Representations of the category of modules over pointed Hopf algebras over $\mathbb{S}_3$ and $\mathbb{S}_4$

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We will recall the basic results on module categories over finite-dimensional Hopf algebras [2], [4] and the classification of finite-dimensional Hopf algebras with coradical $\Bbbk\mathbb{S}_3$ or $\Bbbk\mathbb{S}_4$ from [1], [3], respectively.

Using these results, we will show that if $n = 3, 4$ and $\mathcal{M}$ is an exact indecomposable module category over $\operatorname{Rep}(\mathfrak{B}(X, q)\#\Bbbk\mathbb{S}_n)$, then there exist

- a subgroup $F < \mathbb{S}_n$ and a 2-cocycle $\psi \in Z^2(F, \Bbbk^\times)$,
- a subset $Y \subseteq X$ invariant under the action of F,
- a family of scalars $\{\xi_C\}$ *compatible* with (F, ψ, Y),

such that $\mathcal{M} \simeq {}_{\mathcal{B}(Y,F,\psi,\xi)}\mathcal{M}$, where $\mathcal{B}(Y, F, \psi, \xi)$ is a left $\mathfrak{B}(X, q)\#\Bbbk\mathbb{S}_n$-comodule algebra constructed from data (Y, F, ψ, ξ). We also show a classification of connected homogeneous left coideal subalgebras $\mathcal{B}(Y, F, \psi, \xi)$ of $\operatorname{gr} H$ and together with a presentation by generators and relations.

Finally we prove that if H is a finite-dimensional Hopf algebra with coradical $\Bbbk\mathbb{S}_3$ or $\Bbbk\mathbb{S}_4$ then H and $\operatorname{gr} H$ are cocycle deformations of each other, a result analogous to a theorem of Masuoka for abelian groups. This implies that there is a bijective correspondence between module categories over $\operatorname{Rep}(H)$ and $\operatorname{Rep}(\operatorname{gr} H)$.

Bibliography

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Representations of the category of modules over pointed Hopf algebras over $\mathbb{S}_3$ and $\mathbb{S}_4$

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Let H be a Hopf algebra. Then, $\mathcal{C} = \operatorname{Rep} H$, the category of finite-dimensional modules over H is a tensor category, with

- $\otimes = \otimes_{\Bbbk}$ the usual tensor product: if $M, N \in \mathcal{C}$, then $M \otimes N \in \mathcal{C}$ via

$$h \cdot m \otimes n = \Delta(h) \cdot m \otimes n, \quad h \in H, \ m \in M, \ n \in N$$

- $\mathbf{1} = \Bbbk$: $\Bbbk \in \mathcal{C}$ via

$$h \cdot 1 = \epsilon(h) 1, \quad h \in H.$$

The associativity follows from the fact that $\operatorname{Vect}_{\Bbbk} \supseteq \operatorname{Rep} H$ and the coassociativity of Δ.

Let $\mathcal{C} = (\mathcal{C}, \otimes, \mathbf{1})$ be a tensor category.

- A module category[1] over $\mathcal{C}$ is an abelian category $\mathcal{M}$ equipped with an exact bifunctor $\odot : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ such that, for each $V, W \in \mathcal{C}$, $M \in \mathcal{M}$, there are natural isomorphims

$$(V \otimes W) \odot M \cong V \odot (W \odot U), \qquad \mathbf{1} \odot M \cong M,$$

  subject to natural axioms of associativity and unity.
- A module category is said to be exact if for every projective object $P \in \mathcal{C}$ then $P \odot M$ is projective in $\mathcal{M}$ for every $M \in \mathcal{M}$.

[1] P. Etingof and V. Ostrik, Mosc. Math. J. (2004).

- If $(\mathcal{C}, \otimes, \mathbf{1})$ is a tensor category, then $(\mathcal{C}, \odot)$ is a module category over $\mathcal{C}$, with $\odot = \otimes$.

[2] $\lambda : A \to H \otimes A$, the coaction, is an algebra morphism.

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- Let H be a Hopf algebra, (A, λ)[2] a left H-comodule algebra. The category of A-modules of finite dimension ${}_A\mathcal{M}$ is a representation of $\mathrm{Rep}(H)$.

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- Let H be a Hopf algebra, (A, λ)[2] a left H-comodule algebra. The category of A-modules of finite dimension ${}_A\mathcal{M}$ is a representation of $\text{Rep}(H)$.
  The action $\odot : \text{Rep}\, H \times {}_A\mathcal{M} \to {}_A\mathcal{M}$ is given by

  $$V \odot M = V \otimes M, \qquad V \in \text{Rep}(H),\ M \in {}_A\mathcal{M}$$

  where $V \otimes M \in {}_A\mathcal{M}$ via

  $$a \cdot v \otimes m = \lambda(a) \cdot (v \otimes m), \qquad a \in A,\ v \in V,\ m \in M.$$

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Let H be a Hopf algebra of finite dimension. Let $\mathcal{M}$ be a module category indecomposable and exact over Rep(H).

- There exists a left H-comodule algebra A right H-simple (i.e. with no non-trivial H-costable right ideals) with trivial coinvariants ($A^{\mathrm{co}\,H} = \Bbbk$) such that $\mathcal{M} \simeq {}_A\mathcal{M}$ as modules over Rep(H).[3]

[3] Andruskiewitsch. N. and Mombelli, M., J. Algebra (2007).

Let H be a pointed Hopf algebra.

- Assume $H = \mathfrak{B}(V)\#\Bbbk G$. There exist
  1. a subgroup $F \subseteq G$,
  2. a 2-cocycle $\psi \in Z^2(F, \Bbbk^\times)$,
  3. an homogeneous left coideal subalgebra $\mathcal{K} = \bigoplus_{i=0}^{m} \mathcal{K}^i \subseteq \mathfrak{B}(V)$ such that $\mathcal{K}^1 \subseteq V$ is an F-invariant $\Bbbk G$-subcomodule,

  such that $\operatorname{gr} A \simeq \mathcal{K}\#\Bbbk_\psi F$ as $\operatorname{gr} H$-comodule algebras.[4]

[4] Mombelli, M., J. London Math. Soc., (2010).

[5] G.I., A. and Mombelli, M. Pacific Journal of Math., (2011).

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If A, A' are two H-comodule algebras then

- ${}_A\mathcal{M} \simeq {}_{A'}\mathcal{M}$ as modules over Rep(H) if and only if there exist $g \in G$ such that $gA'g^{-1} \cong A$ as comodule algebras.[5]

[4] Mombelli, M., J. London Math. Soc., (2010).
[5] G.I., A. and Mombelli, M. Pacific Journal of Math., (2011).

Let $\mathcal{H}$ be a finite-dimensional pointed Hopf algebra over $\mathbb{S}_n$, $n = 3, 4$, $\mathcal{U} = \operatorname{gr}\mathcal{H}$.

- $\mathcal{U}$ is the bosonization $\mathfrak{B}(X, q)\#\Bbbk\mathbb{S}_n$, where X is either $\mathcal{O}_2^n$ or $\mathcal{O}_4^4$ (only if $n = 4$) and q is a 2-cocycle.

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Let us denote by $\mathcal{K}_Y$ the subalgebra of $\mathcal{U}$ *generated by* Y, for each subset $Y \subseteq X$ (for instance, $\mathcal{K}_X = \mathfrak{B}(X, q)$).

- Let $\mathcal{K}$ be an homogeneous left coideal subalgebra of $\mathcal{U}$. Then $\mathcal{K}$ is generated in degree one and $\mathcal{K} \cong \mathcal{K}_Y$ for some Y.

We associate an $\mathcal{U}$-comodule algebra $\mathcal{B}(Y, F, \psi, \xi)$ to the data:

- a subgroup $F < \mathbb{S}_n$,
- a cocycle $\psi \in Z^2(F, \Bbbk^\times)$,
- a subset $Y \subseteq X$ such that $F \cdot Y \subseteq Y$,
- a family $\xi = \{\xi_C\}_{C \in \mathcal{R}'} \in \Bbbk$ *compatible*[6] with (Y, F, ψ).

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in such a way that

- $\mathcal{B}(Y, F, \psi, \xi)$ is a right $\mathcal{U}$-simple left $\mathcal{U}$-comodule algebra with trivial coinvariants,
- there is an isomorphism of comodule algebras $\operatorname{gr}\mathcal{B}(Y, F, \psi, \xi) \simeq \mathcal{K}_Y \# \Bbbk_\psi F$.

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Moreover,

- $\mathcal{B}(X, \mathbb{S}_n, \psi, \xi)$ is a $(\mathcal{U}, \mathcal{H})$-biGalois extension.

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- $\mathcal{B}(X, \mathbb{S}_n, \psi, \xi)$ is a $(\mathcal{U}, \mathcal{H})$-biGalois extension.

Therefore,

- $\mathcal{H} = {}^{\sigma}\mathcal{U}$ is a cocycle deformation of $\mathcal{U}$.
- There is a bijective correspondence between equivalence classes of exact module categories over $\operatorname{Rep}\mathcal{U}$ and $\operatorname{Rep}\mathcal{H}$:

$$_{\mathcal{A}}\mathcal{M} \mapsto {}_{\mathcal{A}_\sigma}\mathcal{M}.$$

Let $\mathcal{H}$ be a pointed Hopf algebra over $G = \mathbb{S}_3$ or $\mathbb{S}_4$, $\mathcal{U} = \operatorname{gr}\mathcal{H}$.

1. Let $\mathcal{M}$ be an exact indecomposable module category over $\operatorname{Rep}(\mathcal{U})$, then there exist
   (i) a subgroup $F < G$, and a 2-cocycle $\psi \in Z^2(F, \Bbbk^\times)$,
   (ii) a subset $Y \subset X$ such that $F \cdot Y \subset Y$,
   (iii) a family of scalars $\{\xi_C\}_{C\in\mathcal{R}'}$ compatible with (Y, F, ψ),

   such that there is an equivalence of modules
   $$\mathcal{M} \simeq {}_{\mathcal{B}(Y,F,\psi,\xi)}\mathcal{M}.$$
2. Let (Y, F, ψ, ξ), (Y', F', ψ', ξ') be two families as above. Then there exists an equivalence of module categories ${}_{\mathcal{A}(Y,F,\psi,\xi)}\mathcal{M} \simeq {}_{\mathcal{A}(Y',F',\psi',\xi')}\mathcal{M}$ if and only if there exist an element $h \in G$ such that
   $$F' = hFh^{-1},\ \psi' = \psi^h,\ Y' = h\cdot Y,\ \{\xi'_C\} = \{\xi_{h^{-1}\cdot C}\}.$$

In this case $X = \mathcal{O}_2^3 = \{(12), (13), (23)\}$ and $\mathfrak{B}(\mathcal{O}_2^3, -1)$ is the algebra generated by the set $\{x_{(12)}, x_{(13)}, x_{(23)}\}$ with relations

$$x_{(12)}^2, \ x_{(13)}^2, \ x_{(23)}^2,$$
$$x_{(12)}x_{(13)} + x_{(13)}x_{(23)} + x_{(23)}x_{(12)},$$
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The following are all the proper homogeneous left coideal subalgebras of $\mathfrak{B}(\mathcal{O}_2^3, -1)\#\Bbbk\mathbb{S}_3$:

1. $\mathcal{K}_i = \langle x_i \rangle \cong \Bbbk[x]/\langle x^2 \rangle$, $i \in \mathcal{O}_2^3$;
2. $\mathcal{K}_{i,j} = \langle x_i, x_j \rangle \cong \Bbbk\langle x, y \rangle / \langle x^2, y^2, xyx - yxy \rangle$, $i, j \in \mathcal{O}_2^3$.

Modules categories over $\mathfrak{B}(\mathcal{O}_2^3, -1)\#\Bbbk\mathbb{S}_3$

Let $\mathcal{M}$ be an indecomposable exact module category over $\text{Rep}(\mathfrak{B}(X, -1)\#\Bbbk\mathbb{S}_3)$. Then there is a module equivalence $\mathcal{M} \simeq {}_{\mathcal{A}}\mathcal{M}$ where $\mathcal{A}$ is one (and only one) of the comodule algebras in following list.

1. For any subgroup $F \subseteq \mathbb{S}_3$, $\psi \in Z^2(F, \Bbbk^\times)$, the twisted group algebra $\Bbbk_\psi F$.
2. The algebra $\mathcal{A}(\{i\}, \xi, 1) =< y_i : y_i^2 = \xi 1 >$, with coaction determined by $\lambda(y_i) = x_i \otimes 1 + g_i \otimes y_i$.
3. The algebra $\mathcal{A}(\{i\}, \xi, \mathbb{Z}_2) =< y_i, h : y_i^2 = \xi 1, h^2 = 1, hy_i = -y_i h >$ with coaction determined by $\lambda(y_i) = x_i \otimes 1 + g_i \otimes y_i$, $\lambda(h) = g_i \otimes h$.

4. The algebra $\mathcal{A}(\{i,j\},1) =< y_i, y_j : y_i^2 = y_j^2 = 0,\ y_iy_jy_i = y_jy_iy_j >$ with coaction determined by $\lambda(y_i) = x_i \otimes 1 + g_i \otimes y_i$, $\lambda(y_j) = x_j \otimes 1 + g_j \otimes y_j$.
5. The algebra $\mathcal{A}(\{i,j\},\mathbb{Z}_2) =< y_i, y_j, h : y_i^2 = y_j^2 = 0, h^2 = 1, hy_i = -y_jh, y_iy_jy_i = y_jy_iy_j >$ with coaction determined by $\lambda(y_i) = x_i \otimes 1 + g_i \otimes y_i$, $\lambda(y_j) = x_j \otimes 1 + g_j \otimes y_j$, $\lambda(h) = g_k \otimes h$, where $k \neq i, j$.

6. The algebra $\mathcal{A}(\mathcal{O}_2^3, \xi, 1)$, generated by $\{y_{(12)}, y_{(13)}, y_{(23)}\}$ subject to relations

$$y_{(12)}^2 = y_{(13)}^2 = y_{(23)}^2 = \xi 1,$$
$$y_{(12)}y_{(13)} + y_{(13)}y_{(23)} + y_{(23)}y_{(12)} = 0,$$
$$y_{(13)}y_{(12)} + y_{(23)}y_{(13)} + y_{(12)}y_{(23)} = 0.$$

The coaction is determined by $\lambda(y_s) = x_s \otimes 1 + g_s \otimes y_s$ for any $s \in \mathcal{O}_2^3$.

7. The algebra $\mathcal{A}(\mathcal{O}_2^3, \xi, \mathbb{Z}_2)$, generated by $\{y_{(12)}, y_{(13)}, y_{(23)}, h\}$ subject to relations

$$y_{(12)}^2 = y_{(13)}^2 = y_{(23)}^2 = \xi 1, \quad h^2 = 1,$$
$$hy_{(12)} = -y_{(12)}h, \quad hy_{(13)} = -y_{(23)}h,$$
$$y_{(12)}y_{(13)} + y_{(13)}y_{(23)} + y_{(23)}y_{(12)} = 0.$$

The coaction is determined by $\lambda(h) = g_{(12)} \otimes h$, $\lambda(y_s) = x_s \otimes 1 + g_s \otimes y_s$ for any $s \in \mathcal{O}_2^3$.

8. The algebra $\mathcal{A}(\mathcal{O}_2^3, \xi, \mu, \eta, \mathbb{Z}_3)$, generated by $\{y_{(12)}, y_{(13)}, y_{(23)}, h\}$ subject to relations

$$y_{(12)}^2 = y_{(13)}^2 = y_{(23)}^2 = \xi 1, \quad h^3 = 1,$$
$$hy_{(12)} = y_{(13)}h, \quad hy_{(13)} = y_{(23)}h, \quad hy_{(23)} = y_{(12)}h,$$
$$y_{(12)}y_{(13)} + y_{(13)}y_{(23)} + y_{(23)}y_{(12)} = \mu\, h,$$
$$y_{(13)}y_{(12)} + y_{(23)}y_{(13)} + y_{(12)}y_{(23)} = \eta\, h^2.$$

The coaction is determined by $\lambda(h) = g_{(132)} \otimes h$, $\lambda(y_s) = x_s \otimes 1 + g_s \otimes y_s$, for any $s \in \mathcal{O}_2^3$.

9. The algebras $\mathcal{A}(\mathcal{O}_2^3, \xi, \mu, \mathbb{S}_3, \psi)$, for each $\psi \in Z^2(\mathbb{S}_3, \Bbbk^\times)$, generated by $\{y_{(12)}, y_{(13)}, y_{(23)}, e_h \,:\, h \in \mathbb{S}_3\}$ subject to relations

$$e_h e_t = \psi(h,t)\, e_{ht}, \quad e_h y_s = -y_{h\cdot s} e_h \qquad h, t \in \mathbb{S}_3, s \in \mathcal{O}_2^3,$$
$$y_{(12)}^2 = y_{(13)}^2 = y_{(23)}^2 = \xi 1,$$
$$y_{(12)} y_{(13)} + y_{(13)} y_{(23)} + y_{(23)} y_{(12)} = \mu\, e_{(123)}.$$

The coaction is determined by $\lambda(e_h) = h \otimes e_h$, $\lambda(y_s) = x_s \otimes 1 + g_s \otimes y_s$ for any $s \in \mathcal{O}_2^3$.