BI-POLARIZATION: ENDOGENEOUS CUT-OFF VALUE SEPARATING SUBGROUPS[1]

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ABSTRACT

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I. INTRODUCTION

Suppose we face a bimodal distribution, where there are two segments of the population with the same income within the segments but very different income between them. On top of that, the huge majority of the people concentrate on the lower income segment. In summary, we face a distribution of two spikes with a very different proportion of population. This situation may roughly represent the distribution in many underdeveloped countries. To analyze bipolarization, is it reasonable to select the median or mean incomes as the cut-off values separating the groups? This paper addresses this question proposing an endogenous method to obtain the cut-off subgroups-separating income value. There are two possible extensions within the discrete framework of bipolarization. The first line of study is to make the cut-off endogenous and the second, to select the number of groups. This paper concentrates on the first issue.

The existing literature does not provide an argument for choosing the cut-off separating value when computing bipolarization indices. Wolfson (1994) suggested the median value to be adopted. Esteban and Ray (1994) did not explicitly give a value (as they typically assume pre-agroupated data, although in many applied studies is computed for discrete micro-data distributions) and Esteban *et al.* (1999) proposed the mean value to be considered. They argued that this value minimized the approximation error ε, that is, the within-groups Gini coefficient. We suggest a method that goes a step forward along this line. We select the value that maximizes polarization, a conservative criterion that allows paying attention to the largest possible antagonism. We shall call this income separating value the optimal cut-off.

Moreover, we postulate the following axiom: under these bimodal distributions, the maximum polarization should be attained at the income separating value. We prove that a number of existing indices in the literature: ER, ZK and EGR indices verify this axiom. This axiom is a minimal requirement for any polarization index. Another related result is that the polarization for any cut-off value is maximum at the equal- populated groups case (as in ER, theorem 2, page 837, 1994); and the less equal the population of the groups is, the smaller the (maximum) polarization.

Perhaps this verification is obvious for the polarization indices established in terms of within- and between-groups inequality components, but it is not so obvious for the ER index which is a product of a concave and a convex function. The result can be proved for this case, as the concave component dominates in the end to guarantee the maximum value at the percentile separating value.

Let x = $(x_1,\ldots,x_n)\in\Re^n_{++}$ be an n-dimensional vector of positive incomes, ordered in increasing values such that $x_1 \leq x_2 \leq \ldots \leq x_n$, and letting x_i be the income of the ith person. Vector x is normalized to the mean income value μ, m is the median income value and $F(x)$ is the income distribution function that belongs to the class of all income distribution functions, $\varphi(x)$. Moreover, z is the income that separates the distribution in two different groups, q_z is the corresponding population quantile and $L(q_z)$ is the Lorenz curve at that value.

DEFINITION: q_z^* is the quantile that maximizes a polarization index $P(\cdot)$. That is, $P(F;q_z^*) \geq P(F;q_z) \quad \forall q_z \in (q_a, q_b)$.

NOTE: It does not necessarily coincide with the minimization of within-groups dispersion, as it takes into account also information about the between-groups inequality.

AXIOM 1: Assume the bimodal distribution $x^c = (\underbrace{x_1,...,x_1}_{n_1};\underbrace{x_2,...,x_2}_{n_2}) \in \Re^n_{++}$, where n_1 and n_2 are the subgroup dimensions, μ_1 and μ_2 are the mean income values for each income subgroup and $n = n_1 + n_2$. Then:

$$q_z^* = \frac{n_1}{n}$$

THEOREM 1: The ER polarization index always verifies axiom 1.

Proof: We have to prove that the ER index has a maximum value at q_z^*. As the ER index is not differentiable at q_z^*, it suffices to prove that the ER polarization measure is strictly increasing for all q_z below q_z^* and strictly decreasing for all q_z above q_z^*. We now prove that the ER index is strictly increasing for all q_z below the cut-off value q_z^*. Provided that:

$$P_z^{ER}(F;\alpha) = \left[q_z^\alpha + (1-q_z)^\alpha\right] G_z^B(F)$$

(see Prieto *et al.*, 2005) and

$$G_z^B(F) = q_z - L(q_z)$$
$$= q_z - \frac{1}{\mu}\int_0^{q_z} \mu_1 dF = q_z\left(1 - \frac{\mu_1}{\mu}\right)$$

the ER polarization index (for two income groups separated by the z income value) can be written in the following way:

$$P_z^{ER}(F;\alpha) = \left[q_z^{\alpha+1} + (1-q_z)^{\alpha} q_z\right]\left(1 - \frac{\mu_1}{\mu}\right)$$

Since changes in q_z do not affect μ_1 below the cut-off value q_z^*, the first derivative is:

$$\left.\frac{\partial P_z^{ER}(F;\alpha)}{\partial q_z}\right|_{q_z \in (0, q_z^*)} = \left[(\alpha+1)q_z^{\alpha} - \alpha(1-q_z)^{\alpha-1} q_z + (1-q_z)^{\alpha}\right]\left(1 - \frac{\mu_1}{\mu}\right). \tag{1}$$

The last term on the right-hand side of expression (1) is positive and we prove that the first term in brackets is also positive. That is, we shall verify:[2]

$$\left[(\alpha+1)q_z^{\alpha} - \alpha(1-q_z)^{\alpha-1} q_z + (1-q_z)^{\alpha}\right] > 0 \quad q_z \in (0,1), \quad \alpha \in [1, 1.6]. \tag{2}$$

We make the following transformation: $q_z = \frac{t}{1+t}$ where $t \in (0, \infty)$. Introducing this change of variable into expression (2) we obtain $\frac{1}{(1+t)^{\alpha}}\left[(\alpha+1)t^{\alpha} - \alpha t + 1\right]$. Therefore,

[2] Notice that we consider an open interval for q_z to exclude single income group cases.

$g(t,\alpha) = \left[(\alpha+1)t^{\alpha} - \alpha t + 1\right]$ must be positive for $t \in (0,\infty)$ and $\alpha \in [1,1.6]$. We have for a given α value:

$$\frac{\partial g(t,\alpha)}{\partial t} = \alpha(\alpha+1)\left[t^{\alpha-1} - \frac{1}{\alpha+1}\right] \qquad \text{and}$$

$$\frac{\partial^2 g(t,\alpha)}{\partial t^2} = \alpha(\alpha+1)(\alpha-1)\, t^{\alpha-2} \geq 0$$

therefore $g(t,\alpha)$ is a convex function which has a minimum value for a given α. The minimum value of $g(t,\alpha)$ is attained at t_{min}, that is:

$$t_{\min}(\alpha) = \left(\frac{1}{\alpha+1}\right)^{\frac{1}{\alpha-1}} \quad t \in (0,\infty), \quad \alpha \in [1,1.6]$$

$$\begin{aligned} g(t_{\min}(\alpha),\alpha) &= \left\{(\alpha+1)\left[t_{\min}(\alpha)\right]^{\alpha-1} - \alpha\right\} t_{\min}(\alpha) + 1 \\ &= (1-\alpha)(\alpha+1)^{-\frac{1}{\alpha-1}} + 1 \\ &= 1 - (\alpha-1)(\alpha+1)^{-\frac{1}{\alpha-1}} \end{aligned}$$

We define $a = \alpha - 1$, then

$$g(a) = 1 - a(2+a)^{-\frac{1}{a}}, \quad a \in [0,0.6]$$

We have to prove that $g(a)$ is always positive for $a \in [0,0.6]$. First, we prove that $g(a)$ is a decreasing function. A sufficient condition is that $h(a) = (2+a)^{-\frac{1}{a}}$ is positive and increasing for $a \in [0,0.6]$. The first derivative

$$\begin{aligned}\frac{\partial h(a)}{\partial a} &= -\frac{1}{a}(2+a)^{-\frac{1+a}{a}} + (2+a)^{-\frac{1}{a}} \ln(2+a)\frac{1}{a^2} \\ &= \frac{1}{a^2(2+a)^{\frac{1}{a}+1}}\left[-a + (2+a)\ln(2+a)\right]\end{aligned} \tag{3}$$

The first term on the right-hand side of expression (3) is positive. We define $m(a)$ as the last term of expression (3) in brackets. We have to prove that $m(a)$ is positive for $a \in [0,0.6]$. Provided that $m(a)$ is an increasing function, $\frac{\partial m(a)}{\partial a} = \ln(2+a) > 0$, $m(a) \geq m(0) = 2\ln(2) > 0$. Hence, $g(a)$ is in fact decreasing which implies:

$$g(a) \geq g(0.6) = 1 - 0.6(2+0.6)^{-\frac{1}{0.6}} = 0.878 > 0 \quad \forall a \in [0,0.6]$$

Then, expression (2) is positive and we obtain the first part of the result. We proceed to prove that the ER index is strictly decreasing for all q_z above the cut-off value q_z^*. The first derivative is:

$$\begin{aligned}&\left.\frac{\partial P_z^{ER}(F;\alpha)}{\partial q_z}\right|_{q_z \in (q_z^*,1)} = \\ &\frac{1}{\mu}\left\{\left[(\alpha+1)q_z^{\alpha} - \alpha(1-q_z)^{\alpha-1}q_z + (1-q_z)^{\alpha}\right](\mu-\mu_1) - \left[q_z^{\alpha+1} + (1-q_z)^{\alpha}q_z\right]\frac{\partial \mu_1}{\partial q_z}\right\}\end{aligned} \tag{4}$$

where μ_1 is equal to:

$$\begin{aligned}\mu_1 &= \frac{N}{n_z}\int_0^{q_z} xdF(x)\\ &= \frac{1}{q_z}\left[\int_0^{q_z^*} xdF(x) + \int_{q_z^*}^{q_z} xdF(x)\right]\\ &= \frac{1}{q_z}\left[x_1 q_z^* + (q_z - q_z^*)x_2\right]\\ &= \frac{(x_1 - x_2)q_z^*}{q_z} + x_2\end{aligned}$$

and the first derivative of μ_1 is $\frac{(x_2 - x_1)q_z^*}{q_z^2}$. We obtain the following expression for the income difference between μ and μ_1:

$$\begin{aligned}\mu - \mu_1 &= (q_z - 1)\mu_1 + (1 - q_z)\mu_2\\ &= \left[\mu_2 - \frac{(x_1 - x_2)q_z^*}{q_z} - x_2\right](1 - q_z)\\ &= \left[\mu_2 - x_2 + q_z\frac{\partial \mu_1}{\partial q_z}\right](1 - q_z)\end{aligned}$$

where $\mu_2 - x_2$ is negative. The expression $\left[(\alpha+1)q_z^\alpha - \alpha(1-q_z)^{\alpha-1}q_z + (1-q_z)^\alpha\right](\mu_2 - x_2)$ is also negative, therefore, to end with the demonstration we just need to prove that

$$\left\{\left[(\alpha+1)q_z^\alpha - \alpha(1-q_z)^{\alpha-1}q_z + (1-q_z)^\alpha\right](1-q_z)q_z - \left[q_z^{\alpha+1} + (1-q_z)^\alpha q_z\right]\right\}\frac{\partial \mu_1}{\partial q_z}$$

is negative. Taking into consideration the transformation $q_z = \frac{t}{1+t}$ where $t \in (0, \infty)$ we have to prove:

$$\frac{1}{(1+t)^{\alpha}}\left[(\alpha+1)t^{\alpha}-\alpha t+1\right]\frac{t}{(1+t)^{2}}-\frac{1}{(1+t)^{\alpha+1}}\left[t^{\alpha+1}+t\right]<0 \quad t\in(0,\infty), \quad \alpha\in[1,1.6]$$

After several operations, what we have to prove for a given $\alpha \in [1,1.6]$ is:

$$f(t,\alpha)=-t^{\alpha}+\alpha t^{\alpha-1}-(\alpha+1)<0 \qquad t\in(0,\infty), \quad \alpha\in[1,1.6]$$

It can be shown that the function $f(\cdot)$ is concave with respect t so there is a maximum. Therefore, it is sufficient to prove that $f(\cdot)$ is negative for its maximum value. After equalizing the first derivative of the function $f(\cdot)$ to zero we obtain $t_{\max}=\alpha-1$. The following expression is obtained when $t_{\max}$ is used:

$$\begin{aligned} f(t_{\max}) &= -(\alpha-1)^{\alpha}+\alpha(\alpha-1)^{\alpha-1}-(\alpha+1) \\ &= (\alpha-1)^{\alpha}\left[\frac{\alpha}{\alpha-1}-1\right]-(\alpha+1) \\ &= (\alpha-1)^{\alpha-1}-(\alpha+1) \end{aligned}$$

The function $f(t_{\max})$ is decreasing in α, therefore, whether $f(t_{\max})$ is negative for the lowest value of parameter α, the proof is over. The function $f(t_{\max})$ is not defined in $\alpha=1$ so we calculate the limit of $f(t_{\max})$ for that value. After applying the L'Hopital rule we end with $\lim_{\alpha\to 1} f(t_{\max})=-2<0$.

However, theorem 1 does not tell us anything about how these maximum polarization values are related one another. We know from theorem 2 in ER (1994) that the global maximum polarization value for any n_1 and n_2 must be attained at $n_1 = n_2$. In theorem 2 it is proved that the maximum polarization values are decreasing and converge to zero as the population size of one of the groups tends to zero.

XXXXIn the theorem 1, we n_1 and n_2 are given, and given this restriction the maximum polarization is achieved by setting the cutoff at the kink between n_1 and n_2. We prove that both statements are not inconsistent each otherxxxxx

THEOREM 2: The maximum ER polarization index for any n_1 and n_2 is strictly decreasing with respect to z departing from the median value and converges to zero for z at both extremes of the income distributions.

The interpretation of this theorem is quite intuitive. The closer we are to the extreme of the income distribution, the lower the polarization is. Moreover, at the limit, it reflects the idea that polarization tends to be negligible if the optimal cutoff value is at the extreme, which is the case when there is only one income group in practice.

The following results generalize theorem 1 and 2 for a very general class of bipolarization indices. Define the bipolarization measure $P = p\left(I^B, I_1^W, I_2^W\right)$, where $\partial p/\partial I^B>0$, $\partial p/\partial I_i^W<0$ for i=1,2 and I^B is the standard between-groups S-convex inequality component, and I_i^W is the within-groups S-convex inequality component of the ith-group. The class of these bipolarization measures is denoted by Π.

It is shown in Rodríguez and Salas (2005) that this class of polarization measures is consistent with the two basic axioms in the Foster and Wolfson framework (Foster and

Wolfson, 1992): increasing spreads (IS) and increased bipolarity (IB), see Tsui and Wang (2000) and Chakravarty and Majumder (2001).

THEOREM 3: Any bipolarization index (for two income groups separated by the z income value) $P \in \Pi$ satisfies the Axiom.

Proof:

COROLLARY:

The ZK polarization index applied to a disjoint partition in two groups satisfies the above Axiom.

FALTAN CONCLUSIONES

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