Unified products for Hopf algebras

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Let A be a Hopf algebra and H a coalgebra. We shall describe and classify up to an isomorphism all Hopf algebras E that factorize through A and H: that is E is a Hopf algebra such that A is a Hopf subalgebra of E, H is a subcoalgebra in E with $1_E \in H$ and the multiplication map $A \otimes H \to E$ is bijective. The tool we use is a new product, we call it the unified product, in the construction of which A and H are connected by three coalgebra maps: two actions and a generalized cocycle. Both the crossed product of a Hopf algebra acting on an algebra and the bicrossed product of two Hopf algebras are special cases of the unified product. A Hopf algebra E factorizes through A and H if and only if E is isomorphic to a unified product of A and H. All such Hopf algebras E are classified up to an isomorphism that stabilizes A and H by a Schreier type classification theorem. An equivalent description of the unified product from the extension of Hopf algebras point of view is given. A necessary and sufficient condition for the canonical morphism $i : A \to A \ltimes H$ to be a split monomorphism of Hopf algebras is proved, i.e. a conditions for the unified product $A \ltimes H$ to be isomorphic to a Radford biproduct $L * A$, for some bialgebra L in the category ${}_A^A\mathcal{Y}D$ of Yetter-Drinfel'd modules.

Joint work with A. L. Agore.

Bibliography

[1] A.L. Agore and G. Militaru, *Extending structures II: the quantum version.* To appear in J. Algebra, arXiv:1011.2174.

[2] ——, *Unified products and split extensions of Hopf algebras.* Preprint 2011, arXiv:1105.1474.

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Based on a joint work with Ana Agore

AM 1 A. Agore, G. M. - *Extending structures II: the quantum version*, J. Algebra, 336(2011), 321–341

AM 2 – —— – *Unified products and split extensions of Hopf algebras*, arXiv:1105.1474v1

- **The starting point**:

An elementary question: Let H be a group, E a set s.t. $H \subseteq E$.

Describe the set of all the group structures $(E, \cdot)$ that can be defined on the set E such that $H \leq (E, \cdot)$.

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- **The general context**:

$\mathcal{C}$ = a category whose objects are endowed with various algebraic structures (S).

$\mathcal{D}$ = a category such that there exists a **forgetful functor** $F : \mathcal{C} \to \mathcal{D}$.

Examples:

$$F : \mathcal{G}r \to \mathcal{S}et, \quad F : \mathcal{L}ie \to \mathcal{V}ec, \quad F : \mathcal{H}opf \to \mathcal{C}oAlg$$

$$F : \mathcal{H}opf \to \mathcal{A}lg, \quad F : \mathcal{A}lg \to \mathcal{V}ec, \cdots$$

- **Extending structures (ES) problem:**

Let $C \in \mathcal{C}$, $D \in \mathcal{D}$ be two objects such that $F(C)$ is a subobject of D in $\mathcal{D}$. Describe and classify all mathematical structures (S) that can be defined on D such that D becomes an object of $\mathcal{C}$ and C is a subobject of D in the category $\mathcal{C}$.

The classification – up to an isomorphism that stabilizes C and a certain type of '*fixed quotient*' D/C.

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- **The group case**

Example

Consider the forgetful functor $F : \mathcal{G}r \to \mathcal{S}et$.

(G-S) ES- problem: H = a group, E = a set s.t. $H \subseteq E$ (and $|H| \mid |E|$).
Describe and classify all the group structures $(E, \cdot)$ that can be defined on the set E such that H is a **subgroup** of $(E, \cdot)$.

A. Agore, G.M. - *Extending Structures I: the group case*, arXiv:1011.1633

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Remark

The ES-problem generalizes and unifies the extension problem of Hölder (1895) and the factorization problem of Ore (1937).

Let H be a group, E be a set such that $H \subseteq E$. Then:

- Any group structures '$\cdot$' that can be defined on the set E such that $H \leq E$ is isomorphic to a **unified product.**
- Both the **crossed product** of and the **bicrossed product** of two groups are special cases of the unified product.

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• The unified product $H \ltimes S$: the group level

Let H be a group, $(S, 1_S)$ a pointed set and four maps

$$\star : S \times S \to S, \quad f : S \times S \to H$$

$$\triangleright : S \times H \to H, \quad \triangleleft : S \times H \to S$$

satisfying axioms such that $H \ltimes S := H \times S$ with the multiplication

$$(h_1, s_1) \cdot (h_2, s_2) := \big(h_1 (s_1 \triangleright h_2) f(s_1 \triangleleft h_2, s_2), (s_1 \triangleleft h_2) \star s_2\big)$$

is a group on $H \ltimes S$ with $(1_H, 1_S)$ as a unit.

Remark

Let $H \leq E$ be a subgroup of E. Then there exists a map $\pi : E \to H$ such that

$$\pi(h) = h, \quad \pi(hx) = h\pi(x)$$

for all $h \in H$ and $x \in E$.

Remark

We define:

$$S := \{x \in E \mid \pi(x) = 1_H\}$$

and the well defined maps $\star$, f, $\triangleright$, $\triangleleft$ *given by:*

$$s_1 * s_2 := \pi(s_1 s_2)^{-1} s_1 s_2, \quad f(s_1, s_2) := \pi(s_1 s_2)$$

$$s \triangleright h := \pi(sh), \quad s \triangleleft h := \pi\big(sh\big)^{-1} sh$$

for all s, s_1, $s_2 \in S$ *and* $h \in H$. *Then*

$$\varphi : H \ltimes S \to E, \quad \varphi(h, s) := hs$$

for all $h \in H$ *and* $s \in S$ *is an* ***isomorphism*** *of groups with* $\varphi^{-1}(x) = \big(\pi(x), \pi(x)^{-1} x\big)$.

- **The Hopf algebra case**

Example

Consider the forgetful functor $F : \mathcal{H}opf \to \mathcal{C}oAlg$.

(H-C) ES-problem: Let A be a Hopf algebra and E a coalgebra such that A is a subcoalgebra of E.
Describe and classify all Hopf algebra structures that can be defined on the coalgebra E such that A is a Hopf subalgebra of E.

There is of course a dual version of the ES-problem corresponding to the forgetful functor $F : \mathcal{H}opf \to \mathcal{A}lg$.

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Remark

k = a field, A = a group, E = a set s.t. $A \subseteq E$ and the extension:

$$k[A] \subseteq k[E]$$

where $k[A]$ is the group algebra that is a Hopf algebra and a subcoalgebra in the group-like coalgebra $k[E]$.

Let $(E, \cdot)$ = a group structure on the set E such that A is a subgroup of $(E, \cdot)$.

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Hence we obtain an extension of Hopf algebras $k[A] \subseteq k[E]$ that has a remarkable property:

Let $H \subseteq E$ be a system of representatives for the right cosets of the subgroup A in the group $(E, \cdot)$ such that $1_E \in H$.

Then the multiplication map

$$k[A] \otimes k[H] \to k[E], \quad a \otimes h \mapsto a \cdot h$$

is bijective, i.e. the Hopf algebra $k[E]$ **factorizes** through the Hopf subalgebra $k[A]$ and the subcoalgebra $k[H]$.

We have to restrict the (H-C) extending structures problem as follows:

The restricted (H-C) ES-problem:
Let A be a Hopf algebra and H a coalgebra. Describe and classify up to an isomorphism all Hopf algebras E that factorize through A and H: that is E is a Hopf algebra such that A is a Hopf subalgebra of E, H is a subcoalgebra in E with $1_E \in H$ and the multiplication map $A \otimes H \to E$ is bijective.

We shall give a complete answer below.

Definition

Let A be a bialgebra. An **extending datum** of A is a system $\Omega(A) = \big(H, \triangleleft, \triangleright, f\big)$ where: $\big(H, \Delta_H, \varepsilon_H\big)$ is a coalgebra, $\big(H, 1_H, \cdot\big)$ is an unitary, not necessarily associative k-algebra, $\triangleleft : H \otimes A \to H$, $\triangleright : H \otimes A \to A$, $f : H \otimes H \to A$ are morphisms of coalgebras s.t.

$$\Delta_H(1_H) = 1_H \otimes 1_H, \quad h \triangleright 1_A = \varepsilon_H(h) 1_A$$

$$1_H \triangleright a = a, \quad 1_H \triangleleft a = \varepsilon_A(a) 1_H, \quad h \triangleleft 1_A = h$$

$$f(h, 1_H) = f(1_H, h) = \varepsilon_H(h) 1_A$$

Let $\Omega(A) = \big(H, \triangleleft, \triangleright, f\big)$ be an extending datum of A and $A \ltimes_{\Omega(A)} H = A \ltimes H := A \otimes H$ with the multiplication:

$$(a \ltimes h) \bullet (c \ltimes g) := a(h_{(1)} \triangleright c_{(1)}) f\big(h_{(2)} \triangleleft c_{(2)},\, g_{(1)}\big) \ltimes \big(h_{(3)} \triangleleft c_{(3)}\big) \cdot g_{(2)}$$

Definition

$A \ltimes H$ is called the **unified product** of A and $\Omega(A)$ if $A \ltimes H$ is a bialgebra with the unit $1_A \ltimes 1_H$ and the coalgebra structure given by the tensor product of coalgebras. In this case $\Omega(A) = (H, \triangleleft, \triangleright, f)$ is called a **bialgebra extending structure** of A. A bialgebra extending structure $\Omega(A) = (H, \triangleleft, \triangleright, f)$ is called a **Hopf algebra extending structure** of A if $A \ltimes H$ has an antipode.

Theorem

*$A \ltimes H$ is an unified product **if and only if** $\Delta_H : H \to H \otimes H$, $\varepsilon_H : H \to k$ are k-algebra maps, $(H, \triangleleft)$ is a right A-module coalgebra and the following compatibilities hold:*

(BE1) $(g \cdot h) \cdot l = \big(g \triangleleft f(h_{(1)}, l_{(1)})\big) \cdot (h_{(2)} \cdot l_{(2)})$

(BE2) $g \triangleright (ab) = (g_{(1)} \triangleright a_{(1)})[(g_{(2)} \triangleleft a_{(2)}) \triangleright b]$

(BE3) $(g \cdot h) \triangleleft a = [g \triangleleft (h_{(1)} \triangleright a_{(1)})] \cdot (h_{(2)} \triangleleft a_{(2)})$

(BE4) $[g_{(1)} \triangleright (h_{(1)} \triangleright a_{(1)})] f\Big(g_{(2)} \triangleleft (h_{(2)} \triangleright a_{(2)}),\, h_{(3)} \triangleleft a_{(3)}\Big) = f(g_{(1)}, h_{(1)})[(g_{(2)} \cdot h_{(2)}) \triangleright a]$

(BE5) $\Big(g_{(1)} \triangleright f(h_{(1)}, l_{(1)})\Big) f\Big(g_{(2)} \triangleleft f(h_{(2)}, l_{(2)}),\, h_{(3)} \cdot l_{(3)}\Big) = f(g_{(1)}, h_{(1)}) f(g_{(2)} \cdot h_{(2)}, l)$

(BE6) $g_{(1)} \triangleleft a_{(1)} \otimes g_{(2)} \triangleright a_{(2)} = g_{(2)} \triangleleft a_{(2)} \otimes g_{(1)} \triangleright a_{(1)}$

(BE7) $g_{(1)} \cdot h_{(1)} \otimes f(g_{(2)}, h_{(2)}) = g_{(2)} \cdot h_{(2)} \otimes f(g_{(1)}, h_{(1)})$

Example

Let $\Omega(A) = \big(H, \triangleleft, \triangleright, f\big)$ be an extending datum of A such that the cocyle f is trivial, that is $f(g,\ h) = \varepsilon_H(g)\varepsilon_H(h)1_A$.

Then $\Omega(A) = \big(H, \triangleleft, \triangleright, f\big)$ is a bialgebra extending structure of A if and only if H is a bialgebra and $(A, H, \triangleleft, \triangleright)$ is a **matched pair of bialgebras.**

In this case, the associated unified product $A \ltimes H = A \bowtie H$ is the **bicrossed product of bialgebras.**

Example

Let $\Omega(A) = \big(H, \triangleleft, \triangleright, f\big)$ be an extending datum of A such that the action $\triangleleft$ is trivial, that is $h \triangleleft a = \varepsilon_A(a)h$.

Then $\Omega(A) = \big(H, \triangleleft, \triangleright, f\big)$ is a bialgebra extending structure of A if and only if H is an usual bialgebra and:

(a) The *twisted module condition* and the *cocycle condition* hold (**Blatter, Cohen, Montgomery**);

(b) $g \triangleright (ab) = (g_{(1)} \triangleright a)(g_{(2)} \triangleright b)$

(c) $g_{(1)} \otimes g_{(2)} \triangleright a = g_{(2)} \otimes g_{(1)} \triangleright a$

(d) $g_{(1)}h_{(1)} \otimes f(g_{(2)}, h_{(2)}) = g_{(2)}h_{(2)} \otimes f(g_{(1)}, h_{(1)})$

In this case, the associated unified product $A \ltimes H = A\#_f H$ is the **crossed product of two bialgebras**. Next talk!

- **A challenging problem:** Give an example of an unified product which is neither a crossed product nor a bicrossed product and nor ... a Radford biproduct.

There exists such an example!

Example

Let A_n be the alternating group on a set with n elements. Then $k[A_6]$ is a Hopf algebra which is neither a crossed product nor a bicrossed product of two Hopf algebras and

$$k[A_6] \cong [A_4] \ltimes k[S]$$

where S is a set with thirty elements.

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Proposition

Let A be a Hopf algebra and $\Omega(A) = (H, \triangleleft, \triangleright, f)$ a bialgebra extending structure of A s.t. there exists an antimorphism of coalgebras $S_H : H \to H$ such that

$$h_{(1)} \cdot S_H(h_{(2)}) = S_H(h_{(1)}) \cdot h_{(2)} = \varepsilon_H(h) 1_H$$

Then $A \ltimes H$ is a Hopf algebra with the antipode

$$S(a \ltimes g) := \Big(S_A[f(S_H(g_{(2)}), g_{(3)})] \ltimes S_H(g_{(1)})\Big) \bullet (S_A(a) \ltimes 1_H)$$

Theorem

Let $A \subseteq E$ be an extension of Hopf algebras, H a subcoalgebra of E such that $1_E \in H$. The following are equivalent:

1. *E factorizes through A and H, i.e. the multiplication map $A \otimes H \to E$ is bijective.*
2. *There exists an isomorphism of Hopf algebras*

$$E \cong A \ltimes H$$

for some bialgebra extending structure $\Omega(A) = \big(H, \triangleleft, \triangleright, f\big)$ of A.

- **The classification of unified products**

Definition

A morphism of coalgebras $u : H \to A$ is called a *coalgebra lazy 1-cocyle* if $u(1_H) = 1_A$ and the following compatibility holds:

$$h_{(1)} \otimes u(h_{(2)}) = h_{(2)} \otimes u(h_{(1)})$$

We denote by $H^1_{l,c}(H, A)$ the **group** (with respect to the convolution product) of all coalgebra lazy 1-cocyles of H with coefficients in A.

Theorem

Let $\Omega(A) = \big(H, \triangleleft, \triangleright, f\big)$ and $\Omega'(A) = \big(H, \triangleleft', \triangleright', f'\big)$ be two Hopf algebra extending structures of a Hopf algebra A.
Then there exists $\varphi : A \ltimes' H \to A \ltimes H$ a left A-module, a right H-comodule and a Hopf algebra map ***if and only if*** *$\triangleleft' = \triangleleft$ and there exists a coalgebra lazy 1-cocyle $u \in H^1_{l,c}(H, A)$ such that:*

$$
\begin{aligned}
h \triangleright' c &= u(h_{(1)})(h_{(2)} \triangleright c_{(1)}) S_A\Big(u\big(h_{(3)} \triangleleft c_{(2)}\big)\Big) && (1)\\
f'(h, g) &= u(h_{(1)})(h_{(2)} \triangleright u(g_{(1)})) f(h_{(3)} \triangleleft u(g_{(2)}), g_{(3)}) && (2)\\
& \quad S_A\Big(u\big(h_{(4)} \cdot' g_{(4)}\big)\Big) && (3)\\
h \cdot' g &= \big(h \triangleleft u(g_{(1)})\big) \cdot g_{(2)} && (4)
\end{aligned}
$$

In this case φ is given by $\varphi(a \ltimes h) = a u(h_{(1)}) \ltimes' h_{(2)}$.

Remark

If $\varphi : A \ltimes H \to A \ltimes' H$ is a left A-module, a right H-comodule and Hopf algebra morphism between two unified products then the following diagram

$$\begin{array}{ccccc} A & \xrightarrow{i_A} & A \bowtie H & \xrightarrow{\pi_H} & H \\ \downarrow{\scriptstyle Id_A} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle Id_H} \\ A & \xrightarrow{i_A} & A \bowtie' H & \xrightarrow{\pi_H} & H \end{array}$$

*is commutative and φ is an **isomorphism**.*

Let A be a Hopf algebra, H a coalgebra with a fixed group-like element $1_H \in H$ and $\triangleleft : H \otimes A \to H$ a morphism of coalgebras.

Let $\mathcal{E}S(A, H, \triangleleft)$ be the set of all triples $(\cdot, \triangleright, f)$ such that $\big((H, 1_X, \cdot), \triangleleft, \triangleright, f\big)$ is a Hopf algebra extending structure of A.

Definition

Two elements $(\cdot, \triangleright, f)$, $(\cdot', \triangleright', f')$ of $\mathcal{E}S(A, H, \triangleleft)$ are called **cohomologous** and we denote this by $(\cdot, \triangleright, f) \approx (\cdot', \triangleright', f')$ if there exists a coalgebra lazy 1-cocyle $u \in H^1_{l,c}(H, A)$ such that the compatibility conditions (1) - (4) are fulfilled.

Remark

$\approx$ is an equivalence relation on the set $\mathcal{E}S(A, H, \triangleleft)$.
We denote by $H^2_{l,c}(H, A, \triangleleft)$ the quotient set $\mathcal{E}S(A, H, \triangleleft)/\approx$.

$H^2_{l,c}(H, A, \triangleleft)$ is for the classification of the unified products the counterpart of the second cohomology group for the classification of an extension of an abelian group by a group.

Let $\mathcal{C}(A, H, \triangleleft)$ be the category whose class of objects is the set $\mathcal{E}S(A, H, \triangleleft)$.

A morphism $\varphi : (\triangleright, f) \to (\triangleright', f')$ in $\mathcal{C}(A, H, \triangleleft)$ is a Hopf algebra morphism $\varphi : A \ltimes H \to A \ltimes' H$ that is a left A-module and a right H-comodule map.

Corollary

(Schreier theorem for unified products)

Let A be a Hopf algebra, H a coalgebra with a group-like element 1_H and $\triangleleft : H \otimes A \to H$ a morphism of coalgebras. There exists a bijection between the set of objects of the skeleton of the category $\mathcal{C}(A, H, \triangleleft)$ and the quotient set $H^2_{l,c}(H, A, \triangleleft)$.

- **Split extensions of Hopf algebras**

Radford's biproducts (1985) = smash product algebras + smash coproduct coalgebras.

Theorem

*Let $i : A \to E$ be a split monomorphism of Hopf algebras. Then E is isomorphic as a Hopf algebra to a **Radford biproduct** $G * A$, for a bialgebra G in the braided category ${}_A^A\mathcal{Y}D$ of left-left Yetter-Drinfel'd modules.*

Remark

The theorem of Radford was generalized by:
*• **P. Schauenburg** (1999)*
*• **Ardizzoni, Beatie, Menini, Stefan, Stumbo ...** (2007, 2009, 2010) etc.*

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• Properties of the Hopf algebas extension $A \subset A \ltimes H$. *When is the unified product isomorphic to a Radford biproduct?*

Definition

Let A and E be two bialgebras. A *coalgebra* map $\pi : E \to A$ is called **normal** (Andruskiewitsch and Devoto) if the space

$$\{x \in E \mid \pi(x_{(1)}) \otimes x_{(2)} = 1_A \otimes x\}$$

is a subcoalgebra of E.

Let G and H be two groups. Then any coalgebra map $\pi : k[G] \to k[H]$ is normal. Moreover, assume that G is finite, $H \leq G$ be a subgroup of G. Then the restriction morphism $k[G]^* \to k[H]^*$ is a normal morphism if and only if H is a normal subgroup of G.

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Proposition

Let A be a bialgebra, $\Omega(A) = \big(H, \triangleleft, \triangleright, f\big)$ a bialgebra extending structure of A and the k-linear maps:

$$i_A : A \to A \ltimes H,\ i_A(a) = a \ltimes 1_H,\ \pi_A : A \ltimes H \to A,\ \pi_A(a \ltimes h) = \varepsilon_H(h)a$$

for all $a \in A$, $h \in H$. Then:

1. *i_A is a biagebra map, π_A is a normal left A-module coalgebra morphism and $\pi_A \circ i_A = \mathrm{Id}_A$.*
2. *π_A is a right A-module map if and only if $\triangleright$ is the trivial action.*
3. *π_A is a bialgebra map if and only if $\triangleright$ and f are the trivial maps, i.e. the unified product $A \ltimes H = A \# H$, the right version of the smash product of bialgebras.*

Theorem

*Let $i : A \to E$ be a Hopf algebra morphism such that there exists $\pi : E \to A$ a **normal** left A-module coalgebra morphism for which $\pi \circ i = \mathrm{Id}_A$. Let $H := \{x \in E \mid \pi(x_{(1)}) \otimes x_{(2)} = 1_A \otimes x\}$. Then there exists a bialgebra extending structure $\Omega(A) = \big(H, \triangleleft, \triangleright, f\big)$ of A, given by:*

$$h \cdot g := i\Big(S_A\big(\pi(h_{(1)}g_{(1)})\big)\Big)h_{(2)}g_{(2)}, \qquad f(h,g) := \pi(hg)$$

$$h \triangleleft a := i\Big(S_A\big(\pi(h_{(1)}i(a_{(1)}))\big)\Big)h_{(2)}i(a_{(2)}), \quad h \triangleright a := \pi\big(hi(a)\big)$$

for all $h, g \in H$, $a \in A$ such that

$$\varphi : A \ltimes H \to E, \quad \varphi(a \ltimes h) = i(a)h$$

is an isomorphism of Hopf algebras.

Remark: Any Hopf algebra extending structure of a Hopf algebra A is constructed as in the above theorem.

Corollary

Let A and E be two Hopf algebras. The following are equivalent:

1. *E is isomorphic to a unified product $A \ltimes H$.*
2. *Then there exists a morphism of Hopf algebras $i : A \to E$ which has a retraction $\pi : E \to A$ that is a **normal** left A-module coalgebra morphism.*

Proposition

Let A be a Hopf algebra and $\Omega(A) = \big(H, \triangleleft, \triangleright, f\big)$ a bialgebra extending structure of A. The following are equivalent:

(1) *$i_A : A \to A \ltimes H$ is a split monomorphism in the category of bialgebras;*

(2) *There exists $\gamma : H \to A$ a unitary coalgebra map such that*

$$\begin{aligned} h \triangleright a &= \gamma(h_{(1)})\, a_{(1)}\, \gamma^{-1}(h_{(2)} \triangleleft a_{(2)}) \\ f(h, g) &= \gamma(h_{(1)})\, \gamma(g_{(1)})\, \gamma^{-1}(h_{(2)} \cdot g_{(2)}) \end{aligned}$$

for all $h, g \in H$ and $a \in A$, where $\gamma^{-1} = S_A \circ \gamma$.

In this case, there exists an isomorphism of bialgebras $A \ltimes H \cong L * A$, where $L * A$ is the ***Radford biproduct*** *for a bialgebra L in the braided category ${}_A^A\mathcal{Y}D$ of Yetter-Drinfel'd modules.*

- **Back to the starting point – the conclusion of the talk**:

An answer of a college level question was given!

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Thank you!