Weak bimonads and weak Hopf monads

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An algebra over a commutative ring is known to be a bialgebra if and only if its category of (left or right) modules is monoidal such that the forgetful functor is strong monoidal. By analogy, a monad can be called a "bimonad" whenever its Eilenberg-Moore category is monoidal such that the forgetful functor is strong monoidal. A bimonad in this sense was proved to be the same as an opmonoidal monad, see recent works by Moerdijk, Mc Crudden and others.

More generally, an algebra over a commutative ring is known to be a weak bialgebra if and only if its category of (left or right) modules is monoidal such that the forgetful functor possesses a so-called separable Frobenius monoidal structure. By analogy, we define a "weak bimonad" as a monad with additional structures that are equivalent to the monoidality of its Eilenberg-Moore category such that the forgetful functor is separable Frobenius monoidal. Whenever in the base category idempotent morphisms split, a simple set of axioms is provided, that characterizes the monoidal structure of the Eilenberg-Moore category as a weak lifting of the monoidal structure of the base category. The relation to bimonads, and the relation to weak bialgebras in a braided monoidal category are revealed. We also discuss antipodes, obtaining the notion of weak Hopf monad.

The talk is based on the paper [G. Böhm, S. Lack and R. Street, *Weak bimonads and weak Hopf monads*. J. Algebra **328** (2011), 1-30.]

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Plan.

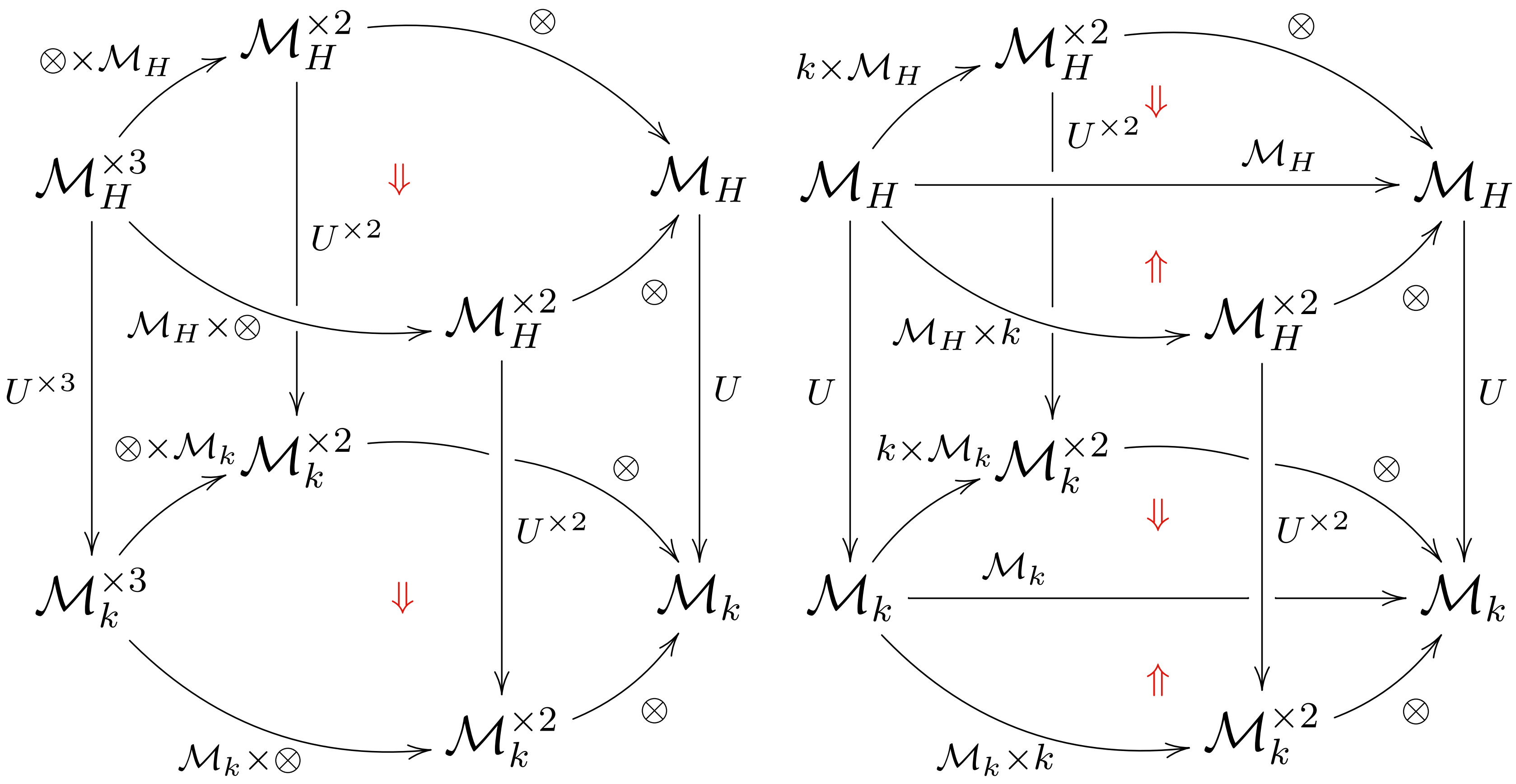
- Preliminaries: (Hopf) bialgebras and (Hopf) bimonads.
- Weak bialgebras and their category of modules.
- Weak bimonads – definition and axioms.
- The category of EM algebras.
- Weak bimonads vs bimonads.
- Weak Hopf monads.

based on: G. Böhm, S. Lack and R. Street.
J. Algebra 328 (2011), 1-30. arXiv:1002.4493.

Bialgebras and their category of modules.

For a k-algebra H, TFAE.

- H is a bialgebra.
- $\mathcal{M}_H$ is monoidal and $\mathcal{M}_H \xrightarrow{U} \mathcal{M}_k$ is strong monoidal:



Hopf algebras.

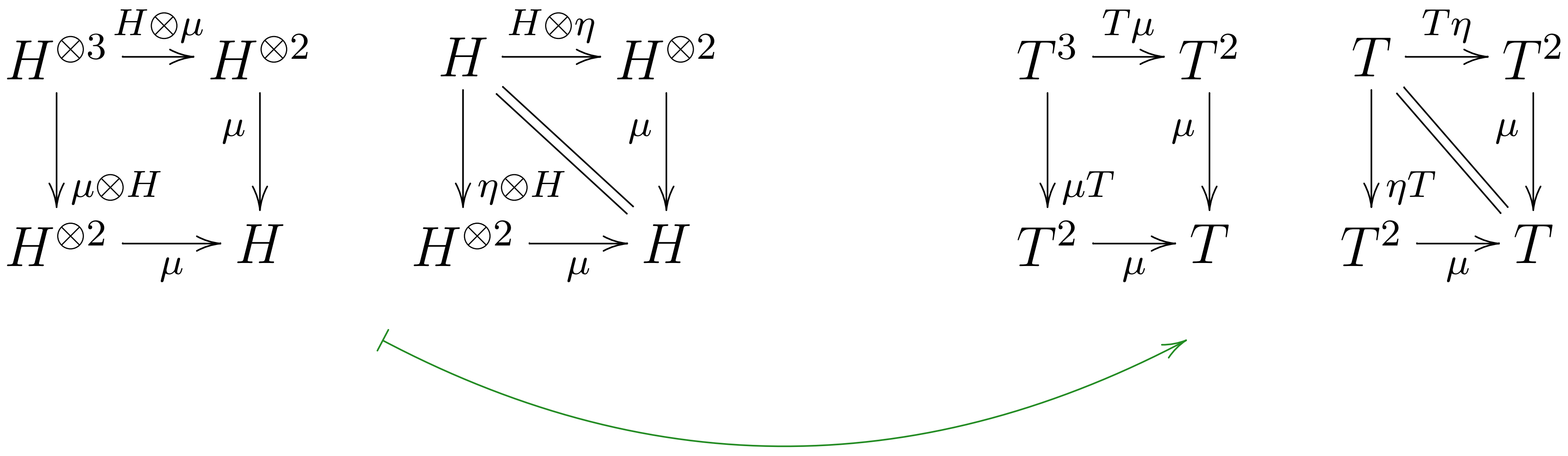
For a k-bialgebra H, TFAE.

- H is a Hopf algebra.
- $H \otimes H \xrightarrow{M \otimes \Delta} H \otimes H \otimes H \xrightarrow{\mu \otimes H} H \otimes H, \;\; h' \otimes h \mapsto h'h_1 \otimes h_2$ is an isomorphism.

Monads and Eilenberg-Moore algebras.

k-algebra ⇝ monad

$$(H, H^{\otimes 2} \xrightarrow{\mu} H, k \xrightarrow{\eta} H) \qquad (\mathcal{M} \xrightarrow{T} \mathcal{M}, T^2 \xrightarrow{\mu} T, \mathcal{M} \xrightarrow{\eta} T)$$



$$(\mathcal{M}_k \stackrel{(-)\otimes H}{\rightarrow} \mathcal{M}_k, (-)\otimes H^{\otimes 2} \stackrel{(-)\otimes \mu}{\rightarrow} (-)\otimes H, \mathcal{M}_k \stackrel{(-)\otimes \eta}{\rightarrow} (-)\otimes H)$$

Monads and Eilenberg-Moore algebras.

k-algebra $\rightsquigarrow$ monad

$$(H, H^{\otimes 2} \xrightarrow{\mu} H, k \xrightarrow{\eta} H) \qquad (\mathcal{M} \xrightarrow{T} \mathcal{M}, T^2 \xrightarrow{\mu} T, \mathcal{M} \xrightarrow{\eta} T)$$

$$\begin{array}{ccc} H^{\otimes 3} & \xrightarrow{H\otimes\mu} & H^{\otimes 2} \\ \downarrow{\scriptstyle \mu\otimes H} & & \downarrow{\scriptstyle \mu} \\ H^{\otimes 2} & \xrightarrow[\mu]{} & H \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{H\otimes\eta} & H^{\otimes 2} \\ \downarrow{\scriptstyle \eta\otimes H} & \searrow\!\!\!= & \downarrow{\scriptstyle \mu} \\ H^{\otimes 2} & \xrightarrow[\mu]{} & H \end{array} \qquad \begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ \downarrow{\scriptstyle \mu T} & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array} \qquad \begin{array}{ccc} T & \xrightarrow{T\eta} & T^2 \\ \downarrow{\scriptstyle \eta T} & \searrow\!\!\!= & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array}$$

H-module $\rightsquigarrow$ EM algebra

$$(N, N \otimes H \xrightarrow{\nu} N) \qquad (X \in |\mathcal{M}|, TX \xrightarrow{\xi} X)$$

$$\begin{array}{ccc} N\otimes H^{\otimes 2} & \xrightarrow{N\otimes\mu} & N\otimes H \\ \downarrow{\scriptstyle \nu\otimes H} & & \downarrow{\scriptstyle \nu} \\ N\otimes H & \xrightarrow[\nu]{} & N \end{array} \qquad \begin{array}{ccc} N & \xrightarrow{N\otimes\eta} & N\otimes H \\ & \searrow\!\!\!= & \downarrow{\scriptstyle \nu} \\ & & N \end{array} \qquad \begin{array}{ccc} T^2X & \xrightarrow{\mu X} & TX \\ \downarrow{\scriptstyle T\xi} & & \downarrow{\scriptstyle \xi} \\ TX & \xrightarrow[\xi]{} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\eta X} & TX \\ & \searrow\!\!\!= & \downarrow{\scriptstyle \xi} \\ & & X \end{array}$$

$\mathcal{M}_H \rightsquigarrow \mathcal{M}_T$

Bimonads.

Definition. A bimonad on a monoidal category $(\mathcal{M}, \otimes, I)$ is a monad $\mathcal{M} \xrightarrow{T} \mathcal{M}$ equipped with the additional str needed for $\mathcal{M}_T$ to be monoidal and $\mathcal{M}_T \xrightarrow{U} \mathcal{M}$ to be strong monoidal.
Explicitly. (e.g. [Moerdijk 02]) An opmonoidal str

$$T(- \otimes -) \xrightarrow{T_2} T(-) \otimes T(-), \qquad TI \xrightarrow{T_0} I$$

compatible with $T^2 \xrightarrow{\mu} T$ and $\mathcal{M} \xrightarrow{\eta} T$. Shortly, an opmonoidal monad.

$$\begin{array}{ccc} T^2(X \otimes Y) & \xrightarrow{\mu(X \otimes Y)} & T(X \otimes Y) \\ \downarrow{\scriptstyle TT_2} & & \\ T(TX \otimes TY) & & \downarrow{\scriptstyle T_2} \\ \downarrow{\scriptstyle T_2} & & \\ T^2X \otimes T^2Y & \xrightarrow[\mu X \otimes \mu Y]{} & TX \otimes TY \end{array} \qquad \begin{array}{ccc} X \otimes Y & \xrightarrow{\eta(X \otimes Y)} & T(X \otimes Y) \\ \| & & \downarrow{\scriptstyle T_2} \\ X \otimes Y & \xrightarrow[\eta X \otimes \eta Y]{} & TX \otimes TY \end{array}$$

$$\begin{array}{ccc} T^2 I & \xrightarrow{\mu I} & TI \\ \downarrow{\scriptstyle TT_0} & & \\ TI & & \downarrow{\scriptstyle T_0} \\ \downarrow{\scriptstyle T_0} & & \\ I & = & I \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{\eta I} & TI \\ \| & & \downarrow{\scriptstyle T_0} \\ I & = & I \end{array}$$

Bimonads.

Definition. A bimonad on a monoidal category $(\mathcal{M}, \otimes, I)$ is a monad $\mathcal{M} \xrightarrow{T} \mathcal{M}$ equipped with the additional str needed for $\mathcal{M}_T$ to be monoidal and $\mathcal{M}_T \xrightarrow{U} \mathcal{M}$ to be strong monoidal.

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For a k-bialgebra H,

$$T = (-) \otimes H : \mathcal{M}_k \to \mathcal{M}_k,$$

$$\begin{aligned} T_2 \quad &: X \otimes Y \otimes H \to X \otimes H \otimes Y \otimes H, \\ &\quad x \otimes y \otimes h \mapsto x \otimes h_1 \otimes y \otimes h_2; \\ T_0 \quad &= \varepsilon : H \to k. \end{aligned}$$

More generally, for a right R-bialgebroid H,

$$T = (-) \otimes_R H : {}_R\mathcal{M}_R \to {}_R\mathcal{M}_R.$$

Hopf monads.

Definition. [Bruguières, Lack, Virelizier]
A (right) Hopf monad is a bimonad $T : \mathcal{M} \to \mathcal{M}$ such that

$$T(TX \otimes Y) \xrightarrow{T_2} T^2X \otimes TY \xrightarrow{\mu X \otimes TY} TX \otimes TY$$

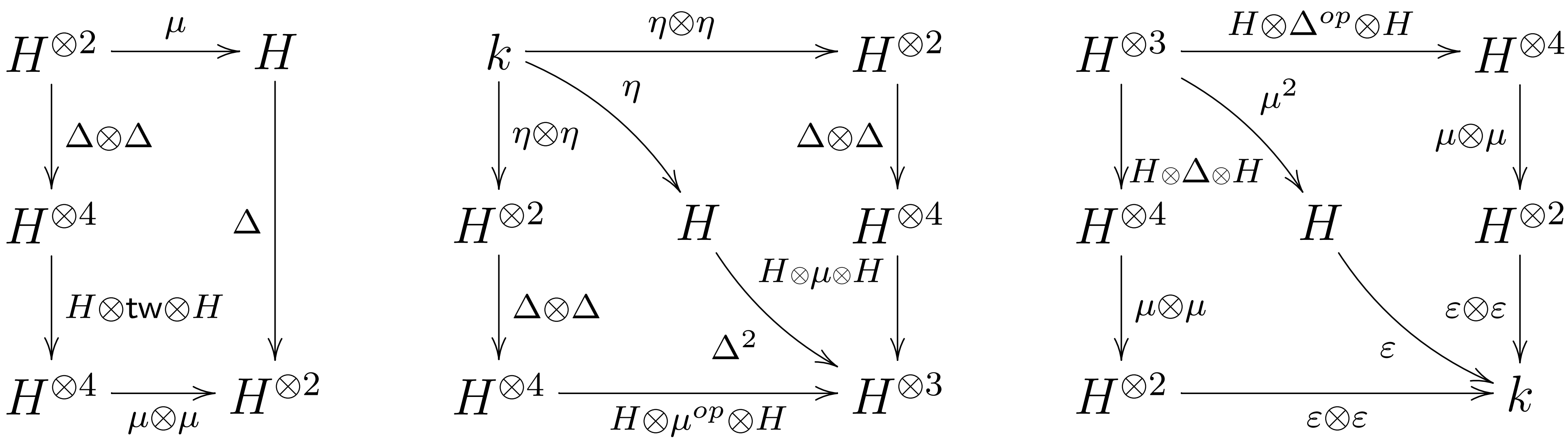
is a natural isomorphism.

For $T = (-) \otimes H : \mathcal{M}_k \to \mathcal{M}_k$,

$$\begin{aligned} X \otimes H \otimes Y \otimes H &\to X \otimes H \otimes Y \otimes H \\ x \otimes h' \otimes y \otimes h &\mapsto x \otimes h'h_1 \otimes y \otimes h_2. \end{aligned}$$

Weak bialgebras.

Definition. A weak bialgebra is a k-algebra and k-coalgebra H s.t.



Theorem. [Szlachányi 05] For a k-algebra H, TFAE.

• H is a weak bialgebra.

• $\mathcal{M}_H$ is monoidal and $\mathcal{M}_H \xrightarrow{U} \mathcal{M}_k$ is both monoidal and op-monoidal such that $U^2 \circ U_2 = U(- \boxtimes -)$ and

$$\begin{array}{ccc} UX \otimes U(Y \boxtimes Z) & \xrightarrow{UX \otimes U_2} & UX \otimes UY \otimes UZ \\ \downarrow U^2 & & \downarrow U^2 \otimes UZ \\ U(X \boxtimes Y \boxtimes Z) & \xrightarrow[U_2]{} & U(X \boxtimes Y) \otimes UZ \end{array} \qquad \begin{array}{ccc} U(X \boxtimes Y) \otimes UZ & \xrightarrow{U_2 \otimes UZ} & UX \otimes UY \otimes UZ \\ \downarrow U^2 & & \downarrow UX \otimes U^2 \\ U(X \boxtimes Y \boxtimes Z) & \xrightarrow[U_2]{} & UX \otimes U(Y \boxtimes Z) \end{array}$$

Such a U is called separable Frobenius monoidal.

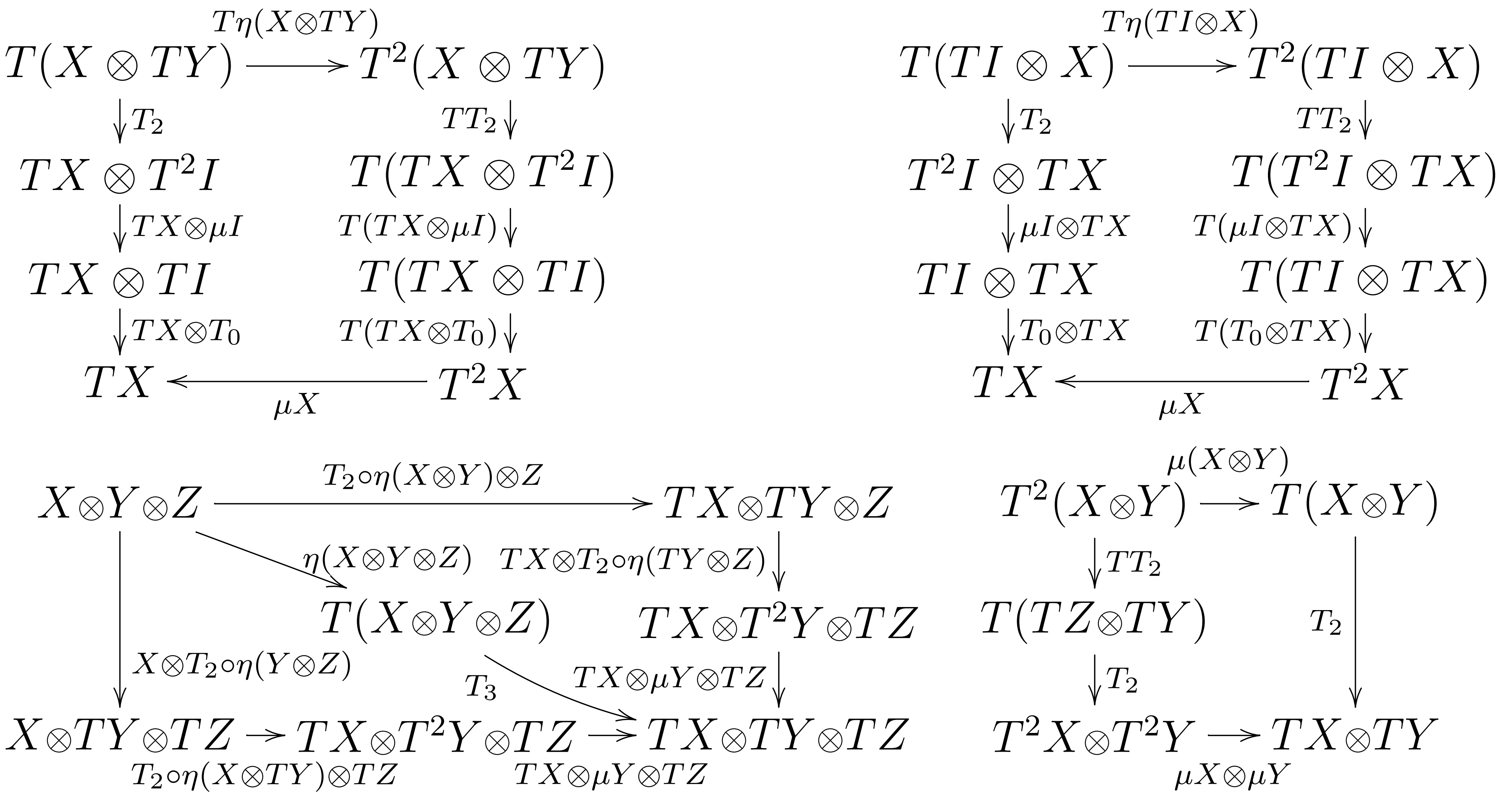
For a WBA H, $X \boxtimes Y = \{x.1_1 \otimes y.1_2 \mid x \in X,\ y \in Y\}$.

Weak bimonads.

Definition. A weak bimonad on a monoidal category $(\mathcal{M}, \otimes, I)$ is a monad $\mathcal{M} \xrightarrow{T} \mathcal{M}$ equipped with the additional str needed for $\mathcal{M}_T$ to be monoidal and $\mathcal{M}_T \xrightarrow{U} \mathcal{M}$ to be separable Frobenius monoidal.

¿ Explicit description ?

Theorem. Let T be a monad on a monoidal category $(\mathcal{M}, \otimes, I)$ in which idempotents split. To give T the str of a weak bimonad is equivalently to give the endofunctor T the str of an opmonoidal functor s.t.



Theorem. Let T be a monad on a monoidal category $(\mathcal{M}, \otimes, I)$ in which idempotents split. To give T the str of a weak bimonad is equivalently to give the endofunctor T the str of an opmonoidal functor s.t. ...

... five axioms (relating the monad str & the opmonoidal str).

For a WBA H (in a braided monoidal category $\mathcal{M}$ with split idempotents), $(-) \otimes H : \mathcal{M} \to \mathcal{M}$ is a weak bimonad.

¿ Interpretation of the axioms via the monoidal str of $\mathcal{M}_T$?

Theorem. For a weak bimonad T on $(\mathcal{M}, \otimes, I)$,

- The monoidal unit $* \xrightarrow{R} \mathcal{M}_T$

is a weak lifting of $* \xrightarrow{I} \mathcal{M} \xrightarrow{T} \mathcal{M}$.

- The monoidal product $\mathcal{M}_T \times \mathcal{M}_T \xrightarrow{\boxtimes} \mathcal{M}_T$

is a weak lifting of $\mathcal{M} \times \mathcal{M} \xrightarrow{\otimes} \mathcal{M}$.

- The associativity constraint $((- \boxtimes -) \boxtimes -) \xrightarrow{\cong} (- \boxtimes (- \boxtimes -))$

is a weak lifting of $((- \otimes -) \otimes -) \xrightarrow{\cong} (- \otimes (- \otimes -))$.

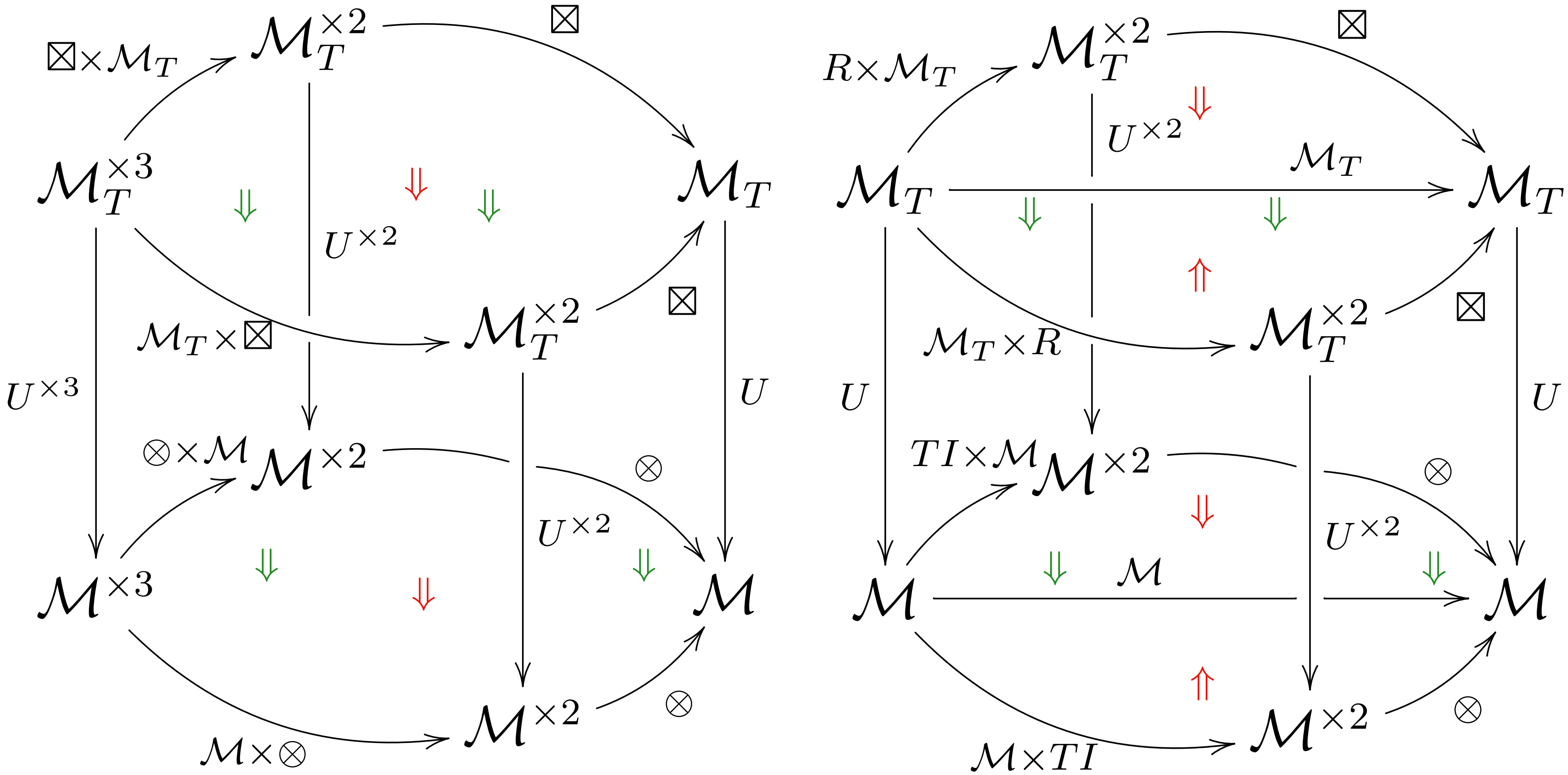
- The left unit constraint $R \boxtimes (-) \xrightarrow{\cong} \mathcal{M}_T$

is a weak lifting of $TI \otimes (-) \xrightarrow{T_0 \otimes (-)} \mathcal{M}$

the right unit constraint $(-) \boxtimes R \xrightarrow{\cong} \mathcal{M}_T$

is a weak lifting of $(-) \otimes TI \xrightarrow{(-) \otimes T_0} \mathcal{M}$

That is, there exist split natural monomorphisms $UR \Rightarrow TI$ and $U(- \boxtimes -) \overset{U_2}{\Rightarrow} U(-) \otimes U(-)$ s.t.



Weak bimonads vs bimonads.

WBA $=$ bialgebroid over a separable Frobenius algebra

¿ Generalization ?

$(\mathcal{M}_T, \boxtimes, R) \stackrel{U}{\to} (\mathcal{M}, \otimes, I)$ is (separable Frobenius) monoidal $\Rightarrow$

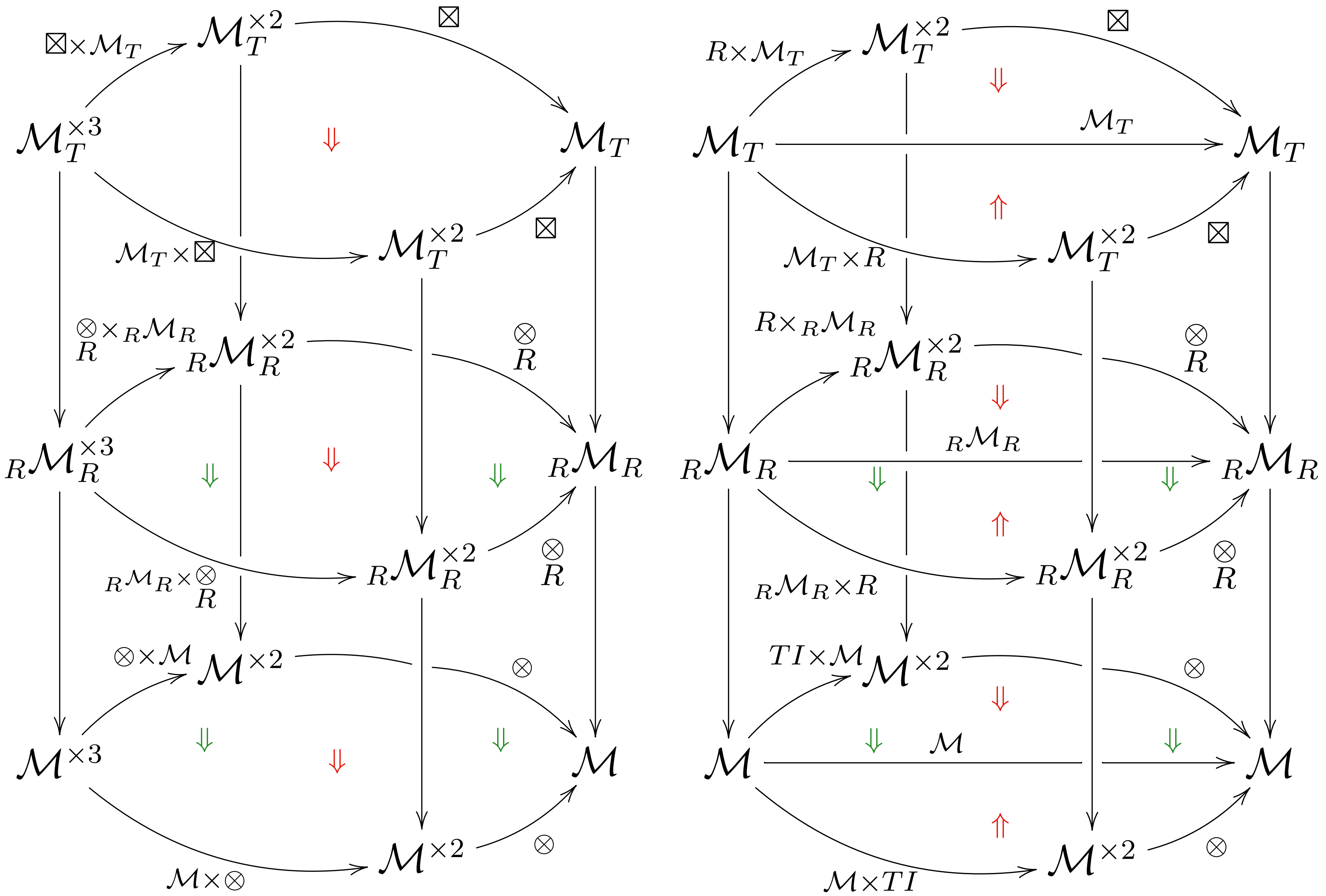
• R is a (separable Frobenius) monoid

• If idempotent morphisms in $\mathcal{M}$ split, then

$$U = (\ \mathcal{M}_T \xrightarrow[\text{monadic, strong monoidal}]{} {}_R\mathcal{M}_R \xrightarrow[\text{forgetful}]{} \mathcal{M}\ ) \qquad \Rightarrow$$

$\exists$ a bimonad $\widetilde{T} : {}_R\mathcal{M}_R \to {}_R\mathcal{M}_R$ s.t. $\mathcal{M}_T \simeq ({}_R\mathcal{M}_R)_{\widetilde{T}}$.

$T \mapsto \widetilde{T}$ is the object map of a category equivalence.



Weak Hopf algebras.

Definition. A WBA H is a weak Hopf algebra if $\exists\ S : H \longrightarrow H$ s.t.

$$
\begin{array}{ccc}
H & \xrightarrow{H\otimes\Delta\circ\eta} & H^{\otimes 3} \\
\downarrow{\scriptstyle \Delta} & & \\
H^{\otimes 2} & & \downarrow{\scriptstyle \varepsilon\circ\mu^{op}\otimes H} \\
\downarrow{\scriptstyle H\otimes S} & & \\
H^{\otimes 2} & \xrightarrow[\mu]{} & H
\end{array}
\qquad
\begin{array}{ccc}
H & \xrightarrow{\Delta\circ\eta\otimes H} & H^{\otimes 3} \\
\downarrow{\scriptstyle \Delta} & & \\
H^{\otimes 2} & & \downarrow{\scriptstyle H\otimes\varepsilon\circ\mu^{op}} \\
\downarrow{\scriptstyle S\otimes H} & & \\
H^{\otimes 2} & \xrightarrow[\mu]{} & H
\end{array}
\qquad
\begin{array}{ccc}
H & \xrightarrow{S} & H \\
\downarrow{\scriptstyle \Delta^2} & & \uparrow{\scriptstyle \mu^2} \\
H^{\otimes 3} & \xrightarrow[S\otimes H\otimes S]{} & H^{\otimes 3}
\end{array}
$$

For a WBA H, there are idempotent maps

$$E : H \otimes H \to H \otimes H, \qquad h' \otimes h \mapsto h'1_1 \otimes h1_2$$
$$F : H \otimes H \to H \otimes H, \qquad h' \otimes h \mapsto \varepsilon(1_1 1_{1'})h'1_2 \otimes 1_{2'}h.$$

Theorem. [Caenepeel & De Groot] A WBA H is a WHA iff for

$\beta := (H \otimes H \xrightarrow{M\otimes\Delta} H \otimes H \otimes H \xrightarrow{\mu\otimes H} H \otimes H)$, $\exists\, \widetilde{\beta} : H \otimes H \to H \otimes H$ s.t. $\widetilde{\beta} \circ E = \widetilde{\beta} = F \circ \widetilde{\beta}$; $\beta \circ \widetilde{\beta} = E$; $\widetilde{\beta} \circ \beta = F$.

¿ Generalization ?

For a weak bimonad T, there are idempotent natural transformations

$$E_{X,Y} : TX \otimes TY \to TX \otimes TY; \quad F_{X,Y} : T(TX \otimes Y) \to T(TX \otimes Y),$$

don't mind their explicit form.

Definition. A weak Hopf monad is a weak bimonad T s.t. for

$$\beta := (\, T(TX \otimes Y) \xrightarrow{T_2} T^2X \otimes TY \xrightarrow{\mu X \otimes TY} TX \otimes TY \,)$$

$\exists\ \widetilde{\beta} : TX \otimes TY \to T(TX \otimes Y)$ s.t. $\widetilde{\beta} \circ E = \widetilde{\beta} = F \circ \widetilde{\beta}$; $\beta \circ \widetilde{\beta} = E$; $\widetilde{\beta} \circ \beta = F$.

Theorem. For a WBA H, $(-) \otimes H$ is a weak Hopf monad iff H is a WHA.

Theorem. If in a monoidal category $\mathcal{M}$ idempotents split, then a weak bimonad T on $\mathcal{M}$ is a weak Hopf monad if $\widetilde{T}$ on ${}_R\mathcal{M}_R$ is a Hopf monad.

Thank you for your attention ☺.