Semisimple Hopf algebras

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Suppose that H is a finite dimensional semisimple Hopf algebra over an algebraically closed field whose characteristic does not divide the dimension of H. We shall assume that for any positive integer $d > 1$ any two irreducible H-modules of dimension d are isomorphic. The category of left H-modules ${}_H\mathcal{M}$ is a monoidal category. In the talk we shall discuss Clebsch-Gordan coefficients in decompositions in ${}_H\mathcal{M}$ of tensor products of irreducible H-modules. Some classifications results are obtained in the case when there exists up to an isomorphism a unique irreducible H-module of dimension greater than 1.

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In the talk we consider a problem of a classification up to an isomorphism of semisimple finite dimensional Hopf algebras H over an algebraically closed field k. We shall assume that either $\operatorname{char} k = 0$ or $\operatorname{char} k > \dim H$.

Dual Hopf algebras

If H has finite dimension then the dual space H^* is again a Hopf algebra with *convolutive* multiplication $l_1 * l_2$, comultiplication Δ^*, counit ε^* and an antipode S^* which are defined as follows:

$$l_1 * l_2 = \mu \cdot (l_1 \otimes l_2) \cdot \Delta, \quad \Delta^*(l)(x \otimes y) = l(xy),$$
$$(S^* l)(x) = l(S(x)), \quad \varepsilon^*(l) = l(1)$$

for all $x, y \in H$.

Group-like elements

An element $g \in H$ is a *group-like element* if $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$. The set $G(H)$ of all group-like elements is a multiplicative group in H.
Elements of $G(H^*)$ of group-like elements in the dual Hopf algebra H^* are just algebra homomorphisms $H \to k$.

There are left and right actions $H^* \rightharpoonup H,\ H \leftharpoonup H^*$ of H^* on H defined as follows: if $f \in H^*,\ x \in H$ and

$$\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)} \in H \otimes H$$

then

$$f \rightharpoonup x = \sum_x x_{(1)} \langle f, x_{(2)} \rangle, \quad x \leftharpoonup f = \sum_x \langle f, x_{(1)} \rangle x_{(2)}$$

In particular if $g \in G(H^*)$ then $g \rightharpoonup x,\ x \leftharpoonup g$ are algebra automorphisms of H.

Direct decomposition of H

In the talk we shall assumed that for any $d > 1$ there exists at most one irreducible H-module of dimension d. It means that H as a semisimple k-algebra has a decomposition

$$H = (\oplus_{g \in G} k e_g) \oplus \mathrm{Mat}(d_1, k) \oplus \cdots \oplus \mathrm{Mat}(d_n, k), \qquad (1)$$

$$1 < d_1 < \cdots < d_n,$$

where $\{e_g \mid g \in G\}$ is a system of central orthogonal idempotents associated with k-algebra homomorphisms $g : H \to k$.

Irreducible H-modules

Let $E_g,\ g \in G$, be the one-dimensional H-module associated with $g \in G$. It means that $hx = \langle h, g\rangle x$ for any $h \in H$.
The number of 1-dimensional non-isomorphic H-modules $E_g,\ g \in G$, is equal to the order of G.
Denote by $M_1, \ldots, M_n$ irreducible H-modules of dimensions $1 < d_1 <, \cdots < d_n$, respectively.
It can be shown that each module M_i is equipped with a non-degenerated (skew-)symmetric bilinear form $\langle x, y\rangle_i$ such that $\langle hx, y\rangle_i = \langle x, S(h)y\rangle_i$ for all $x, y \in M_i$ and for all $h \in H$.

Each matrix component $\mathrm{Mat}(d_i, k)$ in H from (1) is invariant under the antipode S. Let U_i be the Gram matrix of the bilinear form $\langle x, y \rangle_i$ in some base of M_i.

Proposition

$S(x) = U_i {}^t x U_i^{-1}$ *for any* $x \in \mathrm{Mat}(d_i, k)$*.*

Proposition

For any i there exists a faithful projective representation Φ_i of the group G in M_i such that

$$g \rightharpoonup h = \Phi_i(g) h \Phi_i(g)^{-1}, \quad h \leftharpoonup g = S\left(\Phi_i(g)\right) h S\left(\Phi_i(g)\right)^{-1}$$

for any $h \in \mathrm{Mat}(d_i, k)$. Moreover the group commutator $[\Phi_i(g), S\left(\Phi_i(f)\right)] = 1$ in $\mathrm{PGL}(M_i)$ for all $f, g \in G$.

Proposition

If $g \in G$ then there are H-module isomorphisms

$$E_g \otimes M_i \simeq M_i \otimes E_g \simeq M_i,$$
$$M_i \otimes M_j \simeq \delta_{ij} \left(\oplus_{g \in G} E_g\right) \oplus \left(\oplus_{t=1}^{n} m_{ij}^t M_t\right),$$

where $m_{ij}^t \geqslant 0 \in \mathbb{Z}$. In particular

$$d_i d_j = \delta_{ij}|G| + \sum_t m_{ij}^t d_t, \quad |G| \leqslant d_1^2, \quad m_{ij}^s = m_{js}^i = m_{ji}^s.$$

We can identify the space $M_i \otimes M_i$ with the space of matrices $\mathrm{Mat}(d_i, k)$ using the bilinear form $\langle x, y \rangle_i$. Namely if $a, b, c \in M_i$ then $a \otimes b$ is the linear operator on M_i such that

$$(a \otimes b)c = a\langle b, c \rangle_i \in M_i.$$

Proposition

Under this identification the image of the one-dimensional module E_g in $M_i \otimes M_i$ coincides with the linear span of ${}^tS\left({}^t\Phi_i(g)^{-1}\right)$.
Choosing a special base in M_i we can show that the span is equal to $S\left(\Phi_i(g)^{-1}\right)$.

We can associate with H an oriented graph Γ_H whose vertices are indices $\{1, \ldots, n\}$ of irreducible H-modules $M_1, \ldots, M_n$. Two vertrices i, j are connected by an edge $i \to j$ if $m^i_{tj} > 0$ for some $t = 1, \ldots, n$. In other terms the module M_i occurs in $M_t \otimes M_j$ for some index t.

Proposition

Suppose that there is no edge $i \to j$ in Γ_H. Then $i = j = 1$ and $|G| = d_1^2$. Moreover $J = \oplus_{j \geqslant 2} \mathrm{Mat}(d_j, k)$ is a Hopf ideal in H and H/J is the Hopf algebra from Theorems 7 and 8.

Theorem (V.A. Artamonov, R.B. Mukhatov, R. Wisbauer)

Suppose that there exists an index $1 \leqslant i \leqslant n$ such that for any index $j \neq i$ there exists a unique edge $i \to j$. If $i = 1$, then $J = \oplus_{j \geqslant 2} \mathrm{Mat}(d_j, k)$ is a Hopf ideal in H and H/J is the Hopf algebra from Theorems 7 and 8. If $i = n$, then $n = 1$.

Theorem

Let H be a semisimple bialgebra with decomposition (1) *where $n \geqslant 2$. Then $m^t_{n-1,n} \geqslant 2$ for some index $t = 1, \ldots, n$.*

The antipode S

Each matrix constituent $\mathrm{Mat}(d_q, k)$ in (1) is stable under the antipode S. Moreover $S^2 = 1$ and $S(e_g) = e_{g^{-1}}$ for any central idempotent e_g from (1).

Theorem

If the group G is nilpotent then taking an isomorphic copy of each matrix component in (1) *we can assume that the matrices* $\Phi_i(g)$, $S\left(\Phi_i(g)\right)$ *are monomial.*

Theorem

Let H be a semisimple Hopf algebra with semisimple decomposition (1)*.*
Suppose that there exists a matrix constituent $\mathrm{Mat}(d_i, k)$ *which is a Hopf ideal in H. Then $n = 1$.*

Elements $\mathcal{R}_q$

Denote by $\mathcal{R}_q$ the element

$$\mathcal{R}_q = \frac{1}{d_q} \sum_{i,j=1}^{d_q} E_{ij}^{(q)} \otimes E_{ji}^{(q)}$$

in $\mathrm{Mat}(d_q, k)^{\otimes 2}$. Here $E_{**}^{(q)}$ are matrix units from $\mathrm{Mat}(d_q, k)$. The element $\mathcal{R}_q$ is the unique element in $\mathrm{Mat}(d_q, k)^{\otimes 2}$ up scalar multiple such that

$$(A \otimes B)\mathcal{R}_q = \mathcal{R}_q(B \otimes A)$$

for all $A, B \in \mathrm{Mat}(d_q, k)$.

Theorem

Let G be a finite group whose order is coprime with $\operatorname{char} k$*. A projective representation* $\Omega : G \to \mathrm{PGL}(d, k)$ *such that*

$$\Omega(g^{-1}) = \Omega(g)^{-1}, \quad \Omega(E) = E,$$

is irreducible if and only if

$$\mathcal{R}_d = \frac{1}{|G|} \sum_{g \in G} \Omega(g^{-1}) \otimes \Omega(g).$$

Theorem

Let $g \in G$ *and* $x \in \mathrm{Mat}(d_r, k)$. *Put* $\Delta_q = (1 \otimes S)\mathcal{R}_q$. *Then* $\varepsilon(e_g) = \delta_{1,g}$, $\varepsilon(x) = 0$ *and*

$$\Delta(e_g) = \sum_{f \in G} e_f \otimes e_{f^{-1}g} + \sum_{t=1,\ldots,n} (1 \otimes (g \rightharpoonup\ ))\, \Delta_t,$$

$$\Delta(x) = \sum_{g \in G} \left[(g \rightharpoonup x) \otimes e_g + e_g \otimes (x \leftharpoonup g) \right] + \sum_{i,j=1}^{n} \Delta^r_{ij}(x),$$

where $\Delta^r_{ij}(x) \in \mathrm{Mat}(d_i, k) \otimes \mathrm{Mat}(d_j, k)$.

Proposition

For indices i, j the following are equivalent:

1. *there exists an edge $i \to j$;*
2. $\Delta^i_{tj} \neq 0$ *for some* t;
3. $\Delta^t_{ij} \neq 0$ *for some* t.

Hopf algebras with $n = 1$ were considered by several authors. If the order of G has maximal possible value d_1^2 then the group G is Abelian. In the paper

- *Tambara D., Yamagami S., J.Algebra 209 (1998), 692-707, Corollary 3.3.*

Hopf algebra H is classified using monoidal category of its representations in terms of bicharacters of the group G.

If $d_1 = 2$ then there exist up to equivalence four classes of Hopf algebras H, namely group algebras of Abelian groups of order 8, group algebras of dihedral group D_4 and of quaternions Q_8, and G. Kac Hopf algebra H generated by elements x, y, z with defining relations

$$x^2 = y^2 = 1,\ xy = yx,\ zx = yz, zy = xz,$$

$$z^2 = \frac{1}{2}(1 + x + y - xy),$$

$$\varepsilon(z) = 1,\ S(z) = z^{-1},$$

$$\Delta(z) = \frac{1}{2}\left((1 + y) \otimes 1 + (1 - y) \otimes x\right)(z \otimes z),$$

and x, y are group-like elements.

Interesting results were obtained by

- *Masuoka A., Some further classification results on semisimple Hopf algebras, Commun. Algebra, 24(1996),307-329*

Let H be a semisimple Hopf algebra of dimension $2p^2$, where p is an odd integer. Then either H has a semisimple decomposition (1) with $n = 1,\ d_1 = p$ and $|G| = p^2$ or H is its dual and it has a semisimple decomposition with $2p$ one-dimensional components and $\frac{p(p-1)}{2}$ components isomorphic to $\mathrm{Mat}(2, k)$.

Theorem (Artamonov V.A., 2009 — 2010)

Let H be from (1) with $n = 1$ and $G = G(H^*)$. The order of G is divisible by d_1 and is a divisor of d_1^2.
The following conditions are equivalent.

1. *The order of G is equal to d_1^2.*
2. $\Delta^1_{11} = 0$ *in Theorem 6.*
3. Φ_1 *is an irreducible projective representation of G in M_1.*

Under these restrictions $H = \left(\oplus_{g\in G} k e_g\right) \oplus \mathrm{Mat}(d_1, k)$ and $\varepsilon(e_g) = \delta_{1,g},\ \varepsilon(x) = 0$ where $x \in \mathrm{Mat}(d_1, k)$. Moreover

$$\Delta(e_g) = \sum_{f\in G} e_f \otimes e_{f^{-1}g} + \frac{1}{d_1}\sum_{i,j=1}^{d_1} E_{ij} \otimes \left(g^{-1} \rightharpoonup S\left(E_{ji}\right)\right),$$

$$\Delta(x) = \sum_{g\in G}\Big[\left(\Phi_1(g)x\Phi_1(g)^{-1}\right) \otimes e_g$$

$$+ e_g \otimes \left(S\left(\Phi_i(g)\right) x S\left(\Phi_i(g)^{-1}\right)\right)\Big].$$

Theorem (Artamonov V.A., I.A. Chubarov, R. Mukhatov, 2007-2009)

Let H be from (1), $n = 1$ and $G = G(H^*)$. If $\Delta^1_{11} = 0$ then $G = A \times A$ for some Abelian group A of order d_1.

Theorem (Artamonov V.A., I.A. Chubarov, R. Mukhatov, 2007-2009)

Suppose that G is Abelian group of order d^2 with direct decomposition $G \simeq A \times A$ for some Abelian group A of order d. The group G has a faithful irreducible projective representation Φ of degree d. There exists a (skew-)symmetric matrix $U \in \mathrm{GL}(d, k)$ such that $[\Phi(g), S(\Phi(f))] = 1$ in $\mathrm{PGL}(d, k)$ for all $f, g \in G$. Here $S(x) = U{}^t x U^{-1}$ for any $x \in \mathrm{Mat}(d, k)$. Then an algebra H with direct decomposition (1) admits Hopf algebra structure defined in Theorem 6.
There is a group isomorphism $G \simeq G(H^*)$.

Theorem (Puninsky E., 2009)

Under the assumption of Theorem 8 $G(H)$ is a cyclic group of order $2d_1$, provided d_1 is an odd prime.

Theorem (Artamonov V.A., Chubarov I.A., 2008)

Let $n = 1,\ d_1 > 2$ and H from Theorem 8. Then H^* is not isomorphic to any Hopf algebra belonging to the class of Hopf algebras from Theorem 8.

Previous results use

Theorem (R.Frucht, J. Reine Angew. Math. 166(1932), 16-29)

Let G be a finite Abelian group of and let k be an algebraically closed field such that $\operatorname{char} k$ does not divide the order of G. The group G admits a faithful irreducible projective representations of dimension d over k if and only G is a direct product of two isomorphic groups of order d. Dimensions of any irreducible projective representations of the group G are equal either to d or to 1.

Theorem (E. M. Jmud, 1972)

A finite abelian group G of order d^2 has decomposition $G \simeq A \times A$ if and only if it admits a non-degenerate bilinear symmetric form. Any irreducible projective representation of G of degree d is obtained from another one by an automorphism of G.