Preantipodes for dual-quasi bialgebras

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It is known that a dual quasi-bialgebra with antipode H, i.e. a dual quasi-Hopf algebra, fulfils a fundamental theorem for right dual quasi-Hopf H-bicomodules. The converse in general is not true. We prove that, for a dual quasi-bialgebra H, the structure theorem amounts to the existence of a suitable endomorphism S of H that we call a preantipode of H. This is based on joint work with Alice Pavarin (Univ. of Padova, Italy), see arXiv:1012.1956.

Preantipodes for dual quasi-bialgebras

Alessandro Ardizzoni

Hopf algebras and tensor categories,

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This talk is based on a joint work
with Alice Pavarin (Univ. of Padova, Italy):

A. Ardizzoni, A. Pavarin, *Preantipodes for Dual Quasi-Bialgebras*, Israel J. Math., to appear. (arXiv:1012.1956)

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TFAE:

(1) *the functor* $(-) \otimes H : \mathfrak{M} \to \mathfrak{M}_H^H$ *is an equivalence of categories;*

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Implication $(2) \Rightarrow (1)$ is the so-called **structure (or fundamental) theorem for Hopf modules**, which is due, in the finite-dimensional case, to Larson and Sweedler.

R. G. Larson, M. E. Sweedler, *An associative orthogonal bilinear form for Hopf algebras.* Amer. J. Math. **91** (1969), 75–94.

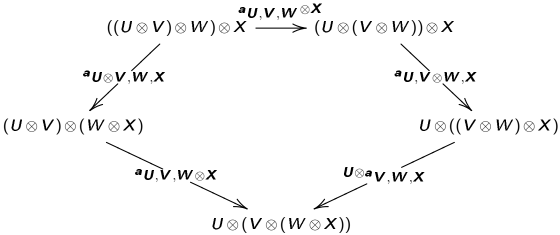
Now, observe that, since H is a bialgebra,

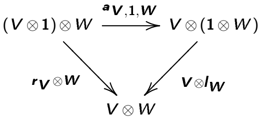
- $\mathfrak{M}^H$ is a monoidal category,
- H is an algebra in this category,
- $\mathfrak{M}_H^H = (\mathfrak{M}^H)_H$ is the category of right H-modules in $\mathfrak{M}^H$.

A monoidal category [Bénabou, 1963] $(\mathscr{M}, \otimes, \mathbf{1}, a, l, r)$ is a category $\mathscr{M}$ endowed with a functor $\otimes : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ (tensor product), an object $\mathbf{1} \in \mathscr{M}$ (unit of $\mathscr{M}$) and functorial isomorphisms:

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z) \quad \text{(associativity constraint,)}$$
$$l_X : \mathbf{1} \otimes X \to X, \quad r_X : X \otimes \mathbf{1} \to X \qquad \text{(unit constraints),}$$

such that the following diagrams commute.





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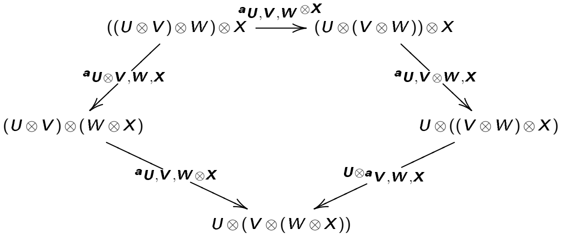

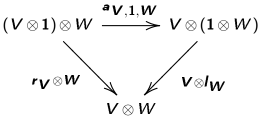

The notions of algebra and module over an algebra can be introduced in the general setting of monoidal categories.

Now consider the following data:

- $(H, \Delta, \varepsilon) =$ coalgebra,
- $m : H \otimes H \to H =$ coalgebra map (multiplication),
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Consider the category $\mathfrak{M}^H$ of right H-comodules. Then,

- $\forall M, N \in \mathfrak{M}^H$, we have that $M \otimes N \in \mathfrak{M}^H$ through

$$x \otimes y \mapsto (x_0 \otimes y_0) \otimes x_1 y_1,$$

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- $\Bbbk \in \mathfrak{M}^H$ with structure $k \mapsto k \otimes 1_H$.

It is well-known that the datum $(\mathfrak{M}^H, \otimes, \Bbbk)$, with the structures above, becomes a monoidal category if and only if H is a dual quasi-bialgebra (Majid, 1990).

This means that there exists a map $\omega : H^{\otimes 3} \to \Bbbk$ such that

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- ω is a unital 3-cocycle i.e. it is convolution invertible and

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- m is quasi-associative and unitary i.e. it satisfies

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This notion is dual to the one of quasi-bialgebra, introduced by Drinfeld in 1989, in connection with some system of partial differential equations.

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There is a functor $F := (-) \otimes H : {}^H\mathfrak{M} \to {}^H\mathfrak{M}^H_H$, with structures, for every $V \in {}^H\mathfrak{M}^H$, $v \in V, h, l \in H$,

$$ \rho^l_{F(V)}(v \otimes h) = v_{-1} h_1 \otimes (v_0 \otimes h_2), $$
$$ \rho^r_{F(V)}(v \otimes h) = (v \otimes h_1) \otimes h_2, $$
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When ω is trivial, i.e. H is an ordinary bialgebra, it can be proved that

F is an equivalence $\Leftrightarrow$ H has an antipode.

Hence the functor F is a natural substitute for the functor $(-) \otimes H : \mathfrak{M} \to \mathfrak{M}^H_H$ considered at the beginning.

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The following result is the right-handed version of Lemma 2.1 in [Sch].

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Proposition

The functor $F : {}^H\mathfrak{M} \to {}^H\mathfrak{M}_H^H$ *has a right adjoint functor* $G := (-)^{\mathrm{co}H}$*.*
The counit and the unit of the adjunction are given by

$$\varepsilon_M : M^{\mathrm{co}H} \otimes H \to M, \quad \varepsilon_M(x \otimes h) := xh \qquad \textit{and by}$$

$$\eta_N : N \to (N \otimes H)^{\mathrm{co}H}, \quad \eta_N(n) := n \otimes 1_H,$$

for every $M \in {}^H\mathfrak{M}_H^H$ *and* $N \in {}^H\mathfrak{M}$*.*
Moreover η_N *is an isomorphism for any* $N \in {}^H\mathfrak{M}$*.*
Equivalently, the functor F *is full and faithful.*

Next result characterizes when the functor F is an equivalence of categories in terms of the existence of a suitable map τ.

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Let $(H, \Delta, \varepsilon, m, u, \omega)$ be a dual quasi-bialgebra. TFAE.

(*i*) *The functor $F : {}^H\mathfrak{M} \to {}^H\mathfrak{M}_H^H$ is an equivalence of categories.*

(*ii*) *$\forall M \in {}^H\mathfrak{M}_H^H$, there exists a $\Bbbk$-linear map $\tau : M \to M^{coH}$ such that:*

$$\begin{aligned} \tau(mh) &= \omega^{-1}[\tau(m_0)_{-1} \otimes m_1 \otimes h]\tau(m_0)_0, \text{ for all } h \in H, m \in M, \\ m_{-1} \otimes \tau(m_0) &= \tau(m_0)_{-1} m_1 \otimes \tau(m_0)_0, \text{ for all } m \in M, \\ \tau(m_0)m_1 &= m \; \forall m \in M. \end{aligned}$$

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Proof.

$(i) \Rightarrow (ii)$. Set $\tau := \left(M \xrightarrow{\varepsilon_M^{-1}} M^{\mathrm{co}H} \otimes H \xrightarrow{M^{\mathrm{co}H} \otimes \varepsilon} M^{\mathrm{co}H} \right)$.

$(ii) \Rightarrow (i)$. Set $\varepsilon_M^{-1} := \left(M \xrightarrow{\rho_M^r} M \otimes H \xrightarrow{\tau \otimes H} M^{\mathrm{co}H} \otimes H \right)$.

□

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Explicitly, the structure of $T(M)$ is given as follows, for all $M \in {}^H\mathfrak{M}^H$:

$$\begin{aligned} \rho^l_{M\otimes H}(x \otimes h) &:= x_{-1} h_1 \otimes (x_0 \otimes h_2), \\ \rho^r_{M\otimes H}(x \otimes h) &:= (x_0 \otimes h_1) \otimes x_1 h_2, \\ (x \otimes h) l &:= \omega^{-1}(x_{-1} \otimes h_1 \otimes l_1) x_0 \otimes h_2 l_2 \omega(x_1 \otimes h_3 \otimes l_3). \end{aligned}$$

Consider the functor $T := (-) \otimes H : {}^H\mathfrak{M}^H \to {}^H\mathfrak{M}^H_H$ as above and set

$$H \widehat{\otimes} H := T({}^\circ H^\bullet) \in {}^H\mathfrak{M}^H_H,$$

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We are now able to state our main theorem.

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(*iii*) *There exists* $\Bbbk$*-linear map* $S : H \to H$ *such that, for all* $h \in H$,

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A map S as above will be called a preantipode for the dual quasi-bialgebra.

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Note that, in order to prove $(iii) \Rightarrow (i)$, for any $M \in {}^{H}\mathfrak{M}_{H}^{H}$, one defines a map $\tau: M \to M^{\mathrm{co}H}$, by setting, for every $m \in M$,

$$\tau(m) := \omega[m_{-1} \otimes S(m_1)_1 \otimes m_2] m_0 S(m_1)_2.$$

Definition (Majid)

A dual quasi-Hopf algebra $(H, m, u, \Delta, \varepsilon, \omega, s, \alpha, \beta)$ is a dual quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \omega)$ endowed with a coalgebra anti-homomorphism

$$s : H \to H$$

and two maps α, β in H^*, such that, for all $h \in H$:

$$\begin{array}{rcl} h_1\beta(h_2)s(h_3) = \beta(h)1_H, & & s(h_1)\alpha(h_2)h_3 = \alpha(h)1_H, \\ \omega(h_1 \otimes \beta(h_2)s(h_3)\alpha(h_4) \otimes h_5) = & \varepsilon(h) & = \omega^{-1}(s(h_1) \otimes \alpha(h_2)h_3\beta(h_4) \otimes s(h_5)). \end{array}$$

The map s is called an antipode for the dual quasi-bialgebra.

Definition (Majid)

A dual quasi-Hopf algebra $(H, m, u, \Delta, \varepsilon, \omega, s, \alpha, \beta)$ is a dual quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \omega)$ endowed with a coalgebra anti-homomorphism

$$s : H \to H$$

and two maps α, β in H^*, such that, for all $h \in H$:

$$\begin{array}{rcl} h_1\beta(h_2)s(h_3) = \beta(h)1_H, & & s(h_1)\alpha(h_2)h_3 = \alpha(h)1_H, \\ \omega(h_1 \otimes \beta(h_2)s(h_3)\alpha(h_4) \otimes h_5) = & \varepsilon(h) & = \omega^{-1}(s(h_1) \otimes \alpha(h_2)h_3\beta(h_4) \otimes s(h_5)). \end{array}$$

The map s is called an antipode for the dual quasi-bialgebra.

Theorem

Let $(H, m, u, \Delta, \varepsilon, \omega, s, \alpha, \beta)$ be a dual quasi-Hopf algebra. Then

$$S := \beta * s * \alpha$$

is a preantipode. Here $*$ denotes the convolution product.

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Let $(H, m, u, \Delta, \varepsilon, \omega, s, \alpha, \beta)$ *a dual quasi-Hopf algebra.*
Then the functor $F : {}^{H}\mathfrak{M} \to {}^{H}\mathfrak{M}_{H}^{H}$ *is an equivalence of categories i.e. the structure theorem for right dual quasi-Hopf H-bicomodules holds.*

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Remark

In the following paper, there is an example of a dual quasi-bialgebra H which has no antipode but such that the functor F is still an equivalence. Thus H has preantipode $\not\Rightarrow$ H has an antipode, in general.

*P. Schauenburg, Hopf algebra extensions and monoidal categories. New directions in Hopf algebras, 321–381, Math. Sci. Res. Inst. Publ.,**43**, Cambridge Univ. Press, Cambridge, 2002.*

Example

Let $\theta : G \times G \times G \to \Bbbk \setminus \{0\}$ be a normalized 3-cocycle on a group G i.e.

$$\begin{aligned} \forall g,h,k,l \in G, \qquad \theta(g,1_G,h) &= 1 \\ \theta(h,k,l)\,\theta(g,hk,l)\,\theta(g,h,k) &= \theta(g,h,kl)\,\theta(gh,k,l). \end{aligned}$$

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$$\alpha(g) := 1_{\Bbbk}, \quad \beta(g) := \left[\omega\left(g, g^{-1}, g\right)\right]^{-1} \text{ and } s(g) := g^{-1}, \quad \forall g \in G.$$

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Note that, unlike the antipode, this preantipode is not a coalgebra anti-homomorphism.

Work in progress

Definition

Let H and A be dual quasi-bialgebras endowed with morphisms of dual quasi-bialgebras

$$\sigma : H \to A \qquad \text{and} \qquad \pi : A \to H$$

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For (A, H, σ, π) as above, one easily proves that $A \in {}^H\mathfrak{M}_H^H$ through

$$\rho_A^r(a) = a_1 \otimes \pi(a_2), \qquad \rho_A^l(a) = \pi(a_1) \otimes a_2, \qquad \mu_A^r(a \otimes h) = a\sigma(h).$$

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Next aim is to investigate the dual quasi-bialgebra structure on $A^{\mathrm{co}H} \otimes H$.