Conjugacy classes and class sums for Hopf algebras

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We extend the notion of conjugacy classes and class sums from finite groups to semisimple Hopf algebras and show that the conjugacy classes are obtained from the factorization of H as an irreducible left $D(H)$-module. For quasitriangular semisimple Hopf algebras H we prove that the product of two class sums is an integral combination of the class sums up to $1/d^2$ where $d = dim(H)$. We show also that in this case the character table is obtained from the S-matrix associated to $D(H)$.

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Along this lecture the base field k is assumed to be algebraically closed of characteristic 0. H is a semisimple algebra of dimension d. We denote by S and s the antipodes of H and H^* respectively. We denote by Λ the (2-sided) idempotent integral of H.

Let $\{V_1, \ldots, V_n\}$ be a full set of non-isomorphic irreducible left H-modules of respective dimension d_j and corresponding characters χ_j. We have:

$$V_i \otimes V_j = \sum_{i,j}^{l} m_{ij}^l V_l,$$

where m_{ij}^l are non-negative integers.

The character algebra $R(H)$ of H is the k-span of all the characters on H. In fact $\{\chi_1, \ldots \chi_n\}$ form a basis for $R(H)$. By Kac (1972) and Zhu (1994) -
$R(H)$ **is a semisimple algebra with involution**.

By Larson (1971), the bilinear form defined on the ring of characters by

$(\chi_i, \chi_j) = \dim_k \hom_H(V_i, V_j) = m_{ij}^1 =< \chi_i s(\chi_j), \Lambda >$ satisfies -

The irreducible characters are orthogonal with respect to this form.

This will be applied in two directions:

(i) $R(H)$ is a symmetric algebra with a symmetric form β defined by

$$\beta(p, q) =< \Lambda, pq >$$

and a Casimir element

$$\sum_{k=1}^{n} \chi_k \otimes s(\chi_k)$$

(ii) Define an inner product on $R(H)$ as follows: For $u = \sum \alpha_i \chi_i$, $v = \sum \beta_j \chi_j$, set

$$(u, v) = \sum \alpha_i \overline{\beta}_i$$

Nichols and Richmond (1998) discussed various properties of that inner product. If we define an involution $*$ by

$$\chi^* = s(\chi)$$

and extend it to $u = \sum \alpha_i \chi_i \in R(H)$ by

$$u^* = \sum \overline{\alpha}_i \chi_i^*$$

Then

$$(uv, w) = (v, u^* w) = (u, wv^*)$$

Let $\{F_1, \ldots F_m\}$ be the set of central primitive idempotents of $R(H)$. Then [NR] proved the following:

$$F_i^* = F_i$$

and for all $x \in R(H)$, μ a character defined on $R(H)$,

$$\mu(x^*) = \overline{\mu(x)}$$

These results will be used later to prove that certain matrices are unitary.

For $1 \leq i \leq m$, define the **Class sum**

$$C_i = dF_i \rightharpoonup \Lambda.$$

Claim: The irreducible character μ_i of $R(H)$ corresponding to F_i can be identified inside $Z(H)$ by

$$\mu_i = q_i C_i,\; q_i \in \mathbb{Q}$$

Proof For $x \in R(H)$,

$$\mu_i(x) = Trace(L_{xF_i})$$

Let $\{f_1, \dots f_m\}$ be a complete set of primitive orthogonal idempotents in $R(H)$ so that $f_i F_j = \delta_{ij} f_i$.

The Casimir element of $R(H)$ satisfies

$$\sum \chi_i \otimes \chi_i^* = \sum_{k=1}^{n} \chi_k \otimes s(\chi_k) = \sum_j n_j F_j$$

where

$$n_j = \frac{d \dim(C(H) f_j)}{\dim(H^* f_j)}$$

The result follows now from the trace formula for symmetric algebras. The coefficient q_i is given by:

$$q_i = \frac{\dim(C(H)f_i)}{\dim(H^* f_i)}$$

As a result we obtain that:

$$< \chi_{i^*}, C_j > = \overline{< \chi_i, C_j >}$$

Recall the left adjoint action of H on itself,

$$h \dot{_{ad}} x = \sum h_1 x S(h_2)$$

Then

$$\Lambda \dot{_{ad}} H = Z(H).$$

(When H is not semisimple then $\Lambda \dot{_{ad}} H$ is a proper ideal of $Z(H)$ - the Higman ideal.

We have also left coadjoint action of H on H^*, $\triangleright$ given by:

$$h \triangleright p = \sum h_2 \rightharpoonup p \leftharpoonup Sh_1$$

When H is semisimple then

$$\Lambda \triangleright H^* = R(H).$$

Recall the Frobenius map $\Psi : H \to H^*$ defined as:

$$\Psi(h) = \lambda \leftharpoonup S(h).$$

We show that Ψ an H-module map from $(H, \dot{_{ad}})$ to $(H^*, \triangleright)$ in the sense that

$$\Psi(h \dot{_{ad}} a) = h \triangleright \Psi(a).$$

Note, $\dot{_{ad}}$ makes H into an H-module algebra, while $\triangleright$ does not make H^* into an H-module algebra. However, it has some nice properties.
(i) For all $h \in H$, $p \in R(H)$, $x \in H^*$,

$$h \triangleright (px) = p(h \triangleright x)$$

If moreover $h \in \text{Coc}(H)$, then

$$h \triangleright (xp) = (h \triangleright x)p.$$

Define the **conjugacy class** $\mathcal{C}_i$ as:

$$\mathcal{C}_i = \Lambda \leftharpoonup f_i H^*.$$

Then by the properties of Ψ mentioned above it is not hard to see that:
$\mathcal{C}_i$ **is stable under the adjoint action of** H.

By definition, $\mathcal{C}_i$ is stable under the right *hit* action of H^* on H.

It is known that H is a left module over $D(H)$ where the H^* part acts by right *hit* and the H part acts by the left adjoint action. We can show that:

$\mathcal{C}_i$ **is a** $D(H)$**-submodule of** H.

But more is true,

Theorem[CW]: Let H be a semisimple Hopf algebra and let $\{f_1, \dots f_m\}$ be idempotents in $R(H)$ so that $\{f_i R(H)\}$ is the complete set of non-isomorphic irreducible $R(H)$-modules. Assume $\dim(f_i R(H)) = m_i$. Then $\mathcal{C}_i$ is an irreducible $D(H)$-module and moreover,

$$H \cong \oplus_{i=1}^{n} \mathcal{C}_i^{\oplus m_i}$$

as $D(H)$-modules.

The proof is based on the properties of the Frobenius maps Ψ.

When $R(H)$ is commutative the central primitive idempotents $\{F_j\}$ form a basis of $R(H)$, hence the class sums $\{C_j\}$ where $C_j = \Lambda \leftharpoonup dF_j$, form a basis for $Z(H)$.

One can check that

$$< F_j, \Lambda >= \frac{\dim(F_j H^*)}{d},$$

hence by definition

$$\{F_i\} \text{ and } \{\frac{C_j}{\dim(F_j H^*)}\} \text{ are dual bases.}$$

We can define a character table for H as follows:

$$\xi_{ij} = \frac{1}{\dim(F_j H^*)} < \chi_i, C_j >$$

The dual bases imply that the character table is actually the change of bases matrix A from $\{\chi_i\}$ to $\{F_j\}$.

Recall that for groups the character table is defined by $\xi_{ij} = \chi_i(g)$ for some g in the conjugacy class $\mathcal{C}_j$. Hence $\chi_i(g) = \chi_i(\frac{C_j}{|\mathcal{C}_j|})$. Thus the definition extends the definition of character tables for groups.

In [CW,3.1] we proved that the inverse change of bases matrix $(\beta_{jk}) = A^{-1}$ satisfies

$$\beta_{jk} = \frac{\text{dim}(F_j H^*)}{d} \alpha_{k^* j}$$

By using this we can show first and second orthogonality relations (as for groups). That is,

$$(a) \qquad \sum_j \text{dim}(F_j H^*) \xi_{mj} \xi_{nj^*} = \delta_{mn} d.$$

$$(b) \qquad \sum_m \xi_{mi} \xi_{mj^*} = \delta_{ij} \frac{d}{\text{dim}(F_i H^*)}$$

By [NR], $\xi_{ij^*} = \overline{\xi_{ij}}$, thus the character table is "almost" unitary.

When H is a factorizable Hopf algebra we have the Drinfeld map $f_Q : H^* \to H$, which is an algebra isomorphism between $R(H)$ and $Z(H)$. In particular, for any primitive idempotent F of $R(H)$, $f_Q(F) = E$ is a primitive central idempotents of H.

Reorder the set $\{F_j\}$ so that for all $1 \leq j \leq m$,

$$f_Q(F_j) = E_j$$

Recall [CW] that for semisimple factorizable Hopf algebra we have:

$$f_Q(\chi_j) = \frac{1}{d_j} C_j.$$

It follows that the S-matrix satisfies

$$s_{ij} = < \chi_i, f_Q(\chi_j) > = \frac{1}{d_j} < \chi_i, C_j > = \frac{\dim(F_j H^*)}{d_j} \xi_{ij}$$

Since $\dim(F_j H^*) = \dim(E_j H) = d_j^2$, we obtain

$$s_{ij} = d_j \xi_{ij}$$

Hence

$$s_{i^*j} = \overline{s_{ij}}$$

Thus we obtain the result of [ENO,2005]

For a factorizable semisimple Hopf algebra, the S-matrix (multiplied by $\frac{1}{\sqrt{d}}$) is unitary.

Unlike for groups, the structure constants for the product of two class sums are not necessarily integers. We can prove integrability up to d^2 in case H is quasitriangular. In this case, H is a Hopf image of $D(H)$ which is a factorizable Hopf algebra. Denote this map by Φ.

The images of the F_i's under $\Phi^* : H^* \to D(H)^*$, are sums of primitive idempotents in $R(D(H))$, and thus induce a partition $\{I_s\}$ on their indexes. All class sums of $D(H)$ belonging to the same I_s are mapped under Φ to the corresponding class sum of H with a certain coefficient.

On the other hand, if $\{\widehat{E}_i\}_{i=1}^m$ is the set of central primitive idempotents of $D(H)$ then

$$\Phi(\widehat{E}_i) = \begin{cases} E^i & 1 \leq i \leq n, \\ 0 & n+1 \leq i \leq m \end{cases}$$

We use these and the fact that $D(H)$ is factorizable to prove:

Let H be a quasitriangular Hopf algebra. Then the product of two class sums is an integral combination up to a factor of d^{-2} of the class sums of H.

The character table of a quasitriangular Hopf algebra H is strongly related to the S-matrix of $D(H)$. We show:

If (H, R) is quasitriangular and (ξ_{sj}) is its character table, then $\xi_{sj} = d_i^{-1} s_{ij}$ for all $i \in I_s$, where s_{it} arise from the $\mathbf{S}$-matrix of $D(H)$.

The factor d^{-2} can not be avoided as will be demonstrated in the next example - the character table of $D(kS_3)$.

Conjugacy classes of S_3 are given by:

$$C_1 = \{1\} \; C_{(12)} = \{(12), (13), (23)\}$$

$$C_{(123)} = \{(123), (132)\}$$

The centralizers are given by:

$$C_G(1) = S_3 \quad C_G(12) = \{1, (12)\} \cong \mathbb{Z}_2$$

$$C_G(123) = \{1, (123), (132)\} \cong \mathbb{Z}_3$$

For $\sigma = 1$ we have 3 irreducible representations of S_3.

- M_1 is the trivial representation of S_3.

- M_2 is the sign representation of S_3.

- M_3 is the 2 irreducible dimension of S_3 with $\chi_{M_3}(123) = -1$, $\chi_{M_3}(12) = 0$.

For $\sigma = (12)$ we have two representations:

- M_4 is the trivial representation of $\mathbb{Z}_2$.
- M_5 the unique non-trivial representation of $\mathbb{Z}_2$.

For $\sigma = (123)$ we have 3 representations:

- M_6 is the trivial representation of $\mathbb{Z}_3$.
- M_7 is the representation with $\chi_{M_7}(123) = \omega$, $\chi_{M_7}(132) = \omega^2$, ω a third root of unity.
- M_8 is the representation with $\chi_{M_8}(123) = \omega^2$, $\chi_{M_8}(132) = \omega$.

We can compute now the $\mathbf{S}$-matrix which is actually well known known (e.g [BK]) Finally, Denote $\frac{1}{d_i}C_i$ by η_i. Then the generalized character table of $D(kS_3)$ is given by

$$\begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \\ \chi_6 \\ \chi_7 \\ \chi_8 \end{array}\begin{array}{c} \begin{array}{cccccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 & \eta_7 & \eta_8 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 & 0 & -1 & -1 & -1 \\ 3 & -3 & 0 & 1 & -1 & 0 & 0 & 0 \\ 3 & -3 & 0 & -1 & 1 & 0 & 0 & 0 \\ 2 & 2 & -1 & 0 & 0 & 2 & -1 & -1 \\ 2 & 2 & -1 & 0 & 0 & -1 & -1 & 2 \\ 2 & 2 & -1 & 0 & 0 & -1 & 2 & -1 \end{pmatrix} \end{array}$$

We can check that:

$$\chi_4\chi_5 = \chi_2 + \chi_3 + \chi_6 + \chi_7 + \chi_8$$

Hence

$$C_4C_5 = 9f_Q(\chi_4\chi_5) = 9C_2 + \frac{9}{2}C_3 + \dots$$