the cofusion operators are invertible.
All results about Hopf monads translate into results about Hopf comonads. In particular, if T is a Hopf comonad on $\mathcal{C}$,

1. the category $\mathcal{C}_T$ of comodules over T is monoidal,
2. we have a Hopf monoidal adjunction: $\begin{tikzcd} \mathcal{D} \arrow[r, bend right, "U_T"'] & \mathcal{C} \arrow[l, bend right, "F_T"'] \end{tikzcd}$

where U_T is the forgetful functor and F_T is its right adjoint, the cofree comodule functor.

Hopf monads from adjunctions

Let $\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "U"] & \mathcal{C} \arrow[l, bend left, "F"] \end{tikzcd}$ be a comonoidal adjunction (meaning $\mathcal{C}$, $\mathcal{D}$ are monoidal and U is strong monoidal)

Then F is comonoidal and $T = UF$ is a bimonad on $\mathcal{C}$.

There are canonical morphisms:

- $F(c \otimes Ud) \to Fc \otimes d$
- $F(Ud \otimes c) \to d \otimes Fc$

and (F, U) is a *Hopf adjunction* if these morphisms are isos.

Proposition

If the adjunction is *Hopf*, T is a Hopf monad. Such is the case if either of the following hold:

- $\mathcal{C}$, $\mathcal{D}$ are rigid;
- $\mathcal{C}$, $\mathcal{D}$ and U are closed.

A bimonad is Hopf iff its adjunction is Hopf!

Hopf monads from Hopf algebras

Hopf monads generalize Hopf algebras in braided categories.

$\mathcal{C}$ monoidal category, (H, σ) a Hopf algebra in $\mathcal{Z}(\mathcal{C})$ (which is braided) $\rightsquigarrow$ a Hopf monad $T = H \otimes_\sigma ?$ on $\mathcal{C}$, defined by $X \mapsto H \otimes X$. The comonoidal structure of T is

$$\Delta_{X,Y} = (H \otimes \sigma_X \otimes Y)(\Delta \otimes X \otimes Y)$$
$$\varepsilon = \text{counit of } H$$

Moreover T is equipped with a Hopf monad morphism

$$e = (\varepsilon \otimes ?) : T \to \mathrm{id}_{\mathcal{C}}$$

Theorem (BVL)

This construction defines an equivalence of categories

$$\{\{\text{Hopf algebras in } \mathcal{Z}(\mathcal{C})\}\} \xrightarrow{\simeq} \{\{\text{Hopf monads on } \mathcal{C}\}\} \,/\, \mathrm{id}_{\mathcal{C}}$$

Hopf monads as ‘quantum groupoids’

Let R be a unitary ring $\rightsquigarrow$ a monoidal category $({}_R Mod_R, \otimes_R, {}_R R_R)$.

Facts

- linear bimonads on ${}_R Mod_R$ with a right adjoint are bialgebroids in the sense of Takeuchi [Szlacháni]
- linear Hopf monads on ${}_R Mod_R$ with a right adjoint are a Hopf algebroids in the sense of Schauenburg.

Hopf algebroids are non-commutative avatars of groupoids. Complicated axioms $\rightsquigarrow$ a Hopf adjunction $\rightsquigarrow$ a Hopf monad (much easier to manipulate). Using Hopf monads one shows:

Theorem (BVL)

A finite tensor category $\mathcal{C}$ over a field $\Bbbk$ is tensor equivalent to the category of A-modules for some bialgebroid A.

Given a $\Bbbk$- equivalence $\mathcal{C} \overset{\Bbbk}{\simeq}_R \, mod$ for some finite dimensional $\Bbbk$- algebra R, one constructs a canonical Hopf algebroid A over R.

Outlook of General Theory of Hopf monads

- Tannaka dictionary
- Hopf modules and Sweedler decomposition theorem
- Existence of universal integrals (with values in a certain autoequivalence of $\mathcal{C}$)
- Semisimplicity, Maschke criterion
- The Drinfeld double of a Hopf monad
- Cross-products
- Bosonization for Hopf monads
- Applications to construction and comparison of quantum invariants (non-braided setting)

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Let $\omega : \mathcal{C} \to \overline{\mathcal{B}}$ be a strong monoidal functor, with $\mathcal{C}$ small, and let $\mathcal{B} \subset \overline{\mathcal{B}}$ be a full monoidal subcategory s. t. $\omega(\mathcal{C}) \subset \mathcal{B}$. Assume that

1. $\mathcal{C}$ has sums of two, coequalizers, fibered products and equalizers, and ω preserves all of them as well as monomorphisms;
2. $\mathcal{C}$ has functorial mono-epi factorizations and $\mathcal{C}$ is coartinian;
3. $\overline{\mathcal{B}}$ has small filtered colimits which commute with equalizers, the tensor product of $\overline{\mathcal{B}}$ preserves them and the objects of $\mathcal{B}$ have finite type in $\overline{\mathcal{B}}$ (e. g. $\overline{\mathcal{B}} = \mathrm{Ind}\mathcal{B}$);
4. ω is conservative.

Then

a) ω extends uniquely to a strong monoidal functor $\Omega : \mathrm{Ind}\mathcal{C} \to \overline{\mathcal{B}}$ which preserves filtered colimits;

b) Ω is monadic, hence, denoting by T its bicomonad on $\overline{\mathcal{B}}$, a monoidal equivalence $\mathrm{Ind}\mathcal{C} \xrightarrow{\sim} \overline{\mathcal{B}}_T$;

c) Moreover we have $\mathcal{C} \xrightarrow{\sim} \mathcal{B}_T$.

$$\begin{tikzcd} \mathrm{Ind}\mathcal{C} \arrow[rr, "\simeq_\otimes"] \arrow[dr, "\Omega"'] & & \overline{\mathcal{B}}_T \arrow[dl] \\ & \overline{\mathcal{B}} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[rr, "\simeq_\otimes"] \arrow[dr, "\omega"'] & & \mathcal{B}_T \arrow[dl] \\ & \mathcal{B} & \end{tikzcd}$$

If in addition

1. $\mathcal{C}$ is rigid (or $\mathcal{C}$ as internal cohoms and ω preserves them),
2. for any mono *i* of $\mathcal{C}$, $\omega(i)$ is a tensor-universal mono of $\overline{\mathcal{B}}$,

then T is a Hopf comonad.

Note that, if ω has a monoidal section, then the Hopf comonad T is co-augmented, so there exists a Hopf algebra (H, σ) in $\mathcal{Z}(\overline{\mathcal{B}})$ such that $T = H \otimes_\sigma ?$ and $\mathcal{C}$ is the category of H-comodules in $\mathcal{B}$.

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Galois-Grothendieck duality

Assume $\omega : \mathcal{C} \to \mathrm{set}$ satisfies Grothendieck's assumptions. Apply the main result in its dual form (for monads); with $\mathcal{B} = \mathrm{set}$, $\overline{\mathcal{B}} = \mathrm{Proset}$ the category of pro-objects of set, that is, the category of Hausdorf compact, totally disconnected topological spaces. Then there exists a bimonad T on Proset such that $\mathcal{C} \simeq \mathrm{set}^T$.

Moreover T preserves finite sums and filtering limits. Such a bimonad is of the form $M \otimes ?$, where M is a monoid in Proset.

From the assumption that monomorphisms in $\mathcal{C}$ are summands, one deduces easily that $\mathcal{C}$ is closed and so is ω, hence T is a Hopf monad, and M is a group in Proset, that is, a profinite group.

Tannaka duality

Let $\mathcal{C}$ be a symmetric tensor category, $\omega : \mathcal{C} \to \mathrm{Mod}B$ a fibre functor. Let $\overline{\mathcal{B}} = \mathrm{Mod}B$, $\mathcal{B} \subset \overline{\mathcal{B}}$ the full subcategory of (projective) modules of finite type.
Then the theorem applies, and one obtains a Hopf monad T on $\overline{\mathcal{B}}$ such that $\mathcal{C} = \mathcal{B}_T$.
We have for $N \in \mathrm{Mod}\, B$:

$$T(N) = \int^{c \in \mathcal{C}} \mathrm{Hom}_B(\omega(c), N) \otimes_{\Bbbk} \omega(c) = L \otimes_B N$$

where $L = \int^{c \in \mathcal{C}} \omega(c)^* \otimes_{\Bbbk} \omega(c)$ is the coend of ω. It is commutative Hopf bialgebroid, that is, an affine groupoid with base $\mathrm{Spec}B$.

Non-commutative tannaka duality

Let $\mathcal{C}$ be a tensor category. In particular, $\mathcal{C}$ is an abelian $\Bbbk$- linear category which is artinian and has finite dimensional Homs. Such a category is equivalent to the category $\mathrm{comod}L$ of finite dimensional right comodules over a coalgebra L.

The category $\mathcal{C}$ acts on itself by tensoring on the left, hence $\mathcal{C}$ acts on $\mathrm{comod}L$. Hence a strong monoidal functor $\omega : \mathcal{C} \to \overline{\mathcal{B}}$, where $\overline{\mathcal{B}} \subset \mathrm{End}(\mathrm{comod}L)$ is the category $L\mathrm{comod}L$ of L bicomodules.

Applying a (variant of) our main theorem, one obtains :

Theorem

Let $\mathcal{C}$ be a tensor category. There exists a Hopf coalgebroid Λ such that $\mathcal{C}$ is tensor equivalent to $\mathrm{comod}\Lambda$ and $\mathrm{Ind}\mathcal{C}$, to $\mathrm{Comod}\Lambda$. The base of Λ is a coalgebra s. t. $\mathcal{C} \equiv_{\Bbbk} \mathrm{comod}L$.

A *Hopf coalgebroid* Λ is the dual notion of a Hopf algebroid (one may also see it as a Hopf algebroid in the category of Pro-objects of the category of finite dimensional vector spaces).

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Hopf modules and Sweedler's Theorem for Hopf Monads

T Hopf monad on $\mathcal{C} \rightsquigarrow T\mathbb{1}$ is a coalgebra in $\mathcal{C}$ (coproduct $\Delta_{\mathbb{1},\mathbb{1}}$, counit ε) $\rightsquigarrow$ lifts to a coalgebra $\hat{C} = F^T(\mathbb{1})$ in $\mathcal{C}^T$. Moreover we have a natural isomorphism

$$\sigma : \hat{C} \otimes ? \to ? \otimes \hat{C}.$$

Proposition (BVL)

σ is a half-braiding and $(\hat{C}, \sigma)$ is a cocommutative coalgebra in $\mathcal{Z}(\mathcal{C}^T)$ called the *induced central coalgebra* of T.

A (right) T-Hopf module is a (right) $\hat{C}$-comodule in $\mathcal{C}^T$, *i. e.* a data (M, r, ∂) with (M, r) a T-module, (M, ∂) a $T\mathbb{1}$-comodule + T-linearity of ∂.

Under suitable exactness conditions (T is conservative, $\mathcal{C}$ has coequalizers and T preserves them):

Theorem (BVL)

The assignment $X \mapsto (TX, \mu_X, \Delta_{X,\mathbb{1}})$ is an equivalence of categories

$$Q : \mathcal{C} \xrightarrow{\simeq} \{\{T\text{-}\,\text{Hopf modules}\}\}$$

with quasi-inverse the functor *coinvariant part*.
Moreover if $\mathcal{C}$ has equalizers and T preserves them, Q is a monoidal equivalence, the category of Hopf modules (i.e. $\hat{C}$- comodules) being endowed with the cotensor product over $\hat{C}$.

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If $\mathcal{C}$ is a tensor category, its Ind-completion $\mathrm{Ind}\mathcal{C}$ is a monoidal abelian category containing $\mathcal{C}$ as a full subcategory and whose objects are formal filtering colimits of objects of $\mathcal{C}$. For instance $\mathrm{Ind}\,\mathrm{vect} = \mathrm{Vect}$, and $\mathrm{Ind}\,\mathrm{comod}H = \mathrm{Comod}H$. Note that these are no longer rigid.

Theorem

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor. There exists a $\Bbbk$- linear left exact comonad on $\mathrm{Ind}\mathcal{C}$ such that we have a commutative diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "\simeq_{\otimes}"'] & & \mathcal{D} \\ & \mathcal{D}_T \arrow[ur] & \end{tikzcd}$$

where $\mathcal{C}_T$ is the category of T-comoduleS whose underlying object is in $\mathcal{C}$.

Proof

The functor $F : \mathcal{C} \to \mathcal{D}$ extends to a linear faithful exact functor $\mathrm{Ind}F : \mathrm{Ind}\mathcal{C} \to \mathrm{Ind}\mathcal{D}$ which preserves colimits and is strong monoidal.
$\mathrm{Ind}F$ has a right adjoint, denoted by R.
It is also a monoidal adjunction, which is Hopf. Its comonad $T = \mathrm{Ind}FR$ is a Hopf comonad on $\mathrm{Ind}\mathcal{C}$.
$\mathrm{Ind}F$ being faithful exact, the adjunction $(\mathrm{Ind}F, R)$ is comonadic by Beck, hence the theorem.

Example

If $\mathcal{D} = \mathrm{vect}$, a linear Hopf comonad on Vect is of the form $H \otimes ?$ for some Hopf algebra H, so we recover the classical tannakian result.

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor. We say that F is *dominant* if the right adjoint R of $\mathrm{Ind}F$ is faithful exact.

Then applying the classification theorem for Hopf modules in its dual form we obtain:

Theorem

If F is dominant, there exists a commutative algebra (A, σ) in $\mathcal{Z}(\mathrm{Ind}\mathcal{C})$ - the induced central algebra of T - such that we have a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F_A"] \arrow[dr, "F"'] & & A\text{-}\ \mathrm{mod}_{\mathcal{C}} \\ & \mathcal{D} \arrow[ur, "\simeq_\otimes"'] & \end{tikzcd}$$

where A- mod is the category of 'finite type' A-modules in $\mathrm{Ind}\mathcal{C}$ (=quotients of $A \otimes X$, $X \in \mathcal{C}$), with tensor product $\otimes_{A,\sigma}$, and F_A is the tensor functor $X \mapsto A \otimes X$.

If $\mathcal{D} = \mathrm{vect}\Bbbk$ and $\mathcal{C}$, F are symmetric, then A is Deligne's trivializing algebra.