Cocycle twisting of $E(n)$-module algebras and applications to the Brauer group

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Abstract

We classify the orbits of coquasi-triangular structures for the Hopf algebra $E(n)$ under the action of lazy cocycles and the Hopf automorphism group. This is applied to detect subgroups of the Brauer group $BQ(k, E(n))$ of $E(n)$ that are isomorphic. For any triangular structure R on $E(n)$ we prove that the subgroup $BM(k, E(n), R)$ of $BQ(k, E(n))$ arising from R is isomorphic to a direct product of $BW(k)$, the Brauer-Wall group of the ground field k, and $Sym_n(k)$, the group of $n \times n$ symmetric matrices under addition. For a general quasi-triangular structure R' on $E(n)$ we construct a split short exact sequence having $BM(k, E(n), R')$ as a middle term and as kernel a central extension of the group of symmetric matrices of order $r < n$ (r depending on R'). We finally describe how the image of the Hopf automorphism group inside $BQ(k, E(n))$ acts on $Sym_n(k)$.

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Introduction

Since the Brauer group of a field was introduced in 1929, several versions and generalizations of this important invariant have been proposed. Apart from its generalization to commutative rings, which plays an important role in Algebraic Geometry, one of the most important generalizations was the one given by Wall in [31]. He introduced a Brauer group for $\mathbb{Z}_2$-graded algebras, nowadays called the Brauer-Wall group, motivated by the study of quadratic forms and Clifford algebras, see [31], [2], or [13]. Wall's construction was extended to cyclic groups and later to abelian groups. A great impulse to the theory of graded Brauer groups was given by Long in 1974 when he constructed a larger Brauer group for any abelian group, which contains all previous Brauer groups as subgroups ([16]). A step forward was taken in [17] by extending his construction to any commutative and cocommutative Hopf algebra. This new group is today referred to as the Brauer-Long group. During the late seventies and the eighties this group was extensively investigated and a satisfactory knowledge of it was reached. A complete treatment of the Brauer-Long group is given in the monograph [5] by S. Caenepeel.

From a Hopf algebra's point of view the commutativity and cocommutativity assumption in Long's construction is very restrictive since many interesting examples of Hopf algebras are non commutative and/or non cocommutative. This difficulty was overcome by Caenepeel, Van Oystaeyen and Zhang in [6] where they constructed a Brauer group for a Hopf algebra with bijective antipode. Many results concerning this group and some distinguished subgroups can also be found in the recent survey [30]. A greater level of generality in the definition of a Brauer group in representation theory was reached in [26] where Pareigis' construction of the Brauer group of a symmetric monoidal category [24] was generalized to the case of braided monoidal categories. Most of known Brauer groups are particular cases of this construction and braided monoidal categories provide the best framework to understand all the previous constructions.

The Brauer group of a Hopf algebra seems to be much more complicated than the Brauer-Long group. Evidences of this fact are provided in [27], [28], [29] and [8] where subgroups of the Brauer group of Sweedler's Hopf algebra are investigated, and in [9] where this is also done for the family of Hopf algebras H_ν introduced by Radford in [25]. In this paper we study the Brauer group of the family of Hopf algebras $E(n)$ (n a natural number)

introduced in [21] and later studied in [3], [22], [23]. Each quasi-triangular structure R of a Hopf algebra H gives rise to a subgroup $BM(k, H, R)$ of the Brauer group $BQ(k, H)$ of H. Dually, each coquasi-triangular structure r on H induces a subgroup $BC(k, H, r)$ in $BQ(k, H)$. In [8], using cocycle twistings of module algebras, a strategy to classify all these subgroups was exhibited. Given a coquasi-triangular Hopf algebra (H, r) one may consider those cocycles $\sigma : H \otimes H \to k$ such that Doi's twisted Hopf algebra $H^\sigma = {}_\sigma H_{\sigma^{-1}}$ coincides with H and the twisted coquasi-triangular structure r_σ in H^σ is easier to manage. Since twisting provides a braided monoidal equivalence of the categories of comodules and since the Brauer group is invariant under such an equivalence, $BC(k, H, r) \cong BC(k, H, r_\sigma)$. This strategy turned out to be fruitful to classify all these subgroups for Sweedler's Hopf algebra.

In this paper we will use again this technique to classify the subgroups corresponding to the (co)quasi-triangular structures of $E(n)$ (Theorem 2.9 and Corollary 2.10). We will focus on those cocycles σ for which $H^\sigma = H$. These are called lazy cocycles and they form a group, denoted by $Z_L(H)$, which acts on the set of coquasi-triangular structures of H ([4]). For two coquasi-triangular structures on H, one is the twist of the other if and only if they are in the same $Z_L(H)$-orbit. In Section 2 we describe the orbits of coquasi-triangular structures on $E(n)$ under the action of $Z_L(E(n))$. Coquasi-triangular structures on $E(n)$ are parametrized by matrices in $M_n(k)$. We prove in Proposition 2.4 that for $A \in M_n(k)$ the corresponding coquasi-triangular structure r_A is a twist of r_B for some $B \in M_n(k)$ if and only if $A - B$ is a symmetric matrix. As a consequence all $Z_L(E(n))$-orbits are parametrized by skew-symmetric matrices (Corollary 2.8) and cotriangular structures lie in the same orbit as r_0 where r_0 is the nontrivial normalized bicharacter of $k\mathbb{Z}_2$ (Corollary 2.6). The abelian group $BC(k, E(n), r_0)$ is computed in the first part of Section 3. Since $E(n)$ is self-dual, the computation of this group is carried out from a dual point of view, by computing $BM(k, E(n), R_0)$ where R_0 is the nontrivial triangular structure in $k\mathbb{Z}_2$. The injection of $k\mathbb{Z}_2$ into $E(n)$ is a quasi-triangular map and it induces a split group homomorphism from $BM(k, E(n), R_0)$ to $BW(k)$, the Brauer-Wall group of k. The kernel of this homomorphism is shown to be isomorphic to $Sym_n(k)$, the group of symmetric matrices of order n over k under addition. Hence all subgroups $BC(k, E(n), r)$ and $BM(k, E(n), R)$ are described for r and R cotriangular and triangular respectively. The above kernel is determined by studying the action of $E(n)$ on central simple alge-

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bras and its interaction with Clifford-type algebras, which is done in Section 3. The non-triangular case is more difficult to manage but we are able to construct in Theorem 4.3 a split short exact sequence whose middle term is $BM(k, E(n), R_A)$ and whose left hand term is described as a nonabelian central extension of $(Sym_r(k), +)$ with $r < \left[\frac{n}{2}\right]$ by $(M_{n-r,r}(k), +)$, the group of $(n - r) \times r$ matrices with coefficients in k under addition.

Finally we compute a subgroup of $BQ(k, E(n))$ that arises from a quotient of the Hopf automorphism group of $E(n)$. There is a homomorphism from the Hopf automorphism group of a Hopf algebra into its Brauer group. Through this homomorphism $GL_n(k)/\mathbb{Z}_2$ is embedded into $BQ(k, E(n))$. The action by conjugation of the former group stabilizes $BM(k, E(n), r_0)$ and we construct a copy of the semidirect product $Sym_n(k) \rtimes (GL_n(k)/\mathbb{Z}_2)$ inside $BQ(k, E(n))$.

1 Notation and preliminaries

Throughout this paper k will denote a fixed ground field of characteristic different from 2 and its group of invertible elements will be denoted by k^*. Unless otherwise stated, all vector spaces, algebras, tensor products, etc will be over k. For vector spaces V and W, $\tau : V \otimes W \to W \otimes V$ stands for the usual flip map. By H we will denote a finite dimensional Hopf algebra with bijective antipode S. For general facts on the theory of Hopf algebras the reader is referred to [14] and [20].

1.1 The Brauer group of a Hopf algebra

In this paragraph we briefly recall from [6] and [7] the construction of the Brauer group of a Hopf algebra and record some of its properties needed in the sequel. Let $R = \sum R^{(1)} \otimes R^{(2)} \in H \otimes H$ be a quasi-triangular structure on H. The *braided product* $A\#B$ of two H-module algebras A, B is the H-module algebra defined as follows: as an H-module $A\#B = A \otimes B$, with multiplication given by

$$(a\#b)(a'\#b') = \sum a(R^{(2)} \cdot a')\#(R^{(1)} \cdot b)b',$$

for all $a, a' \in A, b, b' \in B$. The *H-opposite algebra* of A, denoted by $\overline{A}$, is equal to A as a left H-module but endowed with multiplication: $aa' =$

$\sum (R^{(2)} \cdot a')(R^{(1)} \cdot a)$ for all $a, a' \in A$. The endomorphism algebra $End(M)$ of a left H-module M is a left H-module algebra with the H-module structure defined by

$$(h \cdot f)(m) = \sum h_{(1)} \cdot f(S(h_{(2)}) \cdot m).$$

Similarly, $End(M)^{op}$ equipped with the H-module structure:

$$(h \cdot f)(m) = \sum h_{(2)} \cdot f(S^{-1}(h_{(1)}) \cdot m)$$

becomes a left H-module algebra. A finite dimensional H-module algebra A is called H-*Azumaya* if the following H-module algebra maps are isomorphisms:

$$F : A\#\overline{A} \rightarrow End(A), \ F(a\#\bar{b})(c) = \sum a(R^{(2)} \cdot c)(R^{(1)} \cdot b),$$
$$G : \overline{A}\#A \rightarrow End(A)^{op}, \ G(\bar{a}\#b)(c) = \sum (R^{(2)} \cdot a)(R^{(1)} \cdot c)b.$$

Two H-Azumaya module algebras A, B are called *Brauer equivalent*, denoted by $A \sim B$, if there are finite dimensional H-modules M, N such that $A\#End(M) \cong B\#End(N)$ as H-module algebras. The relation $\sim$ is an equivalence relation on the set $Az(H)$ of isomorphism classes of H-Azumaya module algebras and the quotient set $BM(k, H, R) = Az(H)/\sim$ is a group under the product $[A][B] = [A\#B]$, with identity element $[End(M)]$ for M a finite dimensional H-module, and inverse $[A]^{-1} = [\overline{A}]$. Starting with a coquasi-triangular structure r on H, a dual construction holds. The corresponding Brauer group is denoted by $BC(k, H, r)$. The duality yields an isomorphism $BM(k, H, R) \cong BC(k, H^*, R)$ where $R \in (H^* \otimes H^*)^*$ is viewed as a coquasi-triangular structure on H^*. The Brauer group $BM(k, D(H), \mathcal{R})$ of the Drinfeld double of H with its canonical quasi-triangular structure $\mathcal{R}$ is denoted by $BQ(k, H)$. Identifying the monoidal categories $_H\mathcal{YD}^H$ of Yetter-Drinfeld H-modules and $_{D(H)}\mathcal{M}$, a $D(H)$-module algebra is a Yetter-Drinfeld H-module algebra.

If (H, R) is a quasi-triangular Hopf algebra, then $BM(k, H, R)$ is a subgroup of $BQ(k, H)$. It consists of those classes $[A] \in BQ(k, H)$ having a representative A for which the right H^{op}-comodule algebra structure is of the form:
$$\rho : A \rightarrow A \otimes H^{op}, \ a \mapsto \sum (R^{(2)} \cdot a) \otimes R^{(1)}.$$

Similarly, if (H, r) is a coquasi-triangular Hopf algebra, then $BC(k, H, r)$ is a subgroup of $BQ(k, H)$. A class $[A] \in BQ(k, H)$ belongs to $BC(k, H, r)$

if it has a representative A such that the left H-module structure is of the form:
$$\cdot : H \otimes A \to A, \quad h \cdot a = \sum r(h \otimes a_{(1)})a_{(0)}.$$

1.2 The Hopf algebras $E(n)$

Let n be a natural number and let $E(n)$ be the Hopf algebra over k generated by c and x_i for $i = 1, \dots, n$ with relations
$$c^2 = 1, \quad x_i^2 = 0, \quad cx_i = -x_i c, \quad x_i x_j = -x_j x_i, \quad for \ 1 \le i < j \le n,$$
coproduct
$$\Delta(c) = c \otimes c, \quad \Delta(x_i) = 1 \otimes x_i + x_i \otimes c,$$
and antipode
$$S(c) = c, \quad S(x_i) = cx_i.$$

Each $E(n)$ is a quasi-triangular Hopf algebra. Its quasi-triangular structures were described by Panaite and Van Oystaeyen in [22]. Notation and terminology throughout this paragraph is that of this reference. Since $E(n) \cong E(n)^*$, we have that $E(n)$ is also coquasi-triangular. An isomorphism $\phi : E(n) \to E(n)^*$ is defined by $\phi(1) = 1^* + c^*$, $\phi(x_j) = x_j^* + (cx_j)^*$ and $\phi(c) = 1^* - c^*$. The quasi-triangular structures are parametrized by matrices in $M_n(k)$ and they are given as follows: for a matrix $A = (a_{ij}) \in M_n(k)$ and for s-tuples P, F of increasing elements in $\{1, \dots, n\}$ we define $|P| = |F| = s$ and x_P as the product of the x_j's whose index belongs to P, taken in increasing order. Any bijective map $\eta : F \to F$ may be identified with an element of the symmetric group S_s. Let $sign(\eta)$ denote the signature of η. If $P = \emptyset$ then we take $F = \emptyset$ and $sign(\eta) = 1$. Finally, by $a_{P,\eta(F)}$ we denote the product $a_{p_1, f_{\eta(1)}} \cdots a_{p_s, f_{\eta(s)}}$. For $P = \emptyset$ we define $a_{P,\eta(F)} := 1$. The R-matrix corresponding to A is

$$R_A = \tfrac{1}{2} \sum_P (-1)^{\frac{|P|(|P|-1)}{2}} \sum_{F, \, |F|=|P|, \, \eta \in S_{|P|}} sign(\eta) a_{P,\eta(F)} (x_P \otimes c^{|P|} x_F$$
$$+ cx_P \otimes c^{|P|} x_F + x_P \otimes c^{|P|+1} x_F - cx_P \otimes c^{|P|+1} x_F).$$

In particular, $\frac{1}{2} a_{ij}$ is the coefficient of $(x_i \otimes cx_j + cx_i \otimes cx_j + x_i \otimes x_j - cx_i \otimes x_j)$. The following proposition strengthens [22, Proposition 7].

Proposition 1.1 *The quasi-triangular structure R_A is triangular if and only if A is symmetric.*

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Proof: If (H, R) is a quasi-triangular Hopf algebra, then (H^{op}, R^{-1}) and $(H^{op}, \tau R)$ are quasi-triangular Hopf algebras. Since $E(n) \cong E(n)^{op}$, we have that $(E(n), R_A^{-1})$ and $(E(n), \tau R_A)$ are quasi-triangular Hopf algebras. The quasi-triangular structure of $E(n)^{op}$ corresponding to A^t, the transpose matrix of A, is τR_A. On the other hand, it is well-known that $R^{-1} = (S \otimes \mathrm{id})(R)$ and a simple computation shows that $(S \otimes \mathrm{id})(R_A)$ is the quasi-triangular structure of $E(n)^{op}$ corresponding to A. Hence, $\tau R_A = R_A^{-1}$ if and only if A is a symmetric matrix. $\qquad\square$

By duality, the coquasi-triangular structures of $E(n)$ are parametrized by matrices in $M_n(k)$ and they can be obtained applying the isomorphism $\phi \otimes \phi$ to the quasi-triangular structures. By direct computation one obtains:

$$r_A = \sum_P (-1)^{\frac{|P|(|P|-1)}{2}} \sum_{F, |F|=|P|, \eta \in S_{|P|}} sign(\eta) a_{P, \eta(F)}((x_P)^* \otimes (x_F)^*)$$
$$+ (cx_P)^* \otimes (x_F)^* + (-1)^{|P|}(x_P)^* \otimes (cx_F)^* - (-1)^{|P|}(cx_P)^* \otimes (cx_F)^*).$$

In particular, $r_A(x_i \otimes x_j) = a_{ij}$. Hence A is nothing but the matrix of the bilinear form determined by the restriction of r_A to the span of the x_j's. In other words, the matrix whose entries are the values of a coquasi-triangular structure r on $x_i \otimes x_j$ for $i, j = 1, \ldots, n$ uniquely determines r. As a consequence of the above proposition, r_A is cotriangular if and only if A is symmetric. Observe that this does not mean that the form r_A is symmetric on $E(n) \otimes E(n)$. For instance, as

$$r(x_i \otimes x_j) = r(cx_i \otimes x_j) = -r(x_i \otimes cx_j) = r(cx_i \otimes cx_j)$$

the form r_A cannot be symmetric if $a_{ii} \neq 0$ for some i.

Finally we recall from [23, Lemma 1] that the group $Aut_{Hopf}(E(n))$ of Hopf automorphisms of $E(n)$ is isomorphic to $GL_n(k)$. Any element $f \in Aut_{Hopf}(E(n))$ is of the form

$$f(c) = c, \qquad f(x_i) = \sum_{j=1}^{n} t_{ij} x_j \quad \forall i = 1, \ldots, n,$$

for $T_f = (t_{ij}) \in GL_n(k)$.

2 Cocycle twisting

Recall that a *left 2-cocycle* is a convolution invertible map $\sigma : H \otimes H \to k$ satisfying:

$$\sum \sigma(g_{(1)} \otimes h_{(1)})\sigma(g_{(2)}h_{(2)} \otimes m) = \sum \sigma(h_{(1)} \otimes m_{(1)})\sigma(g \otimes h_{(2)}m_{(2)}),$$

for all $g, h, m \in H$. When $\sigma(h \otimes 1) = \sigma(1 \otimes h) = \varepsilon(h)1$, for all $h \in H$, the cocycle is said to be *normalized*. A new Hopf algebra H^σ, called the σ-*twist* of H, can be associated to σ ([12]). As a coalgebra $H^\sigma = H$ but with multiplication defined by

$$h \cdot_\sigma h' = \sum \sigma(h_{(1)} \otimes h'_{(1)})h_{(2)}h'_{(2)}\sigma^{-1}(h_{(3)} \otimes h'_{(3)})$$

for all $h, h' \in H$. If (H, r) is coquasi-triangular, then (H^σ, r_σ) is coquasi-triangular where $r_\sigma = \sigma\tau * r * \sigma^{-1}$. It is known that $\mathcal{M}^H$ and $\mathcal{M}^{H^\sigma}$ are equivalent as braided monoidal categories. As a consequence, the Brauer groups $BC(k, H, r)$ and $BC(k, H^\sigma, r_\sigma)$ are isomorphic. In this section we study those 2-cocycles σ of $E(n)$ such that $E(n)^\sigma = E(n)$ and we describe how the coquasi-triangular structures of $E(n)$ transform under this twist procedure. This will allow us to detect isomorphism classes of groups of type $BC(k, E(n), r_A)$ for a coquasi-triangular structure r_A on $E(n)$.

Definition 2.1 *A left 2-cocycle σ is called* lazy *if for all $h, l \in H$*

$$\sum \sigma(h_{(1)} \otimes l_{(1)})h_{(2)}l_{(2)} = \sum h_{(1)}l_{(1)}\sigma(h_{(2)} \otimes l_{(2)}). \qquad (2.1)$$

It is shown in [4] that lazy cocycles are also right cocycles and that they form a group $Z_L(H)$ under the convolution product. This group acts on the set $\mathcal{U}$ of coquasi-triangular structures of H by $\sigma \cdot r = (\sigma\tau) * r * \sigma^{-1}$. It is well-known that the group $Aut_{Hopf}(H)$ of Hopf automorphisms acts on cocycles and on coquasi-triangular structures by $\alpha \cdot \sigma = \sigma \circ (\alpha^{-1} \otimes \alpha^{-1})$ and $\alpha \cdot r = r \circ (\alpha^{-1} \otimes \alpha^{-1})$. The semidirect product $\mathcal{H} := Z_L(H) \rtimes Aut_{Hopf}(H)$ acts again on $\mathcal{U}$.

We will analyze the orbits of this action when $H = E(n)$. We first describe $Z_L(E(n))$. To detect orbits, it is enough to describe normalized lazy cocycles. Indeed, if σ is a lazy cocycle which is not normalized, then $\sigma^{-1}(1 \otimes 1)\sigma$ is a normalized lazy cocycle (see [5, Page 231] for the normalization condition) and it acts as σ on $\mathcal{U}$. Therefore we shall focus on normalized lazy cocycles.

Lemma 2.2 *Let σ be a normalized lazy cocycle for $E(n)$. Then $\sigma(c \otimes c) = 1$ and $\sigma(c \otimes x_j) = \sigma(x_j \otimes c) = 0$.*

Proof: Condition (2.1) for $h = c$, $l = x_j$ forces $\sigma(c \otimes c) = 1$ and $\sigma(c \otimes x_j) = 0$. Similarly, for $h = x_j$, $l = c$ condition (2.1) forces $\sigma(x_j \otimes c) = 0$. $\qquad\square$

Lemma 2.3 *For every choice of scalars m_{ij} for $1 \leq j \leq i \leq n$ there exists a normalized lazy cocycle for $E(n)$ with $\omega(x_i \otimes x_j) = m_{ij}$ if $i \geq j$ and zero if $i < j$.*

Proof: By the classification of $E(n)$-cleft extensions of k in [23, Page 593, §4], cohomology classes of normalized cocycles are parametrized by (α, γ, M) where $\alpha \in k^*$, $\gamma \in k^n$ and M is a lower triangular $n \times n$ matrix with entries in k. They are given by:

$$\omega(c \otimes c) = \alpha, \quad \omega(c \otimes x_i) = 0, \quad \omega(x_i \otimes c) = \gamma_i$$

for every $i = 1, \ldots, n$,

$$\omega(x_i \otimes x_j) = \begin{cases} m_{ij} & \text{if } i \geq j; \\ 0 & \text{if } i < j; \end{cases}$$

$$\omega(c \otimes x_P) = \omega(x_P \otimes c) = \omega(c \otimes cx_P) = \omega(cx_P \otimes c) = 0$$

for every P with $|P| > 1$;

$$\omega(x_P \otimes x_Q) = 0$$

whenever $|P| \neq |Q|$, together with recurrence relations. Let us point out that we slightly changed notation in [23] for later convenience.

If $h = 1$ or if $l = 1$, or if $h = l = c$, condition (2.1) is always satisfied. By Lemma 2.2 any lazy cocycle in this family should correspond to a triple with $\alpha = 1$ and $\gamma = 0$. Let $\omega = \omega(M)$ be the cocycle corresponding to $\alpha = 1$, $\gamma = 0$ and M. For such a cocycle condition (2.1) holds also for $h = x_i$ and $l = x_j$. Besides, the recurrence relations in [23, Page 597] that we shall need become

$$\omega(cx_P \otimes x_Q) = \omega(x_P \otimes x_Q) = (-1)^{|P|}\omega(x_P \otimes cx_Q) = (-1)^{|P|}\omega(cx_P \otimes cx_Q)$$

and, for $|Q| = |P| + 1$

$$\omega(x_i x_P \otimes x_Q) = \sum_{j=1}^{|Q|} (-1)^{|Q|-j} m_{iq_j} \omega(x_P \otimes x_{Q \setminus \{q_j\}}).$$

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The elements of $Z_L(E(n))$ are precisely those cocycles which do not change the product in $E(n)$ when we apply Doi's twisting procedure, i.e.,

$$h \cdot_\omega l = \sum \omega(h_{(1)} \otimes l_{(1)}) h_{(2)} l_{(2)} \omega^{-1}(h_{(3)} \otimes l_{(3)}) = hl$$

for all $h, l \in E(n)$. It is not hard to verify that $c \cdot_\omega (c^a x_P) = cc^a x_P$ and $x_j \cdot_\omega (c^a x_P) = x_j c^a x_P$ for every $j = 1, \ldots, n$, every n-tuple P and every $a = 0, 1$. The assertion follows by induction and associativity of the product $\cdot_\omega$. $\qquad\square$

Proposition 2.4 *Two coquasi-triangular structures r_A and r_B on $E(n)$ are in the same $Z_L(E(n))$-orbit if and only if $A - B$ is a symmetric matrix.*

Proof: Let r_A be the coquasi-triangular structure corresponding to the matrix A and let σ be a normalized lazy cocycle. Then $\sigma \cdot r_A = \sigma\tau * r_A * \sigma^{-1} = r_B$ for some matrix B, which is completely determined by: $b_{ij} = \sigma\tau * r_A * \sigma^{-1}(x_i \otimes x_j)$. By direct computation we obtain:

$$b_{ij} = -\sigma(x_j \otimes x_i) + a_{ij} + \sigma^{-1}(x_i \otimes x_j) = a_{ij} - \sigma(x_j \otimes x_i) - \sigma(x_i \otimes x_j).$$

Then $B = A - L$ where L is the *symmetric* matrix with coefficients $l_{ij} = \sigma(x_i \otimes x_j) + \sigma(x_j \otimes x_i)$. Hence, if $r_B \in Z_L(H) \cdot r_A$, then $B - A \in Sym_n(k)$. Conversely, let $A, B \in M_n(k)$ be such that $L = A - B \in Sym_n(k)$. By Lemma 2.3 there exists a lazy cocycle ω such that $\omega(x_i \otimes x_j) + \omega(x_j \otimes x_i) = l_{ij}$. $\quad\square$

Remark 2.5 The classification of cleft extensions of $E(n)$ in [23] yields a classification of 2-cocycles up to cohomologous cocycles. For a 2-cocycle σ, a cocycle is cohomologous to σ ([10]) if it is of the form

$$\sigma^\theta(h \otimes l) = \sum \theta(h_{(1)})\theta(m_{(2)})\sigma(h_{(2)} \otimes m_{(2)})\theta^{-1}(h_{(3)} m_{(3)})$$

for every $h, m \in H$ and for a convolution invertible map $\theta \colon H \to k$. However, if σ is lazy, σ^θ need not be lazy and viceversa. Therefore, in order to describe $Z_L(E(n))$ completely we would need to compute *all* 2-cocycles first.

Corollary 2.6 *All cotriangular structures on $E(n)$ form a single $Z_L(E(n))$-orbit in $\mathcal{U}$.*

Proof: We have seen that a coquasi-triangular structure is cotriangular if and only if its corresponding matrix is symmetric, hence if and only if it lies in the orbit of the trivial matrix. ☐

Remark 2.7 Since the action of a lazy cocycle preserves cotriangularity and since r_0 is cotriangular we get a new proof of the fact that if A is symmetric r_A is cotriangular.

Corollary 2.8 *The $Z_L(E(n))$-orbits on $\mathcal{U}$ are parametrized by skew-symmetric $n \times n$ matrices.*

Proof: They are parametrized by the skew-symmetric part of the corresponding matrix. ☐

Theorem 2.9 *The $\mathcal{H}$-orbits on $\mathcal{U}$ are parametrized by $n \times n$ matrices of the form*

$$J_l := \begin{pmatrix} 0 & I_l & 0 \\ -I_l & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.2}$$

where I_l is the $l \times l$ identity matrix for some l with $0 \leq l \leq \left[\frac{n}{2}\right]$.

Proof: We know that $Aut_{Hopf}(E(n)) \cong GL_n(k)$. By [23, Proposition 3] it acts on R_A (and, dually, on r_A) by $\alpha_M \cdot R_A = R_{M^t AM}$, where α_M is the automorphism associated to the matrix $M \in GL_n(k)$. Hence the $\mathcal{H}$-orbits are parametrized by congruence classes of skew-symmetric matrices. By [15, Theorem XV.8.1] any alternating form is equivalent to (one and only one) form associated to a matrix of type (2.2). ☐

Corollary 2.10 *The possible non isomorphic Brauer groups $BC(k, E(n), r_A)$ for $A \in M_n(k)$ are at most $\left[\frac{n}{2}\right] + 1$. Every group $BC(k, E(n), r_A)$ is isomorphic to a group of type $BC(k, E(n), r_{J_l})$. More precisely, $BC(k, E(n), r_A)$ is isomorphic to that $BC(k, E(n), R_{J_l})$ for which there exist $L \in GL_n(k)$ and $C \in M_n(k)$ such that $C = {}^t L J_l L$ and $C - A$ is symmetric. In particular, the Brauer groups $BC(k, E(n), r_A)$ corresponding to cotriangular structures are all isomorphic to $BC(k, E(n), r_0)$.*

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Remark 2.11 Since $E(n)$ is self-dual, $BM(k, E(n), R_A)$ is isomorphic to $BC(k, E(n), r_A)$. A result similar to Corollary 2.10 holds for the groups $BM(k, E(n), R_A)$.

Example 2.12 i) If $n = 1$ then $E(n)$ is equal to Sweedler's Hopf algebra H_4. There is only one $\mathcal{H}$-orbit because $span(x_j)$ is one-dimensional. All matrices are symmetric so all forms are cotriangular. This case was studied in [8] and its dual version appears in [1, Example 3.4.2]. The Brauer group $BM(k, H_4, R_0) \cong BC(k, H_4, r_0)$ was computed in [28].

ii) For $n = 2$ there are two $\mathcal{H}$-orbits: the symmetric matrices, corresponding to cotriangular structures, and the non-symmetric matrices, with representative

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

iii) If $n = 3$ again $\mathcal{H}$ acts on $\mathcal{U}$ with two orbits, corresponding to the symmetric matrices and the non-symmetric matrices. The second orbit is represented by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We end this section by introducing some special lazy cocycles and their corresponding $E(n)$-cleft extensions of k. Let ω be the lazy cocycle corresponding to the lower triangular matrix M as in Lemma 2.3 and let $\theta = \varepsilon - \frac{1}{2} \sum_{1 \leq i < j \leq n} m_{ji}(x_i x_j)^* \in E(n)^*$. We consider the cocycle $\sigma := \omega^\theta$ cohomologous to ω as in Remark 2.5. Then one has:

$$\sigma(c \otimes c) = 1, \quad \sigma(1 \otimes h) = \sigma(h \otimes 1) = \varepsilon(h)$$

$$\sigma(x_j \otimes c) = 0 = \sigma(c \otimes x_j)$$

$$\sigma(x_i \otimes x_j) = \begin{cases} \frac{1}{2} m_{ij} & \text{if } i > j, \\ \frac{1}{2} m_{ji} & \text{if } i < j, \\ m_{jj} & \text{if } i = j, \end{cases}$$

together with relations depending on the recurrence relations for ω in [23] and the value of θ. Since $\theta \in Z(E(n)^*)$, σ is again a lazy cocycle (see [4]). The matrix L whose (i, j) entry is $\sigma(x_i \otimes x_j)$ is symmetric. Conversely, for

any symmetric matrix $L = (l_{ij})$ we will denote by $\sigma(L)$ the lazy cocycle with $\sigma(x_i \otimes x_j) = l_{ij}$ contructed as above. The corresponding cleft extension is the generalized Clifford algebra with generators u and v_i, for $i = 1, \ldots n$, and relations

$$u^2 = 1, \ uv_i + v_i u = 0, \quad v_j^2 = l_{jj}, \ v_i v_j + v_j v_i = 2l_{ij} \quad \text{for } i \neq j \quad (2.3)$$

and with comodule algebra structure given by

$$\rho(u) = u \otimes c, \quad \rho(v_j) = 1 \otimes x_j + v_j \otimes c. \quad (2.4)$$

For a symmetric matrix $L = (l_{ij})$ we denote by $Cl(L)$ the algebra with generators and relations as in (2.3) associated to the lazy cocycle $\sigma(L)$.

3 $E(n)$-actions on central simple algebras

In this section we analyze the action of $E(n)$ on central simple algebras.

Proposition 3.1 *Let A be a central simple algebra which is an $E(n)$-module algebra with action $\rightharpoonup$. Then, there exist elements $u, w_1, \ldots, w_n \in A$ such that $c \rightharpoonup a = uau^{-1}$ and $x_i \rightharpoonup a = w_i(c \rightharpoonup a) - aw_i$ for $i = 1, \ldots n$, and for all $a \in A$. These elements satisfy the relations*

$$u^2 = \alpha, \quad w_i^2 = l_{ii} \quad w_i u + u w_i = 2\mu_i, \quad w_i w_j + w_j w_i = 2l_{ij}$$

for certain $\alpha \in k^*$, and $\mu_i, l_{ij} \in k$ for $1 \leq i \leq j \leq n$.

Proof: By the Skolem-Noether Theorem for Hopf algebras ([19]), the action of $E(n)$ on A is inner. Hence there exists a convolution invertible map $\pi \colon E(n) \to A$ such that $h \rightharpoonup a = \sum \pi(h_{(1)}) a \pi^{-1}(h_{(2)})$ for all $h \in H, a \in A$. Putting $u := \pi(c)$ and $w_i := -\pi^{-1}(x_i) = \pi(x_i) u^{-1}$ we get

$$c \rightharpoonup a = uau^{-1}, \qquad x_i \rightharpoonup a = w_i uau^{-1} - aw_i,$$

for all $a \in A$. Since the action of c^2 is trivial, u^2 is central in A and therefore it belongs to k. Moreover, $u^2 = \alpha$ is nonzero because u is invertible, so $\alpha \in k^*$.

From $(x_i c + c x_i) \rightharpoonup a = 0$ for all $a \in A$, it follows that $u^{-1} w_i + w_i u^{-1}$ is central in A. Thus, there is $\mu_i \in k$ such that $w_i u + u w_i = 2\mu_i$ for all

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$i = 1, ..., n$. Since $(x_i x_j + x_j x_i) \rightharpoonup a = 0$ for every $a \in A$, one obtains that $w_i w_j + w_j w_i$ belongs to the center of A. Then for every pair i, j there is $l_{ij} \in k$ such that $w_i w_j + w_j w_i = 2l_{ij}$. Finally, since x_i^2 acts as 0 on A, we get $w_i^2 = l_{ii} \in k$. $\qquad\square$

Corollary 3.2 *Let A be a central simple algebra which is an $E(n)$-module algebra. Then, there exists a submodule algebra A' of A which is a quotient of a generalized Clifford algebra.*

Proof: The subalgebra A' of A generated by u and the w_j's of Proposition 3.1 is a quotient of the generalized Clifford algebra $Cl(\alpha, \mu, L)$ costructed from the basis $\{v_0, \ldots, v_n\}$ of the vector space V with bilinear form associated to the matrix

$$M = \begin{pmatrix} \begin{array}{c|cccc} \alpha & \mu_1 & \mu_2 & \cdots & \mu_n \\ \hline \mu_1 & & & & \\ \mu_2 & & & L & \\ \cdots & & & & \\ \mu_n & & & & \end{array} \end{pmatrix}. \tag{3.5}$$

$\qquad\square$

Lemma 3.3 *With notation as in Proposition 3.1, the subalgebra generated by u and the w_j's does not depend on the choice of π. There exists a family $\pi_t \colon E(n) \to A$ with $t \in k^*$ for which $\mu_j = 0$ for every j. For all values of t the matrix L as in (3.5) is uniquely determined.*

Proof: The action of x_j and c does not depend on the choice of the map π. If we choose another map $\pi' \colon E(n) \to A$ inducing the same action, centrality of A forces $u' = \pi'(c) = tu$ for some $t \in k^*$ and $w_j' := -\pi'(x_j) = w_j + s_j u$ for scalars s_j for every $j = 1, \ldots, n$. Conversely, for any $t \in k^*$ and $s_j \in k$ for $j = 1, \ldots, n$ the map $\pi' \colon E(n) \to A$ given by $\pi'(c) = tu$, $\pi'^{-1}(x_j) = -(w_j + s_j u)$ and $\pi'(c^a x_{i_1} \cdots x_{i_s}) = \pi'(u)^a \pi'(x_{i_1}) \cdots \pi'(x_{i_s})$ is well-defined and satisfies $h \rightharpoonup a = \sum \pi'(h_{(1)}) a \pi'^{-1}(h_{(2)})$. Since

$$(w_j + s_j u)tu + tu(w_j + s_j u) = 2(\mu_j + s_j \alpha)$$

the only choice for $-(\pi')^{-1}(x_j)$ that skew-commutes with $\pi'(c)$ is for $s_j = -\mu_j \alpha^{-1}$, with freedom on t. This choice is always possible because α is

invertible. Hence, one has a family $\pi_t \colon E(n) \to A$ given by $\pi_t(c) = tu$ and $\pi_t(x_j) = w_j u + \mu_j \alpha^{-1} u$ and $\pi_t(c^a x_{i_1} \cdots x_{i_s}) = \pi_t(u)^a \pi_t(x_{i_1}) \cdots \pi_t(x_{i_s})$. The relations among the new w_j's do not depend on t. $\qquad \square$

Let us observe that if u and w_j are as in Lemma 3.3 the action of $E(n)$ on the subalgebra generated by u and the w_j's is as follows:

$$c \rightharpoonup u = u, \qquad x_j \rightharpoonup u = -2uw_j;$$
$$c \rightharpoonup w_j = -w_j, \qquad x_j \rightharpoonup w_i = -2l_{ij}.$$

In particular, the subalgebra generated by u and the w_j's is a submodule algebra and it is independent of the choice of π. We shall call it the subalgebra of A *induced by the $E(n)$-action* and we shall denote it by $Ind(A)$. The scalar α and the symmetric matrix L are called the *invariants associated to A*. Lemma 3.3 establishes that the matrix M in (3.5) can always be chosen to be $\begin{pmatrix} \alpha & 0 \\ 0 & L \end{pmatrix}$.

Lemma 3.4 *Let A be a central simple algebra which is an $E(n)$-module algebra. The action of $E(n)$ on A is strongly inner if and only if α is a square in k and the matrix M as in (3.5) has rank 1.*

Proof: The action of $E(n)$ on A is strongly inner if and only if there exists a choice of π which is an algebra map. Such a π has to be found among the π_t's in Lemma 3.3. Suppose that for a suitable $t \in k^*$ the map π_t is an algebra map. Then for this t one has:

$$(tu)^2 = t^2 \alpha = 1; \qquad (w_j + s_j u)^2 = l_{jj} - \mu_j^2 \alpha^{-1} = 0;$$
$$(w_i + s_i u)(w_j + s_j u) + (w_j + s_j u)(w_i + s_i u) = 2(l_{ij} + s_j \mu_i + s_i \mu_j + s_i s_j) = 0.$$

Hence α must be a square, $l_{jj}\alpha = \mu_j^2$ and $l_{ij}\alpha = \mu_i \mu_j$. Thus the action of $E(n)$ on A is strongly inner if and only if α is a nonzero square and all minors of order 2 of M involving the first row and the first column are 0. Since M is symmetric and $\alpha = m_{11} \neq 0$ it follows that the action is strongly inner if and only if $rk(M) = 1$. $\qquad \square$

Let us observe that Lemma 3.3 and Lemma 3.4 imply that if the action is strongly inner we can always make sure that the only nonzero entry in the matrix M in (3.5) is $m_{11} = \alpha = 1$, i.e., that $L = 0$.

Lemma 3.5 *Let H be a Hopf algebra and let A and B be Yetter-Drinfeld H-module algebras. If H acts on B in a strongly inner way, then $A\#B \cong A \otimes B$ as algebras. In particular, if A and A' are Brauer equivalent H-Azumaya algebras and A is central simple then A' is also central simple.*

Proof: Let $f\colon H \to B$ be a convolution invertible algebra map such that $h \rightharpoonup b = \sum f(h_{(1)})bf^{-1}(h_{(2)})$ for all $h \in H, b \in B$. We check that the linear map

$$\Lambda\colon A \otimes B \to A\#B, \quad a \otimes b \mapsto \sum a_{(0)}\#f(a_{(1)})b$$

is an algebra map. For $a,\, c \in A$ and $b,\, d \in B$, we have:

$$
\begin{aligned}
\Lambda(a \otimes b)\Lambda(c \otimes d) &= \left(\sum a_{(0)}\#f(a_{(1)})b\right)\left(\sum c_{(0)}\#f(c_{(1)})d\right) \\
&= \sum a_{(0)}c_{(0)}\#[c_{(1)} \rightharpoonup \left(f(a_{(1)})b\right)]f(c_{(2)})d \\
&= \sum a_{(0)}c_{(0)}\#f(c_{(1)})\left(f(a_{(1)})b\right)f^{-1}(c_{(2)})f(c_{(3)})d \\
&= \sum a_{(0)}c_{(0)}\#f(c_{(1)})f(a_{(1)})bd \\
&= \sum a_{(0)}c_{(0)}\#f(c_{(1)}a_{(1)})bd \\
&= \Lambda((a \otimes b)(c \otimes d)).
\end{aligned}
$$

The inverse of Λ is the map $a\#b \to \sum a_{(0)} \otimes f^{-1}(a_{(1)})b$.

If A and A' are as in the hypothesis, then $A\#End(P) \cong A'\#End(Q)$ for some Yetter-Drinfeld H-modules P and Q. Since the action of $E(n)$ on $End(P)$ and $End(Q)$ is strongly inner we have that $A \otimes End(P) \cong A' \otimes End(Q)$. Then $A' \otimes End(Q)$ is central simple and by the Double Centralizer Theorem, $A' = C(End(Q))$ is central simple. $\qquad\square$

We shall use the above Lemma for the investigation of the Brauer group $BM(k, E(n), R_A)$ in the next section.

Proposition 3.6 *Let $M \in M_n(k)$ and let A be a central simple algebra which is an $(E(n), R_M)$-module algebra. Let u, w_j be the generators of $Ind(A)$ with associated invariants α and L. Let B be a central simple algebra on which $E(n)$ acts in a strongly inner way, endowed with the coaction induced by R_M. Then the associated invariants for $Ind(A\#B)$ are again α and L.*

Proof: By Lemma 3.5, $A\#B \cong A \otimes B$ is also a central simple algebra, so the action of $E(n)$ on $A\#B$ is again inner. Let u, w_j be the generators for $Ind(A)$ and v, v_j be the generators for $Ind(B)$. In particular, $x_j \rightharpoonup v_i = 0$

for every i and j. It is not difficult to check that the map Λ in Lemma 3.5 preserves the c-action. Since

$$\Lambda^{-1}(c \rightharpoonup a\#c \rightharpoonup b) = (u \otimes v)\Lambda^{-1}(a\#b)(u \otimes v)^{-1},$$

it follows that for $U := \Lambda(u \otimes v)$ we have:

$$c \rightharpoonup (a\#b) = U(a\#b)U^{-1}, \quad U^2 = u^2\#v^2 = \alpha.$$

Next we consider the braiding for the elements w_j. Let Ψ_0 denote the braiding of two $E(n)$-modules induced by the quasi-triangular structure R_0. For $b \in B$ we have:

$$\begin{aligned}
\Psi_M(w_j \otimes b) &= \sum R_M^{(2)} \rightharpoonup b \otimes R_M^{(1)} \rightharpoonup w_j \\
&= \Psi_0(w_j \otimes b) + \tfrac{1}{2}\sum_P(-1)^{\frac{|P|(|P|-1)}{2}}\sum_{F,\,|F|=|P|,\,\eta \in S_{|P|}} sign(\eta)m_{P,\eta(F)} \\
&\quad [(c^{|P|}x_F \rightharpoonup b) \otimes (x_P \rightharpoonup w_j) + (c^{|P|}x_F \rightharpoonup b) \otimes (cx_P \rightharpoonup w_j) + \\
&\quad (c^{|P|+1}x_F \rightharpoonup b) \otimes (x_P \rightharpoonup w_j) - (c^{|P|+1}x_F \rightharpoonup b) \otimes (cx_P \rightharpoonup w_j)].
\end{aligned}$$

By the particular module structure of $Ind(A)$ the terms of the form $cx_P \rightharpoonup w_j$ with $|P| > 1$ will vanish. Hence

$$\begin{aligned}
\Psi_M(w_j \otimes b) &= \Psi_0(w_j \otimes b) + \tfrac{1}{2}\sum_{p,f} m_{pf}[(cx_f \rightharpoonup b) \otimes (x_p \rightharpoonup w_j) \\
&\quad + (cx_f \rightharpoonup b) \otimes (cx_p \rightharpoonup w_j) + (x_f \rightharpoonup b) \otimes (x_p \rightharpoonup w_j) \\
&\quad - (x_f \rightharpoonup b) \otimes (cx_p \rightharpoonup w_j)] \\
&= \Psi_0(w_j \otimes b) - 2\sum_{p,f} m_{pf}l_{pj}(cx_f \rightharpoonup b) \otimes 1.
\end{aligned}$$

Analogously,

$$\Psi_M(b \otimes w_j) = \Psi_0(b \otimes w_j) - 2\sum_{p,f} m_{pf}l_{fj}(1 \otimes (x_p \rightharpoonup b)).$$

A similar computation shows that $\Psi_M(b \otimes v_j) = \Psi_0(b \otimes v_j)$ and $\Psi_M(v_j \otimes b) = \Psi_0(v_j \otimes b)$. Let us consider the elements

$$W_i := w_i\#1 + 1\#v_i - 2\sum_{p,f} m_{pf}l_{fi}(1\#v_p)$$

for $i = 1, \ldots, n$. Then, for homogeneous a and b with respect to the $\mathbb{Z}_2$-grading induced by the action of c we have:

$$W_i(c \rightharpoonup (a\#b)) - (a\#b)W_i =$$
$$= (w_i\#1)(c \rightharpoonup a\#c \rightharpoonup b) + (1\#v_i)((-1)^{\deg(a)}a\#c \rightharpoonup b)$$
$$-2\sum_{p,f} m_{pf}l_{fi}(1\#v_p)((-1)^{\deg(a)}a\#c \rightharpoonup b)) - (a\#b)(w_i\#1)$$
$$-a\#bv_i + 2(a\#b)\sum_{p,f} m_{pf}l_{fi}(1\#v_p)$$
$$= w_i(c \rightharpoonup a)\#(c \rightharpoonup b) - (-1)^{\deg(b)}aw_i\#b + a\#v_i(c \rightharpoonup b) - a\#bv_i$$
$$+2\sum_{p,f} m_{pf}l_{fi}a\#[v_p(c \rightharpoonup b) - bv_p - x_p \rightharpoonup b]$$
$$= (x_i \rightharpoonup a)\#(c \rightharpoonup b) + a\#(x_i \rightharpoonup b) = x_i \rightharpoonup (a\#b).$$

Hence $Ind(A\#B)$ is the subalgebra generated by U and the W_i's. Besides, $UW_i + W_iU = ((c \rightharpoonup W_i) + W_i)U = 0$. It may be checked that

$$W_i^2 = -\frac{1}{2}(x_i \rightharpoonup W_i) = l_{ii} \quad and \quad W_iW_j + W_jW_i = -(x_i \rightharpoonup W_j) = 2l_{ij}. \ \square$$

Corollary 3.7 *With notation as in Proposition 3.6 the scalar α and the matrix L are invariant under the Brauer equivalence induced by R_M.*

Proof: Apply Proposition 3.6 with $B = End(P)$ for a $E(n)$-module P. $\quad \square$

The following lemma will be needed in the forthcoming section.

Lemma 3.8 *Let σ be a 2-cocycle of a Hopf algebra H and let f be the map $f: H \rightarrow End(H)$ defined by*

$$(f(h))(a) = \sum \sigma(h_{(1)} \otimes a_{(1)})h_{(2)}a_{(2)}$$

for every $a, h \in H$. Then:

(i) The map f is convolution invertible with inverse

$$f^{-1}(h) = \sum \sigma^{-1}(S(h_{(2)}) \otimes h_{(3)})f(S(h_{(1)}))$$

for every $h \in H$;

(ii) For every $h, l \in H$ there holds $f(h) \circ f(l) = \sum \sigma(h_{(1)} \otimes l_{(1)})f(h_{(2)}l_{(2)})$;

(iii) H measures $End(H)$ by means of $h \rightharpoonup F = \sum f(h_{(1)}) \circ F \circ f^{-1}(h_{(2)})$ for every $F \in End(H)$.

Let now σ be a lazy cocycle. Then:

(iv) The weak action $\rightharpoonup$ is indeed an action;

(v) The induced subalgebra on $End(H)$ is a quotient of the cleft extension of k determined by σ.

Proof: *(i)* This assertion follows from [20, Proposition 7.2.7] which holds for all crossed products. Indeed $f(h)$ is just multiplication by Blattner and Montgomery's $\gamma(h)$ in $_\sigma H = k\#_\sigma H$ in their notation. Since $\gamma\colon H \to k\#_\sigma H$ given by $h \to 1\#h$ is convolution invertible with convolution inverse $\mu(h) = \sum \sigma^{-1}(S(h_{(2)}) \otimes h_{(3)})S(h_{(1)})$, the convolution inverse for f is $g\colon H \to End(H)$ given by $g(h) = \sum \sigma^{-1}(S(h_{(2)}) \otimes h_{(3)})f(S(h_{(1)}))$ for every $h \in H$.

(ii) The second assertion is also essentially contained in [20, Proposition 7.2.7] because of the description of $f(h)$ as multiplication by $\gamma(h)$. We can simply view f as the left regular representation of $_\sigma H$.

(iii) By construction $\rightharpoonup$ defines a measuring.

(iv) It follows from *(i)* and *(ii)* applied to $f(S(h))$ that

$$
\begin{aligned}
f^{-1}(hl) \ &= \sum \sigma^{-1}(S(l_{(2)})S(h_{(2)}) \otimes h_{(3)}l_{(3)})f\big(S(l_{(1)})S(h_{(1)})\big)\\
&= \sum \sigma^{-1}(S(l_{(3)})S(h_{(3)}) \otimes h_{(4)}l_{(4)})\sigma^{-1}(S(l_{(2)}) \otimes S(h_{(2)}))\\
&\quad f(S(l_{(1)})) \circ f(S(h_{(1)})).
\end{aligned}
$$

Since σ^{-1} is a right cocycle

$$
\begin{aligned}
f^{-1}(hl) \ &= \sum \sigma^{-1}(S(h_{(2)}) \otimes h_{(5)}l_{(4)})\sigma^{-1}(S(l_{(2)}) \otimes S(h_{(3)})h_{(4)}l_{(3)})\\
&\quad f(S(l_{(1)})) \circ f(S(h_{(1)}))\\
&= \sum \sigma^{-1}(S(h_{(2)}) \otimes h_{(3)}l_{(2)})f^{-1}(l_{(1)}) \circ f(S(h_{(1)})).
\end{aligned}
$$

Using this equality, we obtain:

$$
\begin{aligned}
\sum f^{-1}(h_{(1)}l_{(1)})\sigma^{-1}(h_{(2)} \otimes l_{(2)}) \ &=\\
&= \sum \sigma^{-1}(S(h_{(2)}) \otimes h_{(3)}l_{(2)})\sigma^{-1}(h_{(4)} \otimes l_{(3)})f^{-1}(l_{(1)}) \circ f(S(h_{(1)}))\\
&= \sum \sigma^{-1}(S(h_{(2)}) \otimes h_{(5)})\sigma^{-1}(S(h_{(3)})h_{(4)} \otimes l_{(2)})f^{-1}(l_{(1)}) \circ f(S(h_{(1)}))\\
&= f^{-1}(l) \circ f^{-1}(h).
\end{aligned}
$$

Therefore,

$$
f^{-1}(hl) = \sum \sigma(h_{(2)} \otimes l_{(2)})f^{-1}(l_{(1)}) \circ f^{-1}(h_{(1)}).
$$

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If σ is lazy we also have:

$$f(hl) = \sum \sigma^{-1}(h_{(2)} \otimes l_{(2)}) f(h_{(1)}) \circ f(l_{(1)}),$$

$$f^{-1}(hl) = \sum \sigma(h_{(1)} \otimes l_{(1)}) f^{-1}(l_{(2)}) \circ f^{-1}(h_{(2)}).$$

Let $\rightharpoonup$ be the weak action defined above. Then $1 \rightharpoonup F = F$ for every $F \in End(H)$. Besides, for $F \in End(H)$ and $h, l \in H$ we have:

$$
\begin{aligned}
(hl) \rightharpoonup F \ &= \sum f(h_{(1)} l_{(1)}) \circ F \circ f^{-1}(h_{(2)} l_{(2)}) \\
&= \sum f(h_{(1)}) \circ f(l_{(1)}) \sigma^{-1}(h_{(2)} \otimes l_{(2)}) \circ F \circ \sigma(h_{(3)} \otimes l_{(3)}) \\
&\quad\ f^{-1}(l_{(4)}) \circ f^{-1}(h_{(4)}) \\
&= \sum (f(h_{(1)}) \circ (f(l_{(1)}) \circ F \circ f^{-1}(l_{(2)})) \circ f^{-1}(h_{(2)})) \\
&= h \rightharpoonup (l \rightharpoonup F).
\end{aligned}
$$

Hence $End(H)$ is an H-module algebra.

(v) Since $End(H)$ is a central simple algebra, the induced subalgebra, i.e., the subalgebra of $End(H)$ generated by the elements of the form $f(h)$ for $h \in H$ is well-defined and by the relations for $f(hl)$ we deduce that it is a quotient of $H_\sigma \cong_\sigma H$, the module algebra twist of H by σ. $\qquad \square$

Given a lazy cocycle σ we shall denote the H-module algebra structure on $End(H)$ above defined by A^σ.

4 Subgroups of $BQ(k, E(n))$

4.1 Subgroups arising from (co)quasi-triangular structures

We first focus on the computation of the Brauer group $BM(k, E(n), R)$. Since $E(n)^* \cong E(n)$, by Corollary 2.10 and Remark 2.11, we can reduce the computation of $BM(k, E(n), R_M)$ for $M \in M_n(k)$ to the computation of $BM(k, E(n), R_{J_l})$ with J_l as in (2.2). In particular we may always assume that the matrix M is skew-symmetric, as we will do in the sequel.

Lemma 4.1 *Let n and r be nonnegative integers with $r \leq n$. Let $M \in M_n(k)$ be skew-symmetric and such that the last r rows and columns are 0. Let $Sym_{M,n,r}(k)$ be the set of symmetric matrices of the form*

$$L = \overbrace{}^{n-r}\overbrace{}^{r} \left(\begin{array}{c|c} 0 & L_1 \\ \hline L_1^t & L_2 \end{array} \right).$$

Then the operation

$$L \oplus N = L + N - 2(NML) - 2(NML)^t = L + N - 2(NML) + 2(LMN)$$

turns $Sym_{M,n,r}(k)$ into a group with unit 0 and with opposite $-L$.

Proof: It is clear that 0 is the neutral element with respect to this operation and that $-L$ is a right and left opposite for L. To check associativity, let L, N and $P \in Sym_{M,n,r}(k)$. Then,

$$\begin{aligned}
(L \oplus N) \oplus P &= (L + N - 2(NML) + 2(LMN)) \oplus P \\
&= L + N - 2(NML) + 2(LMN) + P - 2PM(L + N - 2(NML) \\
&\quad + 2(LMN)) + 2(L + N - 2(NML) + 2(LMN))MP \\
&= L + N + P - 2(NML) + 2(LMN) - 2(PML) \\
&\quad - 2(PMN) + 2(LMP) + 2(NMP) \\
&= L + N + P - 2(PMN) + 2(NMP) - 2((N + P - 2(PMN) \\
&\quad + 2(NMP))ML) + 2(LM(N + P - 2(PMN) + 2(NMP)) \\
&= L \oplus (N + P - 2(PMN) + 2(NMP)) \\
&= L \oplus (N \oplus P).
\end{aligned}$$

We have used that $MTM = 0$ for every $T \in Sym_{M,n,r}(k)$. This holds because of the particular block form of M. Hence $(Sym_{M,n,r}(k), \oplus)$ is a group. $\square$

Lemma 4.2 *With hypothesis as in Lemma 4.1, the group $Sym_{M,n,r}(k)$ is a central extension with kernel isomorphic to $(Sym_r(k), +)$ and quotient isomorphic to $(M_{n-r,n}(k), +)$.*

Proof: Let M' denote the submatrix of M corresponding to the first $(n-r)$ rows and columns. Writing every element $L = \begin{pmatrix} 0 & L_1 \\ L_1^t & L_2 \end{pmatrix} \in Sym_{M,n,r}(k)$ as a pair $(L_1, L_2) \in M_{n-r,r}(k) \times Sym_r(k)$, the product in $Sym_{M,n,r}(k)$ becomes:

$$(L_1, L_2) \oplus (N_1, N_2) = (L_1 + N_1, L_2 + N_2 - 2N_1^t M' L_1 + 2L_1^t M' N_1)$$

so the elements of type $(0, S)$ form a subgroup isomorphic to $(Sym_r(k), +)$. One can directly see that this subgroup is central and that the image of the projection $(L_1, L_2) \mapsto L_1$ is isomorphic to $(M_{n-r,n}(k), +)$. $\square$

Theorem 4.3 *Let $n \geq 1$, let $r \leq n$ and M be an $n \times n$ skew-symmetric matrix whose last r rows and r columns are zero. Let $M' \in M_{n-r}(k)$ be the submatrix of M corresponding to the first $(n-r)$ rows and columns. Then there is a split short exact sequence:*

$$1 \longrightarrow Sym_{M,n,r}(k) \longrightarrow BM(k,\ E(n), R_M) \longrightarrow BM(k,\ E(n-r), R_{M'}) \longrightarrow 1.$$

If $n = r$, we have $E(n-n) = E(0) = k\mathbb{Z}_2$, $M' = 0$ and $BM(k,\ E(0), R_{M'}) = BW(k)$, the Brauer-Wall group of k.

Proof: The natural injection $j_{n-r} \colon (E(n-r), R_{M'}) \to (E(n), R_M)$ is a quasi-triangular map, hence the braidings in the corresponding categories of modules coincide, due to the particular form of M and M'. Indeed, if we look at the action of $\sum R_M^{(2)} \otimes R_M^{(1)}$ on tensor products we see that the action of the x_j's with $j > r$ never occurs because all the monomials involving these elements appear with zero coefficient. Therefore the pull-back along j_{n-r} induces a group homomorphism $j_{n-r}^* \colon BM(k,\ E(n), R_M) \to BM(k,\ E(n-r), R_{M'})$. Since the natural projection $p_{n-r} \colon (E(n), R_M) \to (E(n-r), R_{M'})$ is a quasi-triangular map, the pull-back along p_{n-r} induces a group homomorphism $p_{n-r}^* \colon BM(k,\ E(n-r), R_{M'}) \to BM(k,\ E(n), R_M)$ splitting j_{n-r}^*. Let us compute $Ker(j_{n-r}^*)$. Its elements are represented by endomorphism algebras with strongly inner $E(n-r)$-action. Let $A = End(P)$ be such a representative. The matrix L associated to A is uniquely determined by Lemma 3.3 and by Corollary 3.7 and it is invariant under Brauer equivalence induced either by R_M and by $R_{M'}$. By Lemma 3.4 applied to $E(n-r)$ there holds $l_{ij} = 0$ for $1 \leq i,\ j \leq r$, i.e., L lies in $Sym_{M,n,r}(k)$ and we can make sure that the associated scalar $\alpha = 1$. Hence we have a well-defined map

$$\chi \colon Ker(j_{n-r}^*) \to Sym_{M,n,r}(k)$$

$$[A] \mapsto L.$$

We next check that χ is a group homomorphism. For this purpose let B be a representative of another element in $Ker(j^*_{n-r})$. Assume that $Ind(B)$ is generated by v and v_i as in Lemma 3.3, and let $L' = (l'_{ij})$ be a matrix such that $\chi([B]) = L'$. Then

$$\Psi_M(v_j \otimes a) = \Psi_0(v_j \otimes a) - 2 \sum_{p,f \leq n-r} m_{pf} l'_{pj}(cx_f \rightharpoonup a) \otimes 1,$$

for all $a \in A$. Similarly,

$$\Psi_M(a \otimes v_j) = \Psi_0(a \otimes v_j) - 2 \sum_{p,f \leq n-r} m_{pf} l'_{fj}(1 \otimes (x_p \rightharpoonup a)),$$

while the braiding for the w_j's is as in Proposition 3.6. Again, by Proposition 3.6, the element $U = \Lambda(u \otimes v)$ satisfies: $U^2 = 1$ and $c \rightharpoonup (a \# b) = U(a \# b)U^{-1}$.

Let

$$W_j := w_j \# 1 + 1 \# v_j - 2 \sum_{p,f} m_{pf} l_{fj} \# v_p - 2 \sum_{p,f} m_{pf} l'_{pj} w_f \# 1.$$

We consider the $\mathbb{Z}_2$-grading on A, B induced by the c-action. For homogeneous elements $a \in A, b \in B$ one has:

$$
\begin{aligned}
&W_j(c \rightharpoonup (a \# b)) - (a \# b)W_j = \\
&= w_j(c \rightharpoonup a) \# (c \rightharpoonup b) + a \# v_j(c \rightharpoonup b) \\
&\quad + 2 \sum_{p,f} m_{pf} l'_{pj}((x_f \rightharpoonup a) \# (c \rightharpoonup b)) - 2 \sum_{p,f} m_{pf} l_{fj}(a \# v_p(c \rightharpoonup b)) \\
&\quad - 4 \sum_{p,f} m_{pf} l_{fj} \sum_{r,s} m_{rs} l'_{rp}((x_s \rightharpoonup a) \# (c \rightharpoonup b)) \\
&\quad - 2 \sum_{p,f} m_{pf} l'_{pj}(w_f(c \rightharpoonup a) \# (c \rightharpoonup b)) - a w_j \# (c \rightharpoonup b) \\
&\quad + 2 \sum_{p,f} m_{pf} l_{fj}(a \# (x_p \rightharpoonup b)) - a \# b v_j \\
&\quad + 2 \sum_{p,f} m_{pf} l_{fj}(a \# b v_p) + 2 \sum_{p,f} m_{pf} l'_{pj}(a w_f \# (c \rightharpoonup b)) \\
&\quad - 4 \sum_{p,f} m_{pf} l'_{pj} \sum_{r,s} m_{rs} l_{sf}(a \# (x_r \rightharpoonup b)) \\
&= ((x_j \rightharpoonup a) \# (c \rightharpoonup b) + a \# (x_j \rightharpoonup b)) \\
&\quad + 2 \sum_{p,f} m_{pf} l'_{pj}[(x_f \rightharpoonup a - w_f(c \rightharpoonup a) + a w_f) \# (c \rightharpoonup b)] \\
&\quad + 2 \sum_{p,f} m_{pf} l_{fj}[a \# (x_p \rightharpoonup b - v_p(c \rightharpoonup b) + b v_p)] \\
&\quad - 4 \sum_s (LM^t L'M)_{js}((x_s \rightharpoonup a) \# (c \rightharpoonup b)) - 4 \sum_s (L'MLM^t)_{js}(a \# (x_s \rightharpoonup b)) \\
&= x_j \rightharpoonup (a \# b)
\end{aligned}
$$

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because MTM for every $T \in Sym_{M,n,r}(k)$. Therefore $Ind(A\#B)$ is the subalgebra generated by U and the W_j's. Since $c \rightharpoonup W_j = -W_j$ we have $UW_j + W_jU = 0$. Besides,

$$
\begin{aligned}
W_j^2 &= -\tfrac{1}{2}(x_j \rightharpoonup W_j) \\
&= l_{jj} + l'_{jj} - 2\sum_{p,f} m_{pf} l_{fj} l'_{pj} - 2\sum_{p,f} m_{pf} l'_{pj} l_{fj} \\
&= (l_{jj} + l'_{jj}) - 2(L'ML)_{jj} - 2(L'ML)_{jj}.
\end{aligned}
$$

Similarly,

$$
\frac{1}{2}(W_j W_i + W_i W_j) = -\frac{1}{2}(x_j \rightharpoonup W_i) = (l_{ij} + l'_{ij}) - 2(L'ML)_{ji} - 2(L'ML)_{ij}.
$$

Hence $\chi \colon Ker(j^*_{n-r}) \to Sym_{M,n,r}(k)$ is a group homomorphism. We claim that it is injective. If $\chi([A]) = 0$ then the action of $E(n)$ on the endomorphism algebra A is strongly inner, forcing $[A] = [k]$. We finally show that χ is surjective. To this end let $L \in Sym_{M,n,r}(k)$ and let σ be a lazy cocycle with $\sigma(x_i \otimes x_j) = -l_{ij}$. Such a cocycle exists because of the results in Section 2. Let A^σ be constructed as in Lemma 3.8. For the induced subalgebra $Ind(A^\sigma)$ one has $f(h) \circ f(l) = \sum \sigma(h_{(1)} \otimes l_{(1)}) f(h_{(1)} l_{(1)})$. Then, for every element of $E(n-r)$, we have $f(h)f(k) = f(hk)$, i.e, the action of $E(n-r)$ is strongly inner. Therefore, $E(n)$, as a vector space, is a $E(n-r)$-module (hence, via $R_{M'}$, a Yetter-Drinfeld module). Thus A^σ is a $(E(n-r), R_{M'})$-Azumaya algebra. Since the braiding induced by R_M does not involve the action of the $x'_j s$ for $j > n - r$, the algebra A^σ is also $(E(n), R_M)$-Azumaya. Besides, for $u = f(c)$ we have $u^2 = 1$; for $v_j = f(x_j)$ we have $v_j^2 = -l_{jj}$, and $v_i v_j + v_i v_j = -2l_{ij}$. Then for $w_j = v_j u$ for $j = 1, \ldots, n$ we have $w_j^2 = l_{jj}$ and $w_i w_j + w_j w_i = 2l_{ij}$. Hence $\chi([A^\sigma]) = L$ and χ is surjective. $\qquad\square$

Theorem 4.4 *Let $A \in Sym_n(k)$. Then $BM(k, E(n), R_A)$ is isomorphic to the direct product of $(Sym_n(k), +)$ and $BW(k)$.*

Proof: If A is symmetric, then $BM(k, E(n), R_A) \cong BM(k, E(n), R_0)$ and it is an abelian group because the category of $E(n)$-modules endowed with the braiding stemming from R_A is symmetric. Now apply Theorem 4.3 for $M = 0$ and $r = n$. In this case, $Sym_{M,n,r}(k) \cong Sym_n(k)$, hence the proof. $\square$

Remark 4.5 In [32] an exact sequence for a the Brauer group BC of a coquasi-triangular Hopf algebra (H, R)

$$1 \longrightarrow Br(k) \longrightarrow BC(k, H, R) \longrightarrow Gal(\mathcal{H}_R)$$

is constructed, where $\mathcal{H}_R$ is the braided Hopf algebra constructed from H and R as in [18, Theorem 4.1] and $Gal(\mathcal{H}_R)$ is the group of biGalois objects of $\mathcal{H}_R$ satisfying suitable commuting properties. When $H = E(n)$ and $R = R_0$ the last arrow is surjective, $Gal(\mathcal{H}_R)$ is isomorphic to the direct product of $k^*/k^{*2} \rtimes \mathbb{Z}_2$ and $(k, +)^{(n+1)n/2} \cong Sym_n(k)$ and $BC(k, E(n), R_0) \cong BW(k) \times Sym_n(k)$. Since $BW(k)$ is isomorphic to an extension of $Br(k)$ by $k^*/k^{*2} \rtimes \mathbb{Z}_2$ our sequence corresponds to the injection of $Sym_n(k)$ into $BC(k, E(n), R_0)$. The generalization of this computation for arbitrary finite dimensional cotriangular Hopf algebras (under slight conditions on the base field) is the subject of a forthcoming paper. By the results in [4], $Sym_n(k)$ is the second lazy cohomology group $H_L^2(E(n))$ of $E(n)$ and the second lazy cohomology group $H^2(A)$ always occurs as a normal subgroup of the group $BiGal(A)$ of BiGalois objects of a Hopf algebra A. We expect that, at least for cotriangular Hopf algebras (H, R), $H^2(H)$ will always occur as a subgroup of $BC(k, H, R)$.

Theorem 4.6 *Let $n \geq 1$, $j \leq \left[\frac{n}{2}\right]$ and let J_l and J_l' be the square matrices of order n and $2l$ respectively, defined as in (2.2). Then $BM(k, E(n), R_{J_l})$ is a semidirect product of $Sym_{n,n-2l,J_l}(k)$ and $BM(k, E(2l), R_{J_l'})$.*

Proof: Apply Theorem 4.3 to $M = J_l$ and $r = n - 2l$. $\qquad\qquad\square$

We point out that $BM(k, E(n), R_{J_l})$ for $l > 0$ is not abelian because its subgroup $Sym_{n,n-2l,J_l}(k)$ is not. Theorem 4.6 together with the classification of orbits in Section 2 shows that in order to compute the Brauer groups $BM(k, E(n), R_A)$ for every matrix A it is enough to understand those of type $BM(k, E(2l), R_{J_l})$. This latter group seems to be more complicated than those corresponding to R_0. The group $BM(k, E(2l), R_{J_l})$ contains two copies of $BM(k, E(l), R_0)$, obtained as the images of splitting maps similar to p_{n-r}^* in the proof of Proposition 4.3 isolating either c and $x_1, \ldots, x_l$ or c and $x_{l+1}, \ldots x_{2l}$. The intersection of this two subgroups is the Brauer-Wall group of k because it corresponds to those $E(2l)$-Azumaya algebras with trivial action of all the skew-primitive elements.

4.2 Subgroups arising from the Hopf automorphisms group

In [27, Theorem 5] an exact sequence relating the Brauer group of a Hopf algebra H and its Hopf automorphism group $Aut_{Hopf}(H)$ was constructed. We recall the construction of this sequence and apply it to $E(n)$. Given $\alpha \in Aut_{Hopf}(H)$ let H_α denote the right H-comodule H with the following left H-action:

$$h \cdot m = \sum \alpha(h_{(2)}) m S^{-1}(h_{(1)}) \qquad \forall h \in H, m \in H_\alpha.$$

The algebra $A_\alpha = End(H_\alpha)$ is an H-Azumaya algebra with H-module and H-comodule structures defined by

$$(h \cdot f)(m) = \sum h_{(1)} \cdot f(S(h_{(2)}) \cdot m),$$
$$\rho(f)(m) = \sum f(m_{(0)})_{(0)} \otimes S^{-1}(m_{(1)}) f(m_{(0)})_{(1)}$$

for $h \in H, m \in H_\alpha, f \in A_\alpha$. The map $\pi : Aut_{Hopf}(H) \rightarrow BQ(k, H), \alpha \mapsto [A_{\alpha^{-1}}]$ is a group homomorphism whose kernel is $G(D(H))/G(D(H)^*)$, where $G(H)$ denotes the group of grouplike elements of a Hopf algebra H. It is well-known that $G((D(H)) = G(H) \times G(H^*)$ and

$$G(D(H)^*) =$$
$$\{(g, \lambda) \in G(H) \times G(H^*)| \sum g h_{(1)} \lambda(h_{(2)}) = \sum h_{(2)} g \lambda(h_{(1)}) \ \forall h \in H\}.$$

There is a group homomorphism $\theta : G(D(H)) \rightarrow Aut_{Hopf}(H)$ defined by $\theta((g, \lambda))(h) = \sum \lambda(h_{(1)}) g h_{(2)} g^{-1} \lambda^{-1}(h_{(3)})$ for all $h \in H$. It was proved in [27, Theorem 5] that π and θ fit in the following exact sequence:

$$1 \longrightarrow G(D(H)^*) \longrightarrow G(D(H)) \overset{\theta}{\longrightarrow} Aut_{Hopf}(H) \overset{\pi}{\longrightarrow} BQ(k, H).$$

We compute the first three terms of this sequence for $H = E(n)$ and determine the quotient of $Aut_{Hopf}(E(n))$ embedding in $BQ(k, E(n))$. The case $E(1)$ was treated in [27, Example 7]. Recall that the group $Aut_{Hopf}(E(n))$ is isomorphic to $GL_n(k)$. Any $\alpha \in Aut_{Hopf}(E(n))$ is of the form

$$\alpha(c) = c, \quad \alpha(x_i) = \sum_{j=1}^{n} t_{ij} x_j \quad \forall i = 1, \dots, n,$$

for $T_\alpha = (t_{ij}) \in GL_n(k)$ and the assignment $\alpha \mapsto T_\alpha$ is a group antimorphism. Since $E(n)$ is self-dual, $G(E(n)^*)$ is easily computed. It consists of the elements ε and $C = 1^* - c^*$. Then $G(D(H)) = \{(1, \varepsilon), (c, \varepsilon), (1, C), (c, C)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. It may be checked that $G(D(H)^*) = \{(1, \varepsilon), (c, C)\} \cong \mathbb{Z}_2$. The quotient group $G(D(H))/G(D(H^*)) \cong \mathbb{Z}_2$ is generated by the class of (c, ε) and $\theta(c, \varepsilon)$ is the automorphism corresponding the to matrix $-Id$. Consider $\mathbb{Z}_2$ inside $GL_n(k)$ as the subgroup generated by $-Id$. Then $GL_n(k)/\mathbb{Z}_2$ embeds in $BQ(k, E(n))$. For $T \in GL_n(k)$ let $\overline{T}$ denote the class of T in $GL_n(k)/\mathbb{Z}_2$. The embedding is given by the map $\pi : GL_n(k)/\mathbb{Z}_2 \to BQ(k, E(n)), \overline{T} \mapsto [A_T]$.

The group $Aut_{Hopf}(H)$ acts on $BQ(k, H)$ by setting $[B]^\alpha = [A_\alpha][B][A_{\alpha^{-1}}]$ for any $\alpha \in Aut_{Hopf}(H), [B] \in BQ(k, H)$. In [7, Theorem 4.11], an alternative description of this action is given: $[B]^\alpha = [B(\alpha)]$ where $B(\alpha) = B$ as a k-algebra but with H-module and H-comodule structures given by

$$h \cdot_\alpha b = \alpha(h) \cdot b, \qquad \rho(b) = \sum b_{(0)} \otimes \alpha^{-1}(b_{(1)}),$$

for all $h \in H, b \in B(\alpha)$. We study this action for $H = E(n)$ and show that it stabilizes the subgroup $Sym_n(k)$ found in the previous section. We will prove that this action corresponds to the action of $GL_n(k)$ on $Sym_n(k)$ given by $L^T = TLT^t$ and we will next embed the subgroup $Sym_n(k) \rtimes (GL_n(k)/\mathbb{Z}_2)$ into $BQ(k, E(n))$.

Let $L \in Sym_n(k)$ and let $A^L := A^\sigma$ as in the proof of Theorem 4.3. Then for $f \in A^L$,

$$c \cdot f = U \circ f \circ U, \qquad x_i \cdot f = W_i \circ U \circ f \circ U - f \circ W_i$$

where U and W_i are the linear maps corresponding to the multiplication in $Cl(L)$ by u and w_i respectively. For $T = (t_{ij}) \in GL_n(k)$ we show that $A^L(T)$ is again central and simple and we compute its induced subalgebra by the $E(n)$-action. Since all automorphisms of $E(n)$ fix c, the $E(n)$-comodule structure on $A^L(T)$ is again determined by R_0 so that $A^L(T)$ is again a representative of an element in $BM(k, E(n), R_0)$. Besides, the c-action on $A^L(T)$ coincides with the c-action on A^L so the c-action is again strongly inner and $A^L(T)$ is again the representative of an element in $Ker(j_0^*) = Sym_n(k)$. The element induced by the c-action is again U while the element W_i' induced by the x_i-action is $W_i' = \sum_{j=1}^n t_{ij} W_j$. One may check that

$$(W_i')^2 = l_{ii}', \qquad W_i'W_j' + W_j'W_i' = 2l_{ij}'$$

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where l'_{ij} is the (i,j)-entry of the matrix $L' = TLT^t$. From the computations on the induced subalgebra we get that $[A_T][A^L][A_{T^{-1}}] = [A^L(T)] = [A^{TLT^t}]$.

Theorem 4.7 *The map*

$$\Psi : Sym_n(k) \rtimes (GL_n(k)/\mathbb{Z}_2) \to BQ(k, E(n)), (L, T) \mapsto [A^L][A_T]$$

is an injective group homomorphism.

Proof: It is not difficult to check that Ψ is a group homomorphism. Let $(L, \overline{T}) \in Sym_n(k) \rtimes (GL_n(k)/\mathbb{Z}_2)$ be such that $\Psi[(L, \overline{T})] = [A^L][A_T] = [k]$ in $BQ(k, E(n))$. Then $[A_T] = [A^L]^{-1} = [A^{-L}] \in Sym_n(k)$. So, for every $N \in Sym_n(k)$,

$$[A^{TNT^t}] = [A_T][A^N][A_T]^{-1} = [A^{-L}][A^N][A^L] = [A^N].$$

Since the map χ in Proposition 4.3 is injective $TNT^t = N$ for all $N \in Sym_n(k)$, which implies that $T = \pm Id$ and $\overline{T} = Id$. Then $[A_T] = [A^{-L}]$ is trivial and $(L, \overline{T}) = (0, \overline{Id})$. $\qquad\square$

Acknowledgements

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