ON THE BRAUER GROUP OF A COCOMMUTATIVE COALGEBRA

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1 INTRODUCTION

That the Brauer group of a cocommutative coalgebra need not be equal to the Brauer group of the dual algebra is a good thing because it provides a certain richness to the theory. On the other hand, the relation between the coalgebraic theory and the Brauer group of a local ring is one of the main motivations to further the study in the coalgebra case.

Since a cocommutative coalgebra may be viewed as the direct sum of i-rreducible coalgebras its Brauer group reduces to a product of Brauer groups of irreducible cocommutative coalgebras. Hence it is enough to restrict our attention to the irreducible case. In this paper, for a cocartesian diagram of cocommutative irreducible coalgebras,

$$\begin{tikzcd} \overline{C} \arrow[r, "f_1"] \arrow[d, "f_2"'] & C_1 \arrow[d, "g_1"] \\ C_2 \arrow[r, "g_2"] & C \end{tikzcd}$$

we construct a Mayer-Vietoris exact sequence (Theorem 4.3)

$$0 \longrightarrow Br(C) \longrightarrow Br(C_1) \oplus Br(C_2) \longrightarrow Br(\overline{C}).$$

If we compare to the Mayer-Vietoris sequence in ring theory (see [1]), there is an interesting simplification presented in the coalgebra theory due to the fact (Proposition 4.1) that the Picard group of a cocommutative coalgebra vanishes. The sequence is derived from a dual version of the classical Milnor's theorem (Theorem 3.6). This relates the category of finitely cogenerated injective comodules over the coalgebras appearing in the cocartesian diagram. More concretely, the category of finitely cogenerated C-comodules $I(C)$ is equivalent to the fibre product category $I(C_1) \times_{I(\overline{C})} I(C_2)$. Similar theorems are deduced for the category of finitely cogenerated injective cogenerators (Corollary 3.9) and Azumaya coalgebras (Corollary 3.11).

As application of the Mayer-Vietoris sequence, for a cocommutative irreducible coalgebra C over a field k, its universal connected coalgebra $R(C)$ can be viewed in a cocartesian diagram involving C, its coradical C_0, and the ground field (Example 4.4). An exact sequence relating the Brauer group of the above coalgebras is derived (Corollary 4.5). When the ground field is finite, then $Br(C) \cong Br(R(C))$.

2 NOTATION AND PRELIMINARIES

Throughout k is a fixed field. All algebras, coalgebras, vector spaces and unadorned $\otimes$ are over k. We use the usual sigma notation for coalgebras and comodules. $\mathcal{M}^C$ denotes the category of right C-comodules and for right C-comodules X,Y, $Com_{-C}(X,Y)$ denotes the vector space of all C-colinear maps from X to Y.

Let $U : \mathcal{M}^C \to \mathcal{M}_k$ be the forgetful functor. U has a right adjoint functor $-\otimes C$. For a comodule X_C, with structure map ρ_X, and a vector space V, the adjoint isomorphism

$$\Phi : Com_{-C}(X, V \otimes C) \to Hom_k(X, V) \tag{1}$$

is given by $\Phi(F) = (1 \otimes \varepsilon)F$. The inverse Φ^{-1} is given by $\Phi^{-1}(f) = (f \otimes 1)\rho_X$. In particular, if $V = k$, then

$$Com_{-C}(X, C) \cong X^*.$$

Let $\alpha : C \to D$ be a coalgebra map. Every right C-comodule X may be viewed as a right D-comodule with the structure map:

$$(1 \otimes \alpha)\rho_X : X \to X \otimes C \to X \otimes D.$$

In this case, we will say X_D is induced by X_C via α. A $(C-D)$-bicomodule is a left C-comodule and a right D-comodule X, denoted by ${}_C X_D$, such that the C-comodule structure map $\rho_X : X \to X \otimes C$ is D-colinear.

Cotensor product : Let M be a right C-comodule and N a left C-comodule with structure maps ρ_M and ρ_N respectively. The cotensor product $M \square_C N$ is the kernel of the map

$$\rho_M \otimes 1 - 1 \otimes \rho_N : M \otimes N \to M \otimes C \otimes N.$$

The functors $M \square_C -$ and $- \square_C N$ are left exact and preserve direct sums. If ${}_C M_D$ and ${}_D N_E$ are bicomodules, then $M \square_C N$ is a $(C-E)$-bicomodule with comodule structures induced by those of M and N.

Let $X \in \mathcal{M}^C$, we say that X is finitely cogenerated if it is isomorphic to a subcomodule of $C^{(n)}$ for some $n \geq 1$, where $C^{(n)}$ denotes the direct sum of C n times. X is free if X is isomorphic to $C^{(I)}$ for some set I. X is said to be a cogenerator if for any comodule $M \in \mathcal{M}^C$, $M \hookrightarrow X^{(I)}$, for some set I, as comodules. X is injective if the functor $Com_{-C}(-, X)$ is exact, or equivalently, the functor $X \square_C -$ is exact.

Co-hom functor: A right C-comodule X is quasi-finite if $Com_{-C}(Y, X)$ is finite dimensional for any finite dimensional comodule $Y \in \mathcal{M}^C$. Now, we recall from [2] the definition of the co-hom functor:

LEMMA 2.1 *Let ${}_C X_D$ be a bicomodule. Then X_D is quasi-finite if and only if the functor $-\Box_C X : \mathcal{M}^C \to \mathcal{M}^D$ has a left adjoint functor, denoted by $h_{-D}(X, -)$. That is, for comodules Y_D and Z_C,*

$$Com_{-C}(h_{-D}(X,Y), Z) \cong Com_{-D}(Y, Z\Box_C X) \tag{2}$$

where,

$$h_{-D}(X,Y) = lim_{\to_\mu} Com_{-D}(Y_\mu, X)^*$$

is a right C-comodule and $\{Y_\mu\}$ is a directed family of finite dimensional subcomodules of Y_D such that $Y_D = \cup_\mu Y_\mu$. We denote by θ the canonical D-colinear map $Y \to h_{-D}(X,Y)\Box_C X$ which corresponds to the identity map $h_{-D}(X,Y) \to h_{-D}(X,Y)$ in (2).

If we assume that X_D is a quasi-finite comodule, then $e_{-D}(X) = h_{-D}(X,X)$ is a coalgebra, called the co-endomorphism coalgebra of X. The comultiplication of $e_{-D}(X)$ corresponds to $(1 \otimes \theta)\theta : X \to e_{-D}(X) \otimes e_{-D}(X) \otimes X$ in (2) when $C = k$ and the counit of $e_{-D}(X)$ corresponds to the identity map 1_X.

Let ${}_C X_D$ be a bicomodule such that X_D is quasi-finite. Then there exists a coalgebra map $\lambda : e_{-D}(X) \to C$ such that the left C-comodule structure map equals $(\lambda \otimes 1)\theta$. Conversely, a coalgebra map $\lambda : e_{-D}(X) \to C$ makes X into a $(C - D)$-bicomodule.

In this paragraph we recall the notion of Azumaya coalgebra, the Brauer group of a cocommutative coalgebra and some of its properties. We refer the reader to [3] for the construction of the Brauer group and to [4] for the relative Brauer group and its cohomological interpretation.

Brauer group: A coalgebra map $f : D \to E$ is said to be cocentral if

$$\textstyle\sum_{(c)} f(c_1) \otimes c_2 = \sum_{(c)} f(c_2) \otimes c_1.$$

For a coalgebra D, there exists a cocommutative coalgebra $Z(D)$ with a surjective, cocentral coalgebra map $1^d : D \to Z(D)$ which satisfies the universal property: for any cocentral coalgebra map $f : D \to E$ there is a unique coalgebra map $g : Z(D) \to E$ such that $f = g1^d$. $(Z(D), 1^d)$ is called the cocenter of D. In fact, $Z(D) = h_{-D^e}(D,D) = e_{-D^e}(D)$ where $D^e = D \otimes D^{op}$.

Let C be cocommutative coalgebra. A C-coalgebra D is a k-coalgebra with a cocentral coalgebra map $\epsilon_D : D \to C$, called the C-counit. A k-coalgebra map $f : D \to E$ is a C-coalgebra map if $\epsilon_E f = \epsilon_D$. A C-coalgebra D is said to be cocentral if $Z(D) \cong C$ and D is said to be C-coseparable if there is a D-bicomodule map $\pi : D\Box_C D \to D$ such that $\pi\Delta = 1_D$. An Azumaya C-coalgebra is defined to be a C-cocentral and C-coseparable coalgebra. If P is an injective quasi-finite cogenerator then $e_{-C}(P)$ is an Azumaya coalgebra. Denote by $B(C)$ the set of the isomorphism classes of Azumaya C-coalgebras. An equivalence relation (indeed a Morita-Takeuchi equivalence relation) is introduced in $B(C)$ as follows: if $E, F \in B(C)$, then E is equivalent to F, denoted by $E \sim F$, if there exist two quasi-finite injective cogenerators M, N in $\mathcal{M}^C$ such that

$$E\Box_C e_{-C}(M) \cong F\Box_C e_{-C}(N).$$

The quotient set $B(C)/\sim$, denoted by $Br(C)$, is an abelian group with the multiplication $[E][F] = [E\Box_C F]$, unit element $[C]$ and for $[E]$ the inverse is $[E^{op}]$. The group $Br(C)$ is called the *Brauer group* of the cocommutative coalgebra C.

Let $\eta : D \to C$ a map of cocommutative coalgebras, then η induces a group homomorphism $\eta_* : Br(C) \to Br(D)$ given by $\eta_*([E]) = [E\Box_C D]$ for all $[E] \in Br(C)$. If C is of finite dimension, the Brauer group of C is isomorphic to the Brauer group of the commutative algebra C^*. Since C is cocommutative, C can be expressed as $C = \oplus_{i\in I} C_i$ where each C_i is an irreducible subcoalgebra and we have that $Br(C) \cong \prod_{i\in I} Br(C_i)$. This decomposition has two consequences:

1) In general $Br(C)$ is not torsion. Let $\mathbb{Q}$ be the rational number field and C the group like coalgebra $C = \oplus_{n\in\mathbb{N}}\mathbb{Q}$. It is well-known that for any $n \in \mathbb{N}$ there is $[A_n] \in Br(\mathbb{Q})$ of order n. The coalgebra $A = \oplus_{n\in\mathbb{N}} A_n^*$ is C-Azumaya, cf. [3, Ex. 4.7], and $[A]$ does not have finite order in $Br(C)$.
2) To compute the Brauer group of a cocommutative coalgebra it is enough to compute the Brauer group of irreducible coalgebras.

If C is irreducible, then the map $(-)^* : Br(C) \to Br(C^*), [D] \mapsto [D^*]$ is a group homomorphism. If in addition C is coreflexive then $Br(C) \cong Br(C^*) \cong Br(C_0)$ where C_0 is the coradical of C and $Br(C_0)$ is isomorphic to the classical Brauer group of some finite field extension.

3 MILNOR'S THEOREM

Let C be a cocommutative coalgebra. We consider the following categories of C-comodules which are equipped with a product:

(1) $I(C)$, the category of finitely cogenerated injective left C-comodules and C-colinear maps with product the direct sum. Then $P \in I(C)$ if and only if P is a direct summand of $C^{(n)}$ for some $n \geq 1$.

(2) $CI(C)$, the finitely cogenerated injective cogenerators C-comodules and C-isomorphisms, with product the cotensor product $\square_C$. Then $M \in CI(C)$ if and only if C is a direct summand of $P^{(n)}$ and P is a direct summand of $C^{(m)}$ for some $m, n \geq 1$.

(3) $Az(C)$, the Azumaya C-coalgebras and C-coalgebra maps with product $\square_C$. We recall that a C-coalgebra A is Azumaya if A is C-coseparable and C-cocentral. If $P \in CI(C)$ then $e_{-C}(P) \in Az(C)$, cf. [3, Cor. 4.2].

Let $f : C \to D$ be a map of cocommutative coalgebras. If $P(C)$ is any of these three categories, then the functor $F(-) = -\square_D C : P(D) \to P(C)$ is a product preserving functor.

If we suppose that C is an irreducible cocommutative coalgebra, then quasi-finite injective comodules are finitely cogenerated injective comodules.

Proposition 3.1 *Let C be an irreducible cocommutative coalgebra and P an injective right C-comodule. Then, P is finitely cogenerated if and only if P is quasi-finite.*

Proof. If P is finitely cogenerated then $P \hookrightarrow C^{(n)}$ for some $n \geq 1$. Let F be a finite dimensional C-comodule, we have from (1)

$$Com_{-C}(F, P) \hookrightarrow Com_{-C}(F, C^{(n)}) \cong Hom(F, k^{(n)}). \cong F^{*(n)}$$

Hence P is quasi-finite. This implication is always true, we do not need the injectivity of P and irreducibility of the coalgebra.

Conversely, we suppose that P is quasi-finite. As P is injective and C is irreducible it follows that P is free, cf. [5, A.2.2]. But a comodule that is quasi-finite and free has to be finitely cogenerated. ∎

Definition 3.2 *A diagram of cocommutative coalgebras*

$$\begin{tikzcd} \overline{C} \arrow[r, "f_1"] \arrow[d, "f_2"'] & C_1 \arrow[d, "g_1"] \\ C_2 \arrow[r, "g_2"] & C \end{tikzcd} \qquad (3)$$

is a cocartesian diagram if $C \cong \frac{C_1 \oplus C_2}{Im(f_1 - f_2)}$, *or equivalently, the following sequence is exact:*

$$\overline{C} \xrightarrow{f_1 - f_2} C_1 \oplus C_2 \xrightarrow{g_1 \oplus g_2} C \longrightarrow 0.$$

Any $\overline{C}$-comodule may be viewed as a C_i-comodule and every C_i-comodule is a C-comodule, $i = 1, 2$. For $P \in I(C)$ we apply the functor $P\square_C -$ to (3). So, we have:

$$\begin{tikzcd} P\square_C\overline{C} \arrow[r, "\bar{f}_1"] \arrow[d, "\bar{f}_2"'] & P\square_C C_1 \arrow[d, "\bar{g}_1"] \\ P\square_C C_2 \arrow[r, "\bar{g}_2"] & P\square_C C \end{tikzcd}$$

where $\bar{f}_i = 1\square f_i, \bar{g}_i = 1\square g_i$ for $i = 1, 2$. If P_i denotes $P\square_C C_i$, then $P_i \in I(C_i)$ and we have an isomorphism of $\overline{C}$-comodules

$$P_i\square_{C_i}\overline{C} = P\square_C C_i\square_{C_i}\overline{C} \cong P\square_C\overline{C}.$$

We put $\overline{P_i} = P_i\square_{C_i}\overline{C}$, $i = 1, 2$. Then, $\overline{P_1} \cong P\square_C\overline{C} \cong \overline{P_2}$. Let $\psi : \overline{P_1} \cong \overline{P_2}$ be this isomorphism of $\overline{C}$-comodules. If we consider the P_i as C-comodules then $Im(\bar{f}_1 - \bar{f}_2\psi)$ is a C-subcomodule of $P_1 \oplus P_2$ and P is isomorphic to

$$\frac{P_1 \oplus P_2}{Im(\bar{f}_1 - \bar{f}_2\psi)}.$$

Now, we consider the converse problem, i.e, given $P_i \in I(C_i)$ $i = 1, 2$ such that $\psi : P_1\square_{C_1}\overline{C} \cong P_2\square_{C_2}\overline{C}$ is an isomorphism of $\overline{C}$-comodules, is there a C-comodule $P \in I(C)$ such that $P\square_C C_i \cong P_i$ for $i = 1, 2$? If we impose that either f_1 or f_2 in (3) is injective, then the above question has a positive answer.

Let $P_i \in I(C_i)$ $i = 1, 2$ such that $\psi : P_1\square_{C_1}\overline{C} \to P_2\square_{C_2}\overline{C}$ is an isomorphism of $\overline{C}$-comodules. Let ε denote the counit of the coalgebras appearing in (3), $\bar{\varepsilon} = 1 \otimes \varepsilon$ and $\bar{f}_i : P_i\square_{C_i}\overline{C} \to P_i\square_{C_i}C_i$ to be the map $1\square f_i$, then $\bar{\varepsilon} = \bar{\varepsilon}\bar{f}_i$ for $i = 1, 2$. Moreover, if we identify $P_i\square_{C_i}C_i$ and P_i $i = 1, 2$ via $\bar{\varepsilon}$, $Im(\bar{f}_1 - \bar{f}_2\psi) = Im(\bar{\varepsilon} - \bar{\varepsilon}\psi)$. We define

$$P = \frac{P_1 \oplus P_2}{Im(\bar{\varepsilon} - \bar{\varepsilon}\psi)},$$

P is C-comodule. In fact, P is nothing else but the pushout of

$$\begin{tikzcd} P_1\square_{C_1}\overline{C} \arrow[r, "\psi"] \arrow[d, "\bar{\varepsilon}"'] & P_2\square_{C_2}\overline{C} \arrow[r, "\bar{\varepsilon}"] & P_2 \\ P_1 & & \end{tikzcd}$$

From now on, we denote $P_i\square_{C_i}\overline{C}$ by $\overline{P_i}$ $i=1,2$ and P by (P_1,P_2,ψ). We construct the fibred product category $\mathcal{C}=I(C_1)\times_{I(\overline{C})} I(C_2)$:

$Obj(\mathcal{C})$: C-comodules of the form (P_1,P_2,ψ) with $P_i\in I(C_i)$ $i=1,2$ and $\psi:\overline{P_1}\to\overline{P_2}$ an isomorphism of $\overline{C}$-comodules.

$Hom(\mathcal{C})$: Given $(P_1,P_2,\psi),(Q_1,Q_2,\phi)\in\mathcal{C}$ a morphism $f:(P_1,P_2,\psi)\to(Q_1,Q_2,\phi)$ is a pair of C_i-comodule maps $f_i:P_i\to Q_i$ $i=1,2$ making the following diagram commutative:

$$\begin{tikzcd} \overline{P_1} \arrow[r, "\psi"] \arrow[d, "\bar{f}_1"'] & \overline{P_2} \arrow[d, "\bar{f}_2"] \\ \overline{Q_1} \arrow[r, "\phi"] & \overline{Q_2} \end{tikzcd}$$

With these conventions, we have the following facts:

1) $C\cong(C_1,C_2,1_{\overline{C}})$, where $1_{\overline{C}}$ denotes the identity map in $\overline{C}$.

2) f is an isomorphism if and only if f_i is an isomomorphism for $i=1,2$.

3) $(P_1,P_2,\psi)\oplus(Q_1,Q_2,\phi)\cong(P_1\oplus Q_1,P_2\oplus Q_2,\psi\oplus\phi)$.

Next we define elementary maps for coalgebras. Let C be a coalgebra. If $f:C^{(n)}\to C^{(n)}$ is a C-colinear map then f induces n^2 C-colinear maps $f_{ij}:C\to C$ for $i,j=1,...,n$ defined by the composition:

$$C \xrightarrow{l_i} C^{(n)} \xrightarrow{f} C^{(n)} \xrightarrow{\pi_j} C,$$

where l_i is the injection at position i and π_j is the projection at position j for $i,j=1,...,n$. So, f can be represented as a matrix of C-colinear maps $f=(f_{ij})_{i,j}$. Let $End_{-C}(C^{(n)})$ denote the space of right C-colinear maps from

$C^{(n)}$ to $C^{(n)}$. We define a product in $End_{-C}(C^{(n)})$ as follows: Let $f, g \in End_{-C}(C^{(n)})$ be represented as matrices $f = (f_{ij})_{i,j}, g = (g_{ij})_{i,j}$, we define $h = gf$ to be $h = (h_{ij})_{i,j}$ where $h_{ij} = \sum_{k=1}^{n} g_{ik} f_{kj}$. Using (1), $End_{-C}(C^{(n)})$ with this product is isomorphic to the matrix algebra $M_n(C^*)$. The space $Aut_{-C}(C^{(n)})$ of all bijective C-colinear maps from $C^{(n)}$ to $C^{(n)}$ is a subgroup of $End_{-C}(C^{(n)})$ with respect to the above product. We write 1_C for the identity map of C.

DEFINITION 3.3 *An elementary map in $Aut_{-C}(C^{(n)})$ is a C-colinear map obtained from the identity map ($f_{ii} = 1_C, f_{ij} = 0$ if $i \neq j$) replacing some off-diagonal zero entry by some C-colinear map $f_{uv} : C \to C$.*

Let $E_n(C)$ be the subgroup of $Aut_{-C}(C^{(n)})$ generated by all elementary maps. There is a canonical embedding of $Aut_{-C}(C^{(n)})$ in $Aut_{-C}(C^{(n+1)})$ given by

$$\alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & 1_C \end{pmatrix}.$$

For $\psi \in Aut_{-C}(C^{(n)})$ we obtain the following identity:

$$\begin{pmatrix} \psi & 0 \\ 0 & \psi^{-1} \end{pmatrix} =$$

$$\begin{pmatrix} 1_{C^{(n)}} & \psi^{-1} \\ 0 & 1_{C^{(n)}} \end{pmatrix} \begin{pmatrix} 1_{C^{(n)}} & 0 \\ 1_{C^{(n)}} & 1_{C^{(n)}} \end{pmatrix} \begin{pmatrix} 1_{C^{(n)}} & \psi^{-1} - 1_{C^{(n)}} \\ 0 & 1_{C^{(n)}} \end{pmatrix} \begin{pmatrix} 1_{C^{(n)}} & 0 \\ -\psi & 1_{C^{(n)}} \end{pmatrix}.$$

Since the maps on the right hand side are elementary, it follows that $\psi \oplus \psi^{-1} \in E_2(C^{(n)})$.

We denote $C_i \square_C \overline{C}$ by $\overline{C_i}$, then we have $\overline{C_i} = C_i \square_C \overline{C} \cong \overline{C}$ for $i = 1, 2$. We identify $\overline{C_1}, \overline{C_2}$ and $\overline{C}$.

LEMMA 3.4 *Let $\psi, \psi' \in Aut_{-C}(C^{(n)})$ such that $\psi' = \bar{\psi_1} \psi \bar{\psi_2}$ for some $\psi_i \in Aut_{-C_i}({C_i}^{(n)})$ where $\bar{\psi_i} = \psi_i \square 1$ for $i = 1, 2$. Then there is an isomorphism of C-comodules*

$$({C_1}^{(n)}, {C_2}^{(n)}, \psi) \cong ({C_1}^{(n)}, {C_2}^{(n)}, \psi')$$

Proof. The isomorphism is the map induced by $\psi_1 \oplus {\psi_2}^{-1} : {C_1}^{(n)} \oplus {C_2}^{(n)} \to {C_1}^{(n)} \oplus {C_2}^{(n)}$. This map is an isomorphism since $\psi_1, {\psi_2}^{-1}$ are so. ∎

LEMMA 3.5 *We assume that in the cocartesian diagram* (3) *either* f_1 *or* f_2 *is injective. Then,*

i) For all $\psi \in E_n(\overline{C})$, $({C_1}^{(n)}, {C_2}^{(n)}, \psi) \cong C^{(n)}$.

ii) For all $\psi \in Aut_{-C}(C^{(n)})$ *we have*

$$({C_1}^{(n)}, {C_2}^{(n)}, \psi) \oplus ({C_1}^{(n)}, {C_2}^{(n)}, \psi^{-1}) \cong C^{(2n)}.$$

Hence, $({C_1}^{(n)}, {C_2}^{(n)}, \psi)$ *is a finitely cogenerated injective* C*-comodule, that is,* $({C_1}^{(n)}, {C_2}^{(n)}, \psi) \in I(C)$.

Proof. i) We may suppose that $f_1 : \overline{C} \to C_1$ is injective, then ${f_1}^* : {C_1}^* \to \overline{C}^*$ is surjective. Hence, if Φ is the map defined in (1), the composition

$$Com_{-C_1}(C_1, C_1) \xrightarrow{\Phi} {C_1}^* \xrightarrow{{f_1}^*} \overline{C}^* \xrightarrow{\Phi^{-1}} Com_{-\overline{C}}(\overline{C}, \overline{C})$$

is surjective. Put $h = \Phi^{-1}{f_1}^*\Phi$; h is defined by $\langle h(F), c\rangle = \sum_{(c)}\langle \varepsilon, Ff_1(c_1)\rangle c_2$ for all $F \in Com_{-C_1}(C_1, C_1), c \in \overline{C}$.

Let $\phi = (\phi_{ij})_{i,j}$ be an elementary map in $Aut_{-\overline{C}}(\overline{C}^{(n)})$ with distinguished entry $\phi_{uv},\ u \neq v$. Since h is surjective, there is $g_{uv} \in Com_{-C_1}(C_1, C_1)$ such that $h(g_{uv}) = \phi_{uv}$. Let $g = (g_{ij})_{i,j} \in Aut_{-C_1}({C_1}^{(n)})$ defined by $g_{ii} = 1_{C_i}, g_{uv}$ at position (u, v) and the rest zero. Then, we have:

$$\begin{aligned}
\langle f_1\phi_{uv}, c\rangle &= \langle f_1 h(g_{uv}), c\rangle \\
&= \textstyle\sum_{(c)}\langle \varepsilon, g_{uv}f_1(c_1)\rangle f_1(c_2) \\
&= \textstyle\sum_{(g_{uv}(f_1(c)))}\langle \varepsilon, g_{uv}(f_1(c))_1\rangle g_{uv}(f_1(c))_2 \\
&= g_{uv}(f_1(c)) = \langle g_{uv}f_1, c\rangle
\end{aligned}$$

where we have used the fact that g_{uv} is a C_1-comodule map. Using this equality, we obtain that $\phi = \overline{g}\ 1_{\overline{C}^{(n)}}\overline{1}_{C_1^{(n)}}$ and from the preceding lemma,

$$(C_1^{(n)}, C_2^{(n)}, \phi) \cong (C_1^{(n)}, C_2^{(n)}, 1_{\overline{C}^{(n)}}) \cong C^{(n)}.$$

ii) We have $(C_1^{(n)}, C_2^{(n)}, \phi) \oplus (C_1^{(n)}, C_2^{(n)}, \phi^{-1}) \cong (C_1^{(2n)}, C_2^{(2n)}, \phi \oplus \phi^{-1})$. By the identity preceding Lemma 3.4, $\phi \oplus \phi^{-1} \in E_2(\overline{C}^{(n)})$ and from $i)$ we have,

$$(C_1^{(2n)}, C_2^{(2n)}, \phi \oplus \phi^{-1}) \cong C^{(2n)}.$$

Therefore $(C_1^{(n)}, C_2^{(n)}, \phi)$ is a finitely cogenerated injective C-comodule. ∎

THEOREM 3.6 *Suppose that in the cocartesian diagram* (3), *either f_1 or f_2 is injective. Let $P_i \in I(C_i)$ $i = 1, 2$ and let $\phi : \overline{P_1} \cong \overline{P_2}$ be a $\overline{C}$-isomorphism where $\overline{P_i} = P_i \square_C \overline{C}$. Then the C-comodule*

$$(P_1, P_2, \phi) = \frac{P_1 \oplus P_2}{Im(\bar{\varepsilon} - \bar{\varepsilon}\phi)}$$

belongs to $I(C)$. Moreover,

$$(P_1, P_2, \phi)\square_C C_i \cong P_i \qquad i = 1, 2.$$

Finally, let $Q_i \in I(C_i)$ $i = 1, 2$ and let $\phi' : \overline{Q_1} \cong \overline{Q_2}$ be a $\overline{C}$-isomorphism. Then $(P_1, P_2, \phi) \cong (Q_1, Q_2, \phi')$ as C-comodules if and only if $\phi' = \bar{\phi}_1 \phi \bar{\phi}_2^{-1}$ for some isomorphisms $\phi_1 : P_1 \cong Q_1$ and $\phi_2 : P_2 \cong Q_2$.

Proof. Let $P_i \in I(C_i)$, then there is $Q_i \in I(C_i)$ such that $P_i \oplus Q_i \cong C_i^{(n)}$ for $i = 1, 2$. $\phi : \overline{P_1} \cong \overline{P_2}$ induces an isomorphism of $\overline{C}$-comodules ψ given by the composition:

$$\overline{C_1^{(n)} \oplus Q_1} \cong \overline{P_2 \oplus Q_2 \oplus Q_1} \cong \overline{P_1 \oplus Q_1 \oplus Q_2} \cong \overline{C_2^{(n)} \oplus Q_2}.$$

We have $(P_1, P_2, \phi) \oplus (C_1^{(n)} \oplus Q_1, C_2^{(n)} \oplus Q_2, \psi) \cong (C_1^{(2n)}, C_2^{(2n)}, \phi \oplus \psi)$ and $(C_1^{(2n)}, C_2^{(2n)}, \phi \oplus \psi) \in I(C)$ by the foregoing lemma. Hence, $(P_1, P_2, \phi) \in I(C)$.

For the second claim, first we see that for all $\phi \in Aut_{-\overline{C}}(\overline{C}^{(n)})$ we have:

$$(C_1^{(n)}, C_2^{(n)}, \phi)\square_C C_i \cong C_i^{(n)} \qquad i = 1, 2.$$

Let $I = Im(\bar{\varepsilon} - \bar{\varepsilon}\phi)$ and we suppose $i = 1$. We define,

$$\begin{array}{ll} f : (C_1^{(n)}, C_2^{(n)}, \phi)\square_C C_1 \longrightarrow C_1^{(n)} & \sum_i (x_i \oplus y_i + I) \otimes c_i \mapsto \sum_i x_i \varepsilon(c_i), \\ g : (C_1^{(n)}, C_2^{(n)}, \phi^{-1})\square_C C_1 \longrightarrow C_1^{(n)} & \sum_i (x_i \oplus y_i + I) \otimes c_i \mapsto \sum_i x_i \varepsilon(c_i). \end{array}$$

From the preceding lemma $(C_1^{(n)}, C_2^{(n)}, \phi) \oplus (C_1^{(n)}, C_2^{(n)}, \phi^{-1}) \cong C^{(2n)}$ then,

$$\begin{array}{ccccc} [(C_1^{(n)}, C_2^{(n)}, \phi)\square_C C_1] & \oplus & [(C_1^{(n)}, C_2^{(n)}, \phi^{-1})\square_C C_1] & \cong & C_1^{(2n)} \\ \downarrow f & & \downarrow g & & \downarrow f \oplus g \\ C_1^{(n)} & \oplus & C_1^{(n)} & \cong & C_1^{(2n)} \end{array}$$

and $f \oplus g$ is an isomorphism. Hence f and g are isomorphisms. Now, we consider the identity,

$$(P_1, P_2, \phi) \oplus (C_1^{(n)} \oplus Q_1, C_2^{(n)} \oplus Q_2, \psi) \cong (C_1^{(2n)}, C_2^{(2n)}, \phi \oplus \psi).$$

Cotensoring this identity with $-\Box_C C_i$, defining maps similar to the above and then using the foregoing fact we obtain that $(P_1, P_2, \phi)\Box_C C_1 \cong P_1$.

Finally, let $P = (P_1, P_2, \phi)$ and $Q = (Q_1, Q_2, \phi')$ and $\rho : P \cong Q$ an isomorphism of C-comodules. Let

$$\sigma_i : P\Box_C C_i \cong P_i \qquad \tau_i : Q\Box_C C_i \cong Q_i \qquad i = 1, 2,$$

be the isomorphisms of C_i-comodules. We consider the following commutative diagram:

$$\begin{tikzcd}
\overline{P_1} \arrow[r, "{\sigma_1}^{-1}\Box 1"] \arrow[d, "\phi"'] & (P\Box_C C_1)\Box_{C_1}\overline{C} \arrow[r, "\rho\Box 1\Box 1"] \arrow[d] & (Q\Box_C C_1)\Box_{C_1}\overline{C} \arrow[r, "\tau_1\Box 1"] \arrow[d] & \overline{Q_1} \arrow[d, "\phi'"] \\
\overline{P_2} \arrow[r, "{\sigma_2}^{-1}\Box 1"] & (P\Box_C C_2)\Box_{C_2}\overline{C} \arrow[r, "\rho\Box 1\Box 1"] & (Q\Box_C C_2)\Box_{C_2}\overline{C} \arrow[r, "\tau_2\Box 1"] & \overline{Q_2}
\end{tikzcd}$$

We define

$$\mu = \tau_1(\rho\Box 1){\sigma_1}^{-1} : P_1 \longrightarrow Q_1 \qquad \nu = \tau_2(\rho\Box 1){\sigma_2}^{-1} : P_2 \longrightarrow Q_2.$$

Then μ, ν are isomorphisms such that $\phi'\bar{\mu} = \bar{\nu}\phi$. ∎

Corollary 3.7 *$I(C)$ and $I(C_1) \times_{I(\overline{C})} I(C_2)$ are equivalent categories.*

Theorem 3.8 *Assumptions are as in the above theorem. If $P_i \in CI(C_i)$ $i = 1, 2$ and $\phi : \overline{P_1} \cong \overline{P_2}$ is an isomorphism of $\overline{C}$-comodules, then $(P_1, P_2, \phi) \in CI(C)$. Moreover, if $Q_i \in I(C_i)$ $i = 1, 2$ and $\phi' : \overline{Q_1} \cong \overline{Q_2}$ is a $\overline{C}$-isomorphism, then*

$$(P_1, P_2, \phi)\Box_C(Q_1, Q_2, \phi') \cong (P_1\Box_{C_1}Q_1, P_2\Box_{C_2}Q_2, \phi\Box\phi').$$

Proof. If $P_i \in CI(C_i)$ then there are $Q_i, M_i \in P(C_i)$ such that $P_i \oplus Q_i \cong C_i^{(n)}$ and $C_i \oplus N_i \cong P_i^{(m)}$ for some $m, n \geq 1, i = 1, 2$. We have $\overline{Q_1} \cong \overline{Q_2}$ and if ϕ'

denotes this isomorphism, $(Q_1, Q_2, \phi') \in I(C)$ by the preceding theorem, and $(P_1, P_2, \phi) \oplus (Q_1, Q_2, \phi') \cong C^{(n)}$. Moreover, we have

$$\begin{tikzcd}
\overline{P_1}^{(m)} \arrow[d, "\phi^{(m)}"'] & \cong & \overline{P_1^{(m)}} & \cong & \overline{C_1 \oplus M_1} & \cong & \overline{C} \oplus \overline{M_1} \\
\overline{P_2}^{(m)} & \cong & \overline{P_2^{(m)}} & \cong & \overline{C_2 \oplus M_2} & \cong & \overline{C} \oplus \overline{M_2}
\end{tikzcd}$$

Hence $\overline{M_1} \cong \overline{M_2}$. If ψ denotes this isomorphism, then $(M_1, M_2, \psi) \in I(C)$ by the above theorem and

$$(C_1, C_2, 1_{\overline{C}}) \oplus (M_1, M_2, \psi) \cong (C_1 \oplus M_1, C_2 \oplus M_2, 1_{\overline{C}} \oplus \psi) \cong (P_1, P_2, \phi)^{(m)}.$$

Thus $C \cong (C_1, C_2, 1_{\overline{C}})$ is a direct summand of $(P_1, P_2, \phi)^{(m)}$, so that $(P_1, P_2, \phi) \in CI(C)$. For the second claim, first we prove that if $\phi, \psi \in Aut_{\overline{C}}(\overline{C})$ then

$$(C_1^{(n)}, C_2^{(n)}, \phi)\Box_C(C_1^{(m)}, C_2^{(m)}, \psi) \cong (C_1^{(n)}\Box_{C_1}C_1^{(m)}, C_2^{(n)}\Box_{C_2}C_2^{(m)}, \phi\Box\psi).$$

In order to prove this, we cotensor the equalities below with each other

$$(C_1^{(n)}, C_2^{(n)}, \phi) \oplus (C_1^{(n)}, C_2^{(n)}, \phi^{-1}) \cong C^{(2n)},$$

$$(C_1^{(m)}, C_2^{(m)}, \phi) \oplus (C_1^{(m)}, C_2^{(m)}, \phi^{-1}) \cong C^{(2m)},$$

and define maps as in the proof of the above theorem. Since the claim is true for triples of this form, it is also true for direct summand of such triples by an argument similar to the one given before. ∎

Corollary 3.9 *$CI(C)$ and $CI(C_1) \times_{CI(\overline{C})} CI(C_2)$ are equivalent categories.*

The foregoing result is also true for the category $Az(C)$ when all coalgebras are irreducible. This condition is not restrictive since the computation of the Brauer group of a coalgebra only requires the computation of the Brauer group of its irreducible subcoalgebras, cf. [3, Th. 4.9]. If D_i are C_i-coalgebras $i = 1, 2$ such that $\phi : D_1\Box_{C_1}\overline{C} \cong D_2\Box_{C_2}\overline{C}$ is an isomorphism of $\overline{C}$-coalgebras, it is routine to check that (D_1, D_2, ϕ) is a C-coalgebra.

Lemma 3.10 *Suppose that in the cocartesian diagram (3) coalgebras are irreducible and either f_1 or f_2 is injective. If $P = (P_1, P_2, \phi) \in I(C_1) \times_{I(\overline{C})} I(C_2)$ then $e_{-C}(P) \cong (e_{-C_1}(P_1), e_{-C_2}(P_2), \bar{\phi})$.*

Moreover, $e_{-\overline{C}}(P_i\Box_{C_i}\overline{C}) \cong e_{-C_i}(P_i)\Box_{C_i}\overline{C}$ for $i = 1, 2$.

Proof. From [6, Prop. 3.1] we know that for a finitely cogenerated free C-comodule $Q = C^{(n)}$, $e_{-C}(Q) \cong M^c(C,n)$ the $n \times n$ comatrix coalgebra over C, i.e, $C \otimes k_n^*$. Since $P_i \in I(C_i)$ $i = 1,2$, and the coalgebras are irreducible we have $P_1 \cong {C_1}^{(n)}, P_2 \cong {C_2}^{(m)}$. As $\overline{P_1} \cong \overline{P_2}$, $\overline{C}^{(n)} \cong \overline{C}^{(m)}$ and then $n = m$. Thus, we may suppose that $\phi = 1_{\overline{C}^{(n)}}$. Hence $P \cong C^{(n)}$ and

$$\begin{aligned} e_{-C}(P) &\cong C \otimes k_n^* \cong (C_1, C_2, 1_{\overline{C}}) \otimes k_n^* \\ &\cong (C_1 \otimes k_n^*, C_2 \otimes k_n^*, 1_{\overline{C}^{(n)}}) \cong (e_{-C_1}(P_1), e_{-C_2}(P_2), 1_{\overline{C}^{(n)}}). \end{aligned}$$

Finally,

$$\begin{aligned} e_{-\overline{C}}(P_i \square_{C_i} \overline{C}) &\cong \overline{C} \otimes k_n^* \cong (C_i \square_{C_i} \overline{C}) \otimes k_n^* \\ &\cong (C_i \otimes k_n^*) \square_{C_i} \overline{C} \cong e_{-C_i}(P_i) \square_{C_i} \overline{C} \quad i = 1,2. \quad \blacksquare \end{aligned}$$

We recall from [3, Prop. 3.2] the following characterization of coseparable coalgebras. Let D be a C-coalgebra and let D^e be the C-enveloping coalgebra $D \square_C D^{op}$, then D is C-coseparable if and only if D is injective as a right D^e-comodule.

COROLLARY 3.11 *With the same hypotheses as in the foregoing lemma, the categories $Az(C)$ and $Az(C_1) \times_{Az(\overline{C})} Az(C_2)$ are equivalent.*

Proof. Let $D = (D_1, D_2, \phi) \in Az(C_1) \times_{Az(\overline{C})} Az(C_2)$, then D is a C-coalgebra. We have to check that $D \in Az(C)$. We have the following cocartesian diagram:

$$\begin{tikzcd} D = D \square_C C \arrow[rr, "\bar{f}_1"] \arrow[d, "\bar{f}_2"'] & & D \square_C C_1 = D_1 \arrow[d, "\bar{g}_1"] \\ D_2 = D \square_C C_2 \arrow[r, "\bar{g}_2"] & D_2 \square_{C_2} \overline{C} = \overline{D}_2 \arrow[r, "\phi"] & D_1 \square_{C_1} \overline{C} = \overline{D}_1 \end{tikzcd}$$

Also, we have a similar square for the opposite coalgebras. Hence we obtain the following cocartesian diagram for the enveloping coalgebras.

$$\begin{tikzcd} D^e = D \square_C D^{op} \arrow[r, "\bar{f}_1 \square \bar{f}_1{}^{op}"] \arrow[d, "\bar{f}_2 \square \bar{f}_2{}^{op}"'] & D_1 \square_{C_1} {D_1}^{op} = {D_1}^e \arrow[d] \\ {D_2}^e = D_2 \square_{C_2} {D_2}^{op} \arrow[r] & \overline{D_1} \square_{\overline{C}} \overline{D_1}^{op} = \overline{D_1}^e \end{tikzcd}$$

Since either f_1 or f_2 is injective and the cotensor product is left exact either $\bar{f}_1 \Box \bar{f}_1{}^{op}$ or $\bar{f}_2 \Box \bar{f}_2{}^{op}$ is injective. On the other hand, D_i is always quasi-finite as a D_i^e-comodule. Since C_i is irreducible and D_i is a C_i-Azumaya coalgebra it follows that D_i^e is irreducible, cf [3, Prop. 4.10]. Since D_i is C_i-coseparable, then D_i is injective as a D_i^e-comodule $i = 1, 2$. Hence, D_i is a finitely cogenerated injective D_i^e-comodule $i = 1, 2$. From Theorem 3.6 applied to the square above we obtain that $D = (D_1, D_2, \phi)$ is finitely cogenerated injective as a D^e-comodule, therefore D is C-coseparable. To see that D is cocentral we use the foregoing lemma, $Z(D) = e_{-D^e}(D) \cong (e_{-D_1{}^e}(D_1), e_{-D_2{}^e}(D_2), \bar{\phi})$ and since D_i is C_i-Azumaya it follow that $Z(D_i) = e_{-D_i{}^e}(D_i) \cong C_i$ $i = 1, 2$. So, $Z(D) \cong C$ and D is cocentral. Consequently, D is C-Azumaya.

Conversely, if D is C-Azumaya we always have $D \cong (D_1, D_2, \phi)$ where $D_i = D \Box_C C_i$ $i = 1, 2$ and ϕ is the canonical isomorphism $D_1 \Box_{C_1} \overline{C} \cong D_2 \Box_{C_2} \overline{C}$. ∎

4 A MAYER-VIETORIS TYPE EXACT SEQUENCE

We recall the definition of the Picard group of a cocommutative coalgebra C, cf.[3, page 558], [7]. The Picard group of C, denoted by $Pic(C)$, is defined as the group of all isomorphism classes of invertible C-comodules, where M is an invertible C-comodule if there exists a C-comodule N such that $M \Box_C N \cong C$ as C- comodules. Multiplication is induced by cotensor product, i.e, $[M][N] = [M \Box_C N]$. The class $[C]$ is the identity element, and the inverse is given by $[M]^{-1} = [M^{-1}]$, with $M^{-1} = h_{-C}(M, C)$.

PROPOSITION 4.1 *Let C be a cocommutative coalgebra, then $Pic(C)$ is trivial.*

Proof. If P is an invertible C-comodule, then P defines an equivalence from $\mathcal{M}^C$ to itself. From [2, Th. 3.5] we obtain that P is a quasi-finite injective cogenerator. When C is irreducible, from Proposition 3.1 we retain that quasi-finite injective implies finitely cogenerated injective, hence P is finitely cogenerated injective and invertible, so P is isomorphic to C. If C is cocommutative then $C = \oplus_{i \in I} C_i$ with C_i irreducible for all $i \in I$. Every C-comodule P is isomorphic to $\oplus_{i \in I} P_i$ with $P_i = P \Box_C C_i$ and P is invertible if and only if P_i is invertible as C_i-comodule. Since C_i is irreducible then $P_i \cong C_i$ and $P \cong \oplus_{i \in I} P_i \cong \oplus_{i \in I} C_i = C$. ∎

Proposition 4.2 *Let C be a cocommutative coalgebra, $P, Q \in CI(C)$ and $\alpha : e_{-C}(P) \to e_{-C}(Q)$ an isomorphism of C-coalgebras. Then α is induced by an isomorphism of C-comodules $f : P \to Q$.*

Proof. Put $D = e_{-C}(P)$, then Q is a left D-comodule with structure map,

$$Q \xrightarrow{\ \theta\ } e_{-C}(Q) \otimes Q \xrightarrow{\ \alpha \otimes 1\ } D \otimes Q.$$

If P^* denotes $h_{-C}(P, C)$, from [2, Th. 3.5] we have $Q \cong P\square_C(P^*\square_D Q)$ as left D-comodules. Let $I = P^*\square_D Q$ and $Q^* = h_{-C}(Q, C)$ then $I \in Pic(C)$ since,

$$\begin{aligned}(Q^*\square_D P)\square_C(P^*\square_D Q) &\cong Q^*\square_D(P\square_C P^*)\square_D Q\\ &\cong Q^*\square_D D\square_D Q \cong Q^*\square_D Q \cong C.\end{aligned}$$

Since $Pic(C)$ is trivial, $I \cong C$ and we have an isomorphism of left D-comodules $f : Q \to P$, i.e., the diagram

$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[d, "(\alpha \otimes 1)\theta"'] & P \arrow[d, "\theta"] \\ D \otimes Q \arrow[r, "1 \otimes f"] & D \otimes P \end{tikzcd}$$

is commutative. So, $(1 \otimes f^{-1})(\theta \otimes 1)f = (\alpha \otimes 1)\theta$. Hence Q is a D-comodule with structure map $(1 \otimes f^{-1})(\theta \otimes 1)f$ and from [2, 1.18] it follows that α is induced by this map. ∎

Theorem 4.3 *Suppose that in the cocartesian diagram (3) all coalgebras are irreducible and either f_1 or f_2 is injective. If we consider the induced maps on the Brauer group level, the following sequence is exact:*

$$0 \longrightarrow Br(C) \longrightarrow Br(C_1) \oplus Br(C_2) \longrightarrow Br(\overline{C}).$$

Proof. Let f, g denote the group homomorphisms $f : Br(C) \to Br(C_1) \oplus Br(C_2), g : Br(C_1) \oplus Br(C_2) \to Br(\overline{C})$. Explicitely, this maps are given by:

$$\begin{aligned}&f([D]) = (g_{1*}, g_{2*})([D]) = ([D\square_C C_1], [D\square_C C_2]),\\ &g(([D_1],[D_2])) = (f_{1*}, f_{2*}^{op})([D_1],[D_2]) = [D_1\square_{C_1}\overline{C}]\square_{\overline{C}}[D_2\square_{C_2}\overline{C}]^{op},\end{aligned}$$

for all $[D] \in Br(C)$, $([D_1],[D_2]) \in Br(C_1) \oplus Br(C_2)$. First we prove that f is injective. By Corollary 3.11, if $D = (D_1, D_2, \phi) \in Az(C)$ is such that $[D] \in Ker(f)$, then $[D\Box_C C_i] = [D_i] = [C_i]$ in $Br(C_i)$ for $i = 1,2$. Hence there is a $P_i \in CI(C_i)$ such that $D_i \cong e_{-C_i}(P_i)$ $i = 1,2$. Using Lemma 3.10 we obtain,

$$\begin{aligned} e_{-\overline{C}}(P_1\Box_{C_1}\overline{C}) &\cong e_{-C_1}(P_1)\Box_{C_1}\overline{C} \cong D_1\Box_{C_1}\overline{C} \\ &\cong D_2\Box_{C_2}\overline{C} \cong e_{-C_2}(P_2)\Box_{C_2}\overline{C} \cong e_{-\overline{C}}(P_2\Box_{C_2}\overline{C}). \end{aligned}$$

Let $\phi' : e_{-\overline{C}}(P_1\Box_{C_1}\overline{C}) \to e_{-\overline{C}}(P_2\Box_{C_2}\overline{C})$ denote this isomorphism. By the foregoing proposition, ϕ' is induced by a $\overline{C}$-isomorphism $\psi : P_1\Box_{C_1}\overline{C} \to P_2\Box_{C_2}\overline{C}$. Therefore,

$$D = (D_1, D_2, \phi) \cong (e_{-C_1}(P_1), e_{-C_2}(P_2), \phi') \cong e_{-C}((P_1, P_2, \psi)).$$

By Theorem 3.8, $(P_1, P_2, \psi) \in CI(C)$. So $[D] = [C]$ in $Br(C)$.

Next, we check the exactness of the sequence. It is clear that $Im(f) \subseteq Ker(g)$. Let $([E_1],[E_2]) \in Br(C_1) \oplus Br(C_2)$ be such that $([E_1],[E_2]) \in Ker(g)$, i.e., $[E_1\Box_{C_1}\overline{C}] = [E_2\Box_{C_2}\overline{C}]$ in $Br(\overline{C})$. Then, there are $P, Q \in CI(\overline{C})$ such that

$$(E_1\Box_{C_1}\overline{C})\Box_{\overline{C}}e_{-\overline{C}}(P) \cong (E_2\Box_{C_2}\overline{C})\Box_{\overline{C}}e_{-\overline{C}}(P).$$

Since $\overline{C}$ is irreducible we may suppose $P \cong \overline{C}^{(n)}, Q \cong \overline{C}^{(m)}$ and in view of Lemma 3.10 the above equality transforms to:

$$E_1\Box_{C_1}M^c(\overline{C}, n) \cong E_2\Box_{C_2}M^c(\overline{C}, m).$$

Put $P_1 = C_1^{(n)}$ and $P_2 = C_2^{(m)}$, then

$$\begin{aligned} (E_1\Box_{C_1}e_{-C_1}(P_1))\Box_{C_1}\overline{C} &\cong (E_1\Box_{C_1}M^c(C_1, n))\Box_{C_1}\overline{C} \\ &\cong E_1\Box_{C_1}M^c(\overline{C}, n) \\ &\cong E_2\Box_{C_2}M^c(\overline{C}, m) \\ &\cong (E_2\Box_{C_2}M^c(C_2, m))\Box_{C_2}\overline{C} \\ &\cong (E_2\Box_{C_2}e_{-C_2}(P_2))\Box_{C_2}\overline{C}. \end{aligned}$$

Let ψ be this isomorphism and $E = (E_1\Box_{C_1}e_{-C_1}(P_1), E_2\Box_{C_2}e_{-C_2}(P_2), \psi)$. By Corollary 3.11, $E \in Az(C)$ and $[E\Box_C C_i] = [E_i\Box_{C_i}e_{-C_i}(P_i)] = [E_i]$ $i = 1,2$. Therefore $([E_1],[E_2]) \in Im(f)$. ∎

Example 4.4 Let C be a cocommutative irreducible coalgebra, and C_0 its coradical. The universal connected coalgebra associated to C, is defined as $R(C) = C/C_0^+$ where $C_0^+ = C_0 \cap Ker\varepsilon$. We are going to relate $Br(C)$ and $Br(R(C))$ via the Mayer-Vietoris sequence. Let $i : C_0 \to C$ the inclusion map, $I = Im(i, -\varepsilon)$ and $D = C \oplus k/I$. We define $q : k \to D,\ \lambda \mapsto (0, \lambda) + I$, and $p : C \to D,\ c \mapsto (c, 0) + I$ for all $\lambda \in k, c \in C$. The commutative diagram

$$\begin{array}{ccc} C_0 & \xrightarrow{\varepsilon} & k \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle q} \\ C & \xrightarrow{p} & D \end{array}$$

is cocartesian. Since ε is surjective, any element in D can be taken of the form $(c, 0) + I$ for $c \in C$. It is trivial to prove that q is injective, p is surjective and the map $D \to R(C),\ (c, 0) + I \mapsto c + C_0^+$ for any $c \in C$ is an isomorphism of coalgebras.

As a consequence of Theorem 4.3 we obtain,

Corollary 4.5 *Let C be a cocommutative irreducible coalgebra. There is an exact sequence:*

$$0 \longrightarrow Br(R(C)) \longrightarrow Br(k) \oplus Br(C) \longrightarrow Br(C_0).$$

Taking into account that the Brauer group of a finite field is trivial, we obtain:

Corollary 4.6 *If C is a cocommutative irreducible coalgebra over a finite field k, then $Br(C) \cong Br(R(C))$.*

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