Deformations of a class of graded Hopf algebras with quadratic relations

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We consider a special class of graded Hopf algebras, which are finitely generated quadratic algebras with anti-symmetric generating relations. We discuss the automorphism group and Calabi-Yau property of a PBW-deformation of such a Hopf algebra. We show that the Calabi-Yau property of a PBW-deformation of such a Hopf algebra is equivalent to that of the corresponding augmented PBW-deformation under some mild conditions.

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Hopf algebras and tensor categories
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Outline

(I) Hopf algebras with quadratic relations

(II) Poincaré-Birkhoff-Witt (PBW) deformation

(III) Calabi-Yau algebras

(IV) Main results

(I) Hopf algebras with quadratic relations

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- **Example.** The polynomial algebra $U = \Bbbk[x_1, \ldots, x_n]$ is a quadratic algebra, its quadratic dual is the exterior algebra $U^! = \bigwedge\{y_1, \ldots, y_n\}$.

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- An antisymmetric element may be written as $r = \mathbf{x}^t M \mathbf{x}$, where $\mathbf{x}^t = (x_1, \ldots, x_n)$ and M is an antisymmetric $n \times n$-matrix.
- Let $U = T(V)/(r_1, \ldots, r_m)$ be a quadratic algebra with antisymmetric generating relations $r_i \in V \otimes V$ for $1 \leq i \leq m$.

  We call such a quadratic algebra U as a weakly symmetric algebra.

Weakly symmetric algebras

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  (i) [Dubois-Violette, 2007] U is a Koszul algebra.

  (ii) [Berger, 2009] U is a Calabi-Yau algebra of dimension 2.

  (iii) [Berger, Bocklandt] Any (connected graded) Calabi-Yau algebra of dimension 2 is obtained in this way.

(II) PBW-deformations

- Let $U = \bigoplus_{n\geq 0} U_n$ be a positively graded algebra. A PBW-deformation of U is a filtered algebra A with filtration $0 \subseteq F_0A \subseteq F_1A \subseteq \cdots \subseteq F_nA \subseteq \cdots$, together with a graded algebra isomorphism $p : U \longrightarrow gr(A)$.

PBW-deformations

- A PBW-deformation A of a quadratic algebra $U = T(V)/(R)$ is determined by two linear maps:

$$\varphi : R \to V \text{ and } \theta : R \to \mathbb{k},$$

so that

$$A = T(V)/(I_2), \text{ where } I_2 = \{r - \varphi(r) - \theta(r) | r \in R\}.$$

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- It is more convenient to consider the augmented PBW-deformations than the nonaugmented cases.

  Especially, when we consider the PBW-deformations of a graded Hopf algebra, we have the tool homological integrals to do with the homological properties of augmented PBW-deformations.

- **Examples.** (i) A universal enveloping algebra a finite dimensional algebra is an augmented PBW-deformation of a polynomial algebra.

  (ii) Weyl algebra A_1 is a PBW-deformation of the polynomial algebra $\Bbbk[x_1, x_2]$.

  (iii) Sridharan enveloping algebras: $\mathfrak{g}$ is a finite dimensional algebra, $f : \mathfrak{g} \times \mathfrak{g} \longrightarrow \Bbbk$ is a 2-cocycle of $\mathfrak{g}$, then

$$U_f(\mathfrak{g}) = T(\mathfrak{g})/I,$$

  where the ideal I is generated by

$$x \otimes y - y \otimes x - [x, y] - f(x, y), \text{ for all } x, y \in \mathfrak{g}.$$

Augmented PBW-deformations

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Theorem (Polishchuk-Positselski)

The dual map $\phi^* : V^* \to R^*$ induces a differential d on the quadratic dual $U^!$ of U so that $(U^!, d)$ is a differential graded algebra.

Moreover, the set of possible augmented PBW-deformations of U is in one-to-one correspondence with the set of all the possible differential structures on $U^!$.

(III) Calabi-Yau algebras

Calabi-Yau algebras

- For the background of Calabi-Yau algebra, see Xiaolan Yu's talk yesterday.

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- **Definition.** [Ginzburg] An algebra A is said to be a Calabi-Yau algebra of dimension d (CY-d, for short) if
  (i) A is homologically smooth, that is; A has a bounded resolution of finitely generated projective A-A-bimodules,
  (ii) $\mathrm{Ext}^i_{A^e}(A, A^e) = 0$ if $i \neq d$ and $\mathrm{Ext}^d_{A^e}(A, A^e) \cong A$ as A-A-bimodules, where $A^e = A \otimes A^{op}$ is the enveloping algebra of A.

  We call d the Calabi-Yau dimension of A.

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- [Berger, 2009] The Weyl algebra A_n is CY-$2n$.
- An interesting question is to find out the relation between the global dimension and the CY dimension of a CY algebra.

(IV) Main results

- **Theorem.** [Yekutieli] If A is a (positively) filtered algebra such that $gr(A)$ is a Calabi-Yau algebra, then A differs from being Calabi-Yau by a filtration-preserving automorphism σ: that is, $\mathrm{RHom}_{A^e}(A, A^e) \cong {}^1A^\sigma[d]$.

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- Denote by $Aut_{filt}(A)$ the group of automorphisms of A which preserve the filtration of A.

Main results

Theorem (H-Zhang)

Let $U = T(V)/(R)$ be a weakly symmetric algebra, and let $A = T(V)/(r - \varphi(r) : r \in R)$ be an augmented PBW-deformation of U. Then $Aut_{filt}(A) \cong Z^1(U^!, d)$, where $Z^1(U^!, d)$ is the group of 1-cocycles of the differential graded algebra $(U^!, d)$.

Moreover, if the quadratic algebra U is Koszul then $Aut_{filt}(A) \cong \mathrm{Ext}^1_A({}_A\Bbbk, {}_A\Bbbk)$.

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- **Corollary.** [Well known] *Any universal enveloping algebra of a finite dimensional semisimple Lie algebra is Calabi-Yau.*

A lemma

- Let $U = T(V)/(R)$ be a weakly symmetric algebra, and $\varphi : R \to V$ and $\theta : R \to \Bbbk$ be linear maps.

  Set
  $$I_2 = \{r - \varphi(r) | r \in R\},$$
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- Assume that both $A = T(V)/(I_2)$ and $A' = T(V)/(I_2')$ are PBW-deformations of U.

  Define
$$D : T(V) \to A' \otimes A'^{op},$$
$$D(x) = x \otimes 1 - 1 \otimes x, \qquad \text{for all } x \in V.$$

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- D induces an algebra morphism (also denoted by D)
  $$D : A \to A' \otimes A'^{op}.$$

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- **Lemma.** *$A' \otimes A'^{op}$ is projective either as a left A-module or as a right A-module.*
- The key point to prove the lemma is that U is a graded Hopf algebra. Then $U \otimes U^{op}$ is a free module either as a left U-module or as a right U-module.

Main results

Theorem (H-Zhang)

Let $U = T(V)/(R)$ be a weakly symmetric algebra. Assume that both $A = T(V)/(r - \varphi(r) : r \in R)$ and $A' = T(V)/(r - \varphi(r) - \theta(r) : r \in R)$ are PBW-deformations of U. If A is CY-d, then so is A'.

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Conversely, assume further that U is a noetherian domain and Artin-Schelter regular. If A' is CY-d, then so is A.

Main results

Theorem (H-Van Oystaeyen-Zhang)

Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then for any 2-cocycle $f \in Z^2(\mathfrak{g}, \Bbbk)$, the following statements are equivalent.

(i) *The Sridharan enveloping algebra $U_f(\mathfrak{g})$ is CY-d.*

(ii) *The universal enveloping algebra $U(\mathfrak{g})$ is CY-d.*

(iii) $\dim \mathfrak{g} = d$ *and $\mathfrak{g}$ is unimodular, that is, for any $x \in \mathfrak{g}$,* $\mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(x)) = 0$.

Main results

Theorem (H-Van Oystaeyen-Zhang)

Let A be a noetherian CY filtered algebra of dimension 3 such that $gr(A)$ is commutative and generated in degree 1, then A is isomorphic to $\Bbbk\langle x, y, z\rangle/(R)$ with the commuting relations R listed in the following table:

Case	$\{x, y\}$	$\{x, z\}$	$\{y, z\}$
1	z	$-2x$	$2y$
2	y	$-z$	0
3	z	0	0
4	0	0	0
5	y	$-z$	1
6	z	1	0
7	1	0	0

where $\{x, y\} = xy - yx$.

Remarks.

- This is a small step towards our aim to find all the possible noetherian connected filtered Calabi-Yau algebras of dimension 3.

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- The results can be generalized without too much difficulty to the nonquadratic algebras. That is, if the graded Hopf algebra U is N-homogeneous with some "anti-symmetric" relations, then the same results still hold.

  For example, $U = T(V)/(r)$, where

$$r = \sum_{\sigma \in S_n} sgn(\sigma) x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}.$$

Thank you!