Crossed product of Hopf algebras

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The main properties of the crossed product in the category of Hopf algebras are investigated. Let A and H be two Hopf algebras connected by two morphism of coalgebras $\triangleright : H \otimes A \to A$, $f : H \otimes H \to A$. The crossed product $A\#_f^{\triangleright} H$ is a new Hopf algebra containing A as a normal Hopf subalgebra. In fact, we prove that a Hopf algebra E is isomorphic as a Hopf algebra to a crossed product of Hopf algebras $A\#_f^{\triangleright} H$ if and only if E factorizes through a normal Hopf subalgebra A and a subcoalgebra H such that $1_E \in H$. The universality of the construction, the existence of integrals, commutativity or involutivity of the crossed product are studied. The crossed product $A\#_f^{\triangleright} H$ is a semisimple Hopf algebra if and only if both Hopf algebras A and H are semisimple. Looking at the quantum side of the construction we shall give necessary and sufficient conditions for a crossed product to be a coquasitriangular (braided) Hopf algebra. In particular, all braided structures on the monoidal category of $A\#_f^{\triangleright} H$-comodules are explicitly described in terms of their components.

Bibliography

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The crossed product of Hopf algebras is a special case of the unified product by considering the right action $h \triangleleft a = \varepsilon_A(a)h$, for all $h \in H$ and $a \in A$

Let H be a Hopf algebra, A a k-algebra and two k-linear maps $\cdot : H \otimes A \to A$, $f : H \otimes H \to A$ such that:

$$h \cdot 1_A = \varepsilon_H(h) 1_A$$

$$1_H \cdot a = a$$

$$h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b)$$

$$f(h, 1_H) = f(1_H, h) = \varepsilon_H(h) 1_A$$

for all $h \in H$, a, $b \in A$.

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for all $h \in H$, a, $b \in A$.
The *crossed product* $A \#^{\cdot}_f H$ of A with H is the k-module $A \otimes H$ with the multiplication given by:

$$(a \otimes h) * (c \otimes g) := a(h_{(1)} \cdot c) f\left(h_{(2)}, g_{(1)}\right) \otimes h_{(3)} g_{(2)} \qquad (1)$$

Blattner, Cohen, Montgomery – Doi, Takeuchi:

Theorem

$(A\#^{\cdot}_{f} H, *)$ is an associative algebra with identity element $1_A\#1_H$ if and only if the following two compatibility conditions hold:

$$[g_{(1)} \cdot (h_{(1)} \cdot a)]f(g_{(2)}, h_{(2)}) = f(g_{(1)}, h_{(1)})((g_{(2)}h_{(2)}) \cdot a)$$

$$(g_{(1)} \cdot f(h_{(1)}, l_{(1)}))f(g_{(2)}, h_{(2)}l_{(2)}) = f(g_{(1)}, h_{(1)})f(g_{(2)}h_{(2)}, l)$$

for all $a \in A$, g, h, $l \in H$.

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$$(g_{(1)}\cdot f(h_{(1)},l_{(1)}))f(g_{(2)},h_{(2)}l_{(2)}) = f(g_{(1)},h_{(1)})f(g_{(2)}h_{(2)},l)$$

for all $a\in A$, $g,h,l\in H$. In this case $(A\#^{\cdot}_{f}H, *)$ is called the crossed product of H acting on A.

Let A and H be two Hopf algebras and $\cdot : H \otimes A \to A$ and $f : H \otimes H \to A$ two coalgebra morphisms.

Proposition

Then the crossed product $A \#_f^{\cdot} H$ has a Hopf algebra structure with the coalgebra structure given by the tensor product of coalgebras if and only if the following two compatibility conditions hold:

$$g_{(1)} \otimes g_{(2)} \cdot a = g_{(2)} \otimes g_{(1)} \cdot a$$

$$g_{(1)} h_{(1)} \otimes f(g_{(2)}, h_{(2)}) = g_{(2)} h_{(2)} \otimes f(g_{(1)}, h_{(1)})$$

for all $g, h \in H$ and $a, b \in A$

Remark

The antipode of the Hopf algebra $A \#^{\cdot}_{f} H$ is given by the formula:

$$S(a\#g) := \Big(S_A\big[f\big(S_H(g_{(2)}), g_{(3)}\big)\big]\#S_H(g_{(1)})\Big)\cdot\big(S_A(a)\#1_H\big)$$

for all $a \in A$ and $g \in H$.

Definition

A quadruple $(A, H, \cdot, f)$, where A and H are Hopf algebras and $\cdot : H \otimes A \to A$, $f : H \otimes H \to A$ are two coalgebra maps such that $A\#^{\cdot}_{f} H$ is a Hopf algebra with the coalgebra structure given by the tensor product of coalgebras is called *crossed system of Hopf algebras*. The corresponding Hopf algebra $A\#^{\cdot}_{f} H$ will be called *crossed product of Hopf algebras*

A special case of **the cocycle bicrossproduct bialgebra**

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1. Constructed for bialgebras by Majid and Soibelman (1994)
2. Constructed for Hopf algebras by Andruskiwitsch and Devoto (1995)

Examples

1) Let A, H be two Hopf algebras and $\cdot$, f be the trivial action respectively the trivial cocycle, that is: $a \cdot h = \varepsilon(a)h$ and $f(g,h) = \varepsilon(g)\varepsilon(h)$ for all $a \in A$ and $g, h \in H$. Then the associated crossed product is exactly the tensor product of Hopf algebras $A \otimes H$.

Examples

2) Let A, H be two Hopf algebras and f be the trivial cocycle, that is $f(g,h)=\varepsilon(g)\varepsilon(h)$ for all $g,h\in H$. If A is an H-module algebra via the coalgebra map $\cdot$ and relation

$$g_{(1)}h_{(1)}\otimes f(g_{(2)},\, h_{(2)}) = g_{(2)}h_{(2)}\otimes f(g_{(1)},\, h_{(1)})$$

is fulfilled then the crossed product has the algebra structure given by the smash product (Molnar – cocommutative case) and it will be denoted by $A\#^{\cdot}H$.

Examples

3) $(H, G, f, \cdot)$ be a normalized crossed system of groups then $(kH, kG, \widetilde{f}, \widetilde{\cdot})$ is a crossed system of Hopf algebras where $\widetilde{f}$ and $\widetilde{\cdot}$ are obtained by linearizing the maps f and $\cdot$. Moreover, there exists an isomorphism of Hopf algebras

$$k[H \times_f^{\cdot} G] \cong kH \#_{\widetilde{f}}^{\widetilde{\cdot}} kG$$

and any crossed product of Hopf algebras between two group algebras arises in this way.

Examples

4) Let A and H be two Hopf algebras such that H is cocommutative and $\gamma : H \to A$ a unitary coalgebra map. Define:

$$\cdot := \cdot_\gamma : H \otimes A \to A, \qquad h \cdot a := \gamma(h_{(1)}) a \gamma^{-1}(h_{(2)})$$

$$f := f_\gamma : H \otimes H \to A, \qquad f(h, g) = \gamma(h_{(1)})\gamma(g_{(1)})\gamma^{-1}(h_{(2)}g_{(2)})$$

Then $(A, H, \cdot, f)$ is a crossed system of Hopf algebras and, moreover, the map given by:

$$\varphi : A \#_f^{\cdot} H \to A \otimes H, \quad \varphi(a \otimes h) = a\gamma(h_{(1)}) \otimes h_{(2)}$$

for all $a \in A$ and $h \in H$ is an isomorphism of Hopf algebras

Definition

We say that a Hopf algebra E factorizes through a Hopf subalgebra A and a subcoalgebra H if the multiplication map $u : A \otimes H \to E$, $u(a \otimes h) = ah$, for all $a \in A$, $h \in H$ is bijective.

The main property of the crossed product of Hopf algebras is the following reconstruction type theorem:

Definition

We say that a Hopf algebra E factorizes through a Hopf subalgebra A and a subcoalgebra H if the multiplication map $u : A \otimes H \to E$, $u(a \otimes h) = ah$, for all $a \in A$, $h \in H$ is bijective.

The main property of the crossed product of Hopf algebras is the following reconstruction type theorem:

Theorem

A Hopf algebra E is isomorphic as a Hopf algebra to a crossed product $A \#^{\cdot}_{f} H$ of Hopf algebras if and only if E factorizes through a normal Hopf subalgebra A and a subcoalgebra H such that $1_E \in H$.

S. Burciu 2011:

Corollary

If a Hopf algebra E factorizes through two Hopf subalgebras A and H, with A normal in E, then E is isomorphic as a Hopf algebra to a smash product.

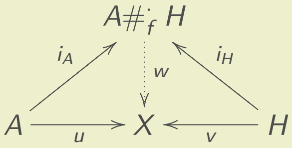
Universal properties: Let $(A, H, \cdot, f)$ be a crossed system of Hopf algebras.

Proposition

(1) *For any Hopf algebra X, any Hopf algebra map $u : A \longrightarrow X$ and any coalgebra map $v : H \longrightarrow X$ such that the following compatibilities hold for all $h, g \in H$, $b \in A$:*

$$
\begin{aligned}
u\big(f(h_{(1)}, g_{(1)})\big)v(h_{(2)}g_{(2)}) &= v(h)v(g) \\
u(h_{(1)} \cdot b)v(h_{(2)}) &= v(h)u(b)
\end{aligned}
$$

there exists a unique Hopf algebra map $w : A\#_f^{\triangleright} H \longrightarrow X$ such that the following diagram commutes:



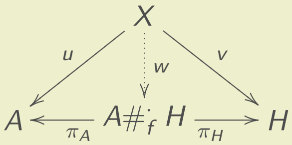
Proposition

(2) *For any Hopf algebra X, any Hopf algebra map $v : X \to H$ and any coalgebra map $u : X \to A$ such that the following compatibilities hold for all $x, y \in X$:*

$$u(x_{(1)}) \otimes v(x_{(2)}) = u(x_{(2)}) \otimes v(x_{(1)})$$

$$u(xy) = u(x_{(1)})\big[v(x_{(2)}) \cdot u(y_{(1)})\big] f\big(v(x_{(3)}),\, v(y_{(2)})\big)$$

there exists a unique Hopf algebra map $w : X \to A\#_f^{\cdot} H$ such that the following diagram commutes:



Classification

Definition

Let A and H be two Hopf algebras, a coalgebra map $u : H \longrightarrow A$ is called a coalgebra lazy 1-cocyle if $u(1_H) = 1_A$ and the following compatibility holds:

$$h_{(1)} \otimes u(h_{(2)}) = h_{(2)} \otimes u(h_{(1)})$$

for all $h \in H$.

Proposition

Let A, H be two Hopf algebras and $A\#^{\cdot}_{f} H$, $A\#^{\cdot'}_{f'} H$ be two crossed products of Hopf algebras. The following are equivalent:
(1) $A\#^{\cdot}_{f} H \approx A\#^{\cdot'}_{f'} H$ *(isomorphism of Hopf algebras, left A-modules and right H-comodules);*
(2) *There exists a coalgebra lazy 1-cocyle $u : H \to A$ such that:*

(1) $h \cdot' a = u^{-1}(h_{(1)})(h_{(2)} \cdot a)u(h_{(3)})$

(2) $f'(h, k) = u^{-1}(h_{(1)})\big(h_{(2)} \cdot u^{-1}(k_{(1)})f(h_{(3)}, k_{(2)})\big)u(h_{(4)}k_{(3)})$

for all $a \in A$ and $h, g \in H$.

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for all $a \in A$ and $h, g \in H$. In this case we shall say that the crossed systems $(A, H, \cdot, f)$ and $(A, H, \cdot', f')$ are cohomologous and we denote the equivalence classes of crossed systems modulo such transformations by $\mathcal{H}^2(H, A)$.

Remark

A crossed system of Hopf algebras which is equivalent to the trivial one is called a coboundary. An example of a coboundary is given in Example 4.

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Problem: *Let "P" be an algebraic property. Give necessary and sufficient conditions for a crossed product of Hopf algebras* $A\#^{\cdot}_{f} H$ *to have the property "P" in terms of his components.*

The general problem that we are interested in related to the crossed product is the following:

Problem: *Let "P" be an algebraic property. Give necessary and sufficient conditions for a crossed product of Hopf algebras* $A\#_f^{\cdot} H$ *to have the property "P" in terms of his components.*

In what follows, we give complete answers for "P" equal to:

- Commutative
- Involutory
- Coquasitriangular

Proposition

Let $A\#^{\cdot}_{f} H$ be a crossed product of Hopf algebras. Then $A\#^{\cdot}_{f} H$ is commutative if and only if A and H are commutative, $\cdot$ is trivial and f is symmetric (i.e. $f(g, h) = f(h, g)$ for all $g, h \in H$).

Proposition

Let $A\#^{\cdot}_{f} H$ be a crossed product of Hopf algebras.

(1) *If $A\#^{\cdot}_{f} H$ is involutory then both A and H are involutory Hopf algebras.*

(2) *Suppose that H is cocommutative. Then $A\#^{\cdot}_{f} H$ is involutory if and only if A is involutory and $g_{(1)} \cdot f\big(S_H(g_{(2)}),\, g_{(3)}\big) = f\big(g_{(1)},\, S_H(g_{(2)})\big)$ for all $g \in H$.*

Definition

Recall that if A and H are two Hopf algebras and $\alpha : A \otimes H \longrightarrow k$ is a k-linear map which fulfills the compatibilities:

(BR1) $\alpha(xy, z) = \alpha(x, z_{(1)})\alpha(y, z_{(2)})$

(BR2) $\alpha(1, x) = \varepsilon(x)$

(BR3) $\alpha(x, yz) = p(x_{(1)}, z)p(x_{(2)}, y)$

(BR4) $\alpha(y, 1) = \varepsilon(y)$

for all x, $y \in A$, $z \in H$, then α is called *skew pairing* on (A, H)

Definition

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for all x, $y \in A$, $z \in H$, then α is called *skew pairing* on (A, H)

Definition

A Hopf algebra H is called *coquasitriangular* if there exists a linear map $p : H \otimes H \longrightarrow k$ such that relations $(BR1) - (BR4)$ are fulfilled and

(BR5) $p(x_{(1)}, y_{(1)})x_{(2)}y_{(2)} = y_{(1)}x_{(1)}p(x_{(2)}, y_{(2)})$

holds for all x, y, $z \in H$.

Definition

Let A, H be two Hopf algebras, $f : H \otimes H \to A$ a coalgebra map and $p : A \otimes A \to k$ a coquasitriangular structure on A. A linear map $u : A \otimes H \to k$ is called *(p,f) - right skew pairing on* (A, H) if the following compatibilities are fulfilled for any a, $b \in A$, g, $t \in H$:

(RS1) $u(ab, t) = u(a, t_{(1)})u(b, t_{(2)})$

(RS2) $u(1, h) = \varepsilon(h)$

(RS3) $u(a_{(1)}, g_{(2)}t_{(2)})p\big(a_{(2)}, f(g_{(1)}, t_{(1)})\big) = u(a_{(1)}, t)u(a_{(2)}, g)$

(RS4) $u(a, 1) = \varepsilon(a)$

Definition

Let A, H be two Hopf algebras, $f : H \otimes H \longrightarrow A$ a coalgebra map and $p : A \otimes A \longrightarrow k$ a coquasitriangular structure on A. A linear map $v : H \otimes A \longrightarrow k$ is called *(p,f) - left skew pairing on* (H, A) if the following compatibilities are fulfilled for any b, $c \in A$, h, $g \in H$:

(LS1) $p\big(f(h_{(1)}, g_{(1)}), c_{(1)}\big)v\big(h_{(2)}g_{(2)}, c_{(2)}\big) = v(h, c_{(1)})v(g, c_{(2)})$

(LS2) $v(1, a) = \varepsilon(a)$

(LS3) $v(h, bc) = v(h_{(1)}, c)v(h_{(2)}, b)$

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(LS2) $v(1, a) = \varepsilon(a)$

(LS3) $v(h, bc) = v(h_{(1)}, c)v(h_{(2)}, b)$

(LS4) $v(h, 1) = \varepsilon(h)$

Example

If $f = \varepsilon_H \otimes \varepsilon_H$ then the notions of (p,f) - right skew pairing and (p,f) - left skew pairing coincide with the notion of skew pairing on (A, H) respectively (H, A).

Example

Let $L =< t \mid t^n = 1 >$ and $G =< g \mid g^m = 1 >$ be two cyclic groups of orders n respectively m and consider the group Hopf algebras $A = k[L]$ and $H = k[G]$.

Example

Let $L =< t \mid t^n = 1 >$ and $G =< g \mid g^m = 1 >$ be two cyclic groups of orders n respectively m and consider the group Hopf algebras $A = k[L]$ and $H = k[G]$.
In this setting a coalgebra map $f : k[G] \otimes k[G] \longrightarrow k[L]$ is completely determined by a map
$\alpha : \{0, 1, ..., m-1\} \times \{0, 1, ..., m-1\} \longrightarrow \{0, 1, ..., n-1\}$ such that $f(g^i,\ g^j) = t^{\alpha(i,\ j)}$.

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The coquasitriangular structures on $k[L]$ are given by:

$$p : k[L] \otimes k[L] \rightarrow k, \qquad p(t^a,\ t^b) = \xi^{ab}$$

where $a, b \in \overline{0, n-1}$ and $\xi \in k$ such that $\xi^n = 1$.

Example

Then there exists $u : k[L] \otimes k[G] \longrightarrow k$ a (p, f) - right skew pairing if and only if α is a symmetric map and there exists $\upsilon \in k$ such that $\upsilon^n = 1$ and $\upsilon^m = \xi^{\alpha(1,\ m-1)}$. In this case the (p, f) - right skew pairing $u : k[L] \otimes k[G] \longrightarrow k$ is given by:

$$u(t^a,\ g^b) = \upsilon^{ab} \xi^{-\alpha(1,\ b-1)}, \qquad a \in \overline{0, n-1}, b \in \overline{0, m-1}.$$

Definition

Let A and H be two Hopf algebras, $p : A \otimes A \longrightarrow k$ a coquasitriangular structure on A, $u : A \otimes H \longrightarrow k$ a (p, f) - right skew pairing on (A, H) and $v : H \otimes A \longrightarrow k$ a (p, f) - left skew pairing on (H, A). A linear map $\tau : H \otimes H \longrightarrow k$ is called (u, v) - *skew coquasitriangular structure on* H if the following compatibilities are fulfilled for all h, g, $t \in H$:

(SB1) $u\big(f(h_{(1)}, g_{(1)}), t_{(1)}\big)\tau(h_{(2)}g_{(2)}, t_{(2)}) = \tau(h, t_{(1)})\tau(g, t_{(2)})$

(SB2) $\tau(1, h) = \varepsilon(h)$

(SB3) $\tau(h_{(1)}, g_{(2)}t_{(2)})v\big(h_{(2)}, f(g_{(1)}, t_{(1)})\big) = \tau(h_{(1)}, t)\tau(h_{(2)}, g)$

(SB4) $\tau(g, 1) = \varepsilon(g)$

(SB5) $\tau(h_{(1)}, g_{(1)})h_{(2)}g_{(2)} = g_{(1)}h_{(1)}\tau(h_{(2)}, g_{(2)})$

Definition

Let A and H be two Hopf algebras, $p : A \otimes A \longrightarrow k$ a coquasitriangular structure on A, $u : A \otimes H \longrightarrow k$ a (p, f) - right skew pairing on (A, H) and $v : H \otimes A \longrightarrow k$ a (p, f) - left skew pairing on (H, A). A linear map $\tau : H \otimes H \longrightarrow k$ is called (u, v) - *skew coquasitriangular structure on* H if the following compatibilities are fulfilled for all h, g, $t \in H$:

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(SB2) $\tau(1, h) = \varepsilon(h)$

(SB3) $\tau(h_{(1)}, g_{(2)}t_{(2)})v\big(h_{(2)}, f(g_{(1)}, t_{(1)})\big) = \tau(h_{(1)}, t)\tau(h_{(2)}, g)$

(SB4) $\tau(g, 1) = \varepsilon(g)$

(SB5) $\tau(h_{(1)}, g_{(1)})h_{(2)}g_{(2)} = g_{(1)}h_{(1)}\tau(h_{(2)}, g_{(2)})$

Remark

If $f = \varepsilon_H \otimes \varepsilon_H$ then the (u, v) - skew coquasitriangular structure τ is a coquasitriangular structure on H in the classical sense.

Theorem

Let $(A, H, \cdot, f)$ be a crossed system of Hopf algebras. The following are equivalent:

1) $(A \#_f^{\cdot} H, \sigma)$ *is a coquasitriangular Hopf algebra*
2) *There exists four linear maps* $p : A \otimes A \longrightarrow k$, $\tau : H \otimes H \longrightarrow k$, $u : A \otimes H \longrightarrow k$, $v : H \otimes A \longrightarrow k$ *such that* (A, p) *is a coquasitrangular Hopf algebra,* u *is a* (p, f) *- right skew pairing on* (A, H), v *is a* (p, f) *- left skew pairing on* (H, A), τ *is a* (u, v) *- skew coquasitriangular structure on* H *and the following compatibilities are fulfilled:*

Theorem

$$v(h_{(1)}, b_{(1)})(h_{(2)} \cdot b_{(2)}) \otimes h_{(3)} = b_{(1)} \otimes h_{(1)} v(h_{(2)}, b_{(2)})$$

$$(g_{(1)} \cdot a_{(1)}) \otimes g_{(2)} u(a_{(2)}, g_{(3)}) = u(a_{(1)}, g_{(1)}) a_{(2)} \otimes g_{(2)}$$

$$\tau(h_{(1)}, g_{(1)}) f(h_{(2)}, g_{(2)}) = f(g_{(1)}, h_{(1)}) \tau(h_{(2)}, g_{(2)})$$

$$u(a_{(1)}, g_{(2)}) p(a_{(2)}, g_{(1)} \cdot c) = p(a_{(1)}, c) u(a_{(2)}, g)$$

$$\tau(h_{(1)}, g_{(2)}) v(h_{(2)}, g_{(1)} \cdot c) = v(h_{(1)}, c) \tau(h_{(2)}, g)$$

$$p(h_{(1)} \cdot b, c_{(1)}) v(h_{(2)}, c_{(2)}) = v(h, c_{(1)}) p(b, c_{(2)})$$

$$u(h_{(1)} \cdot b, t_{(1)}) \tau(h_{(2)}, t_{(2)}) = \tau(h, t_{(1)}) u(b, t_{(2)})$$

Theorem

and the coquasitriangular structure $\sigma : (A \#_f^{\cdot} H) \otimes (A \#_f^{\cdot} H) \longrightarrow k$ *is given by:*

$$\sigma(a\#h, b\#g) = u(a_{(1)}, g_{(1)})p(a_{(2)}, b_{(1)})\tau(h_{(1)}, g_{(2)})v(h_{(2)}, b_{(2)})$$

for all $a, b, c \in A$ *and* $h, g, t \in H$.

Corollary

Let $(A, H, \cdot, f)$ be a crossed system of Hopf algebras with A a commutative Hopf algebra, $\cdot$ the trivial action and (H, τ) a coquasitriangular Hopf algebra such that:

$$\tau(h_{(1)}, g_{(1)})f(h_{(2)}, g_{(2)}) = f(g_{(1)}, h_{(1)})\tau(h_{(2)}, g_{(2)})$$

for all $h, g \in H$. Then $A\#_f H$ is a coquasitriangular Hopf algebra with the coquasitriangular structure given by:

$$\sigma(a\#h, b\#g) = \varepsilon(a)\varepsilon(b)\tau(h, g)$$

for all $h, g \in H$.

Corollary

Let A and H be Hopf algebras. The following are equivalent:

1) $(A \otimes H, \sigma)$ *is a coquasitriangular Hopf algebra*
2) *There exists four linear maps* $p : A \otimes A \to k$, $\tau : H \otimes H \to k$, $u : A \otimes H \to k$, $v : H \otimes A \to k$ *such that* (A, p) *and* (H, τ) *are coquasitriangular Hopf algebras, u and v are skew pairings on* $(A,\ H)$ *respectively* $(H,\ A)$ *and the following compatibilities are fulfilled:*

Corollary

$$v(h_{(1)}, b_{(1)})b_{(2)} \otimes h_{(2)} = b_{(1)}v(h_{(2)}, b_{(2)}) \otimes h_{(1)}$$

$$a_{(1)} \otimes g_{(1)}u(a_{(2)}, g_{(2)}) = a_{(2)} \otimes u(a_{(1)}, g_{(1)})g_{(2)}$$

$$u(a_{(1)}, g)p(a_{(2)}, c) = p(a_{(1)}, c)u(a_{(2)}, g)$$

$$\tau(h_{(1)}, g)v(h_{(2)}, c) = v(h_{(1)}, c)\tau(h_{(2)}, g)$$

$$p(b, c_{(1)})v(h, c_{(2)}) = v(h, c_{(1)})p(b, c_{(2)})$$

$$u(b, t_{(1)})\tau(h, t_{(2)}) = \tau(h, t_{(1)})u(b, t_{(2)})$$

In what follows k is a field such that 2 is invertible in k. Let $H = k[C_3] = k < a \mid a^3 = 1 >$ be the group Hopf algebra and $A = H_4 =< g, x | g^2 = 1,\ x^2 = 0,\ xg = -gx >$ be Sweedler's Hopf algebra.

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$$a \cdot g = a^2 \cdot g = g,\ a \cdot x = a^2 \cdot x = -x,\ a \cdot gx = a^2 \cdot gx = -gx$$

For any $\alpha \in k$, $(H_4,\ p)$ is a coquasitriangular Hopf algebra, where $p : H_4 \otimes H_4 \longrightarrow k$ is given by:

p	1	g	x	gx
1	1	1	0	0
g	1	-1	0	0
x	0	0	α	α
gx	0	0	α	α

The linear map $u : H_4 \otimes k[C_3] \to k$ defined below is a (p, f) - right skew pairing on $(H_4, k[C_3])$:

$$u(g,\ a) = u(g,\ a^2) = -1$$

$$u(x,\ a) = u(x,\ a^2) = u(gx,\ a) = u(gx,\ a^2) = 0$$

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The linear map $v : k[C_3] \otimes H_4 \longrightarrow k$ defined below is a (p, f) - left skew pairing on $(k[C_3], H_4)$:

$$v(a,\ g) = v(a^2,\ g) = -1$$

$$v(a,\ x) = v(a^2,\ x) = v(a,\ gx) = v(a^2,\ gx) = 0$$

For any $\gamma \in k$ such that $\gamma^3 = 1$, the linear map $\tau : k[C_3] \otimes k[C_3] \to k$ defined below is a (u, v) - skew coquasitriangular structure on $k[C_3]$:

τ	1	a	a^2
1	1	1	1
a	1	γ	$-\gamma^2$
a^2	1	$-\gamma^2$	$-\gamma$

Remark

The maps p, u, v and τ satisfy the seven conditions in our theorem. Thus, $\sigma : (H_4 \#^{\cdot}_f k[C_3]) \otimes (H_4 \#^{\cdot}_f k[C_3]) \rightarrow k$ is a coquasitriangular structure on the crossed product $H_4 \#^{\cdot}_f k[C_3]$, where:

$$\sigma(b \otimes y,\ c \otimes z) = u(b_{(1)}, z_{(1)}) p(b_{(2)}, c_{(1)}) \tau(y_{(1)}, z_{(2)}) v(y_{(2)}, c_{(2)})$$

is given by:

σ	$1\#1$	$1\#a$	$1\#a^2$	$g\#1$	$g\#a$	$g\#a^2$
$1\#1$	1	1	1	1	1	1
$1\#a$	1	γ	$-\gamma^2$	-1	$-\gamma$	γ^2
$1\#a^2$	1	$-\gamma^2$	$-\gamma$	-1	γ^2	γ
$g\#1$	1	-1	-1	-1	1	1
$g\#a$	1	$-\gamma$	γ^2	1	$-\gamma$	γ^2
$g\#a^2$	1	γ^2	γ	1	γ^2	γ
$x\#1$	0	0	0	0	0	0
$x\#a$	0	0	0	0	0	0
$x\#a^2$	0	0	0	0	0	0
$gx\#1$	0	0	0	0	0	0
$gx\#a$	0	0	0	0	0	0
$gx\#a^2$	0	0	0	0	0	0

σ	$x\#1$	$x\#a$	$x\#a^2$	$gx\#1$	$gx\#a$	$gx\#a^2$
$1\#1$	0	0	0	0	0	0
$1\#a$	0	0	0	0	0	0
$1\#a^2$	0	0	0	0	0	0
$g\#1$	0	0	0	0	0	0
$g\#a$	0	0	0	0	0	0
$g\#a^2$	0	0	0	0	0	0
$x\#1$	α	$-\alpha$	$-\alpha$	$-\alpha$	α	α
$x\#a$	α	$-\alpha\gamma$	$\alpha\gamma^2$	α	$-\alpha\gamma$	$\alpha\gamma^2$
$x\#a^2$	α	$\alpha\gamma^2$	$\alpha\gamma$	α	$\alpha\gamma^2$	$\alpha\gamma$
$gx\#1$	α	α	α	α	α	α
$gx\#a$	α	$\alpha\gamma$	$-\alpha\gamma^2$	$-\alpha$	$-\alpha\gamma$	$\alpha\gamma^2$
$gx\#a^2$	α	$-\alpha\gamma^2$	$-\alpha\gamma$	$-\alpha$	$\alpha\gamma^2$	$\alpha\gamma$

Thank You!