Extensions of Rational Modules

J. Cuadra*
Dept. Álgebra y Análisis Matemático
Universidad de Almería
E-04120 Almería, Spain
email: jcdiaz@ual.es

Abstract

For a coalgebra C the rational functor $Rat(-) : \mathcal{M}_{C^*} \to \mathcal{M}_{C^*}$ is a left exact preradical whose associated linear topology is the family $\mathcal{F}_C$ consisting of all closed and cofinite right ideals of C^*. It was proved in [8] that if C is right $\mathcal{F}$-noetherian (every $I \in \mathcal{F}_C$ is finitely generated), then $Rat(-)$ is a radical. We show that the converse follows if C_1, the second term of the coradical filtration, is right $\mathcal{F}$-noetherian. This is a consequence of our main result on $\mathcal{F}$-noetherian coalgebras which states that the following assertions are equivalent: i) C is right $\mathcal{F}$-noetherian; ii) C_n is right $\mathcal{F}$-noetherian for all $n \in \mathbb{N}$; iii) $\mathcal{F}_C$ is closed under products and C_1 is right $\mathcal{F}$-noetherian. New examples of right $\mathcal{F}$-noetherian coalgebras are provided.

1 Introduction

Let C be a coalgebra over a field k and C^* its dual algebra. Let ${}^C\mathcal{M}$ denote the category of left C-comodules and $\mathcal{M}_{C^*}$ the category of right C^*-modules. It is well-known that ${}^C\mathcal{M}$ is isomorphic to the subcategory $Rat(\mathcal{M}_{C^*})$ of all rational right C^*-modules. Indeed, $Rat(\mathcal{M}_{C^*})$ is an hereditary pretorsion class in $\mathcal{M}_{C^*}$ (i.e., a class closed under subobjects, quotients, and arbitrary direct sums). The linear topology $\mathcal{F}_C$ on C^* associated to $Rat(\mathcal{M}_{C^*})$ consists of all closed and

*Supported by grant BFM2002-02717 from DGES.
AMS Subject Classification: 16W30, 18E40.

cofinite right ideals of C^*. The problem of when $Rat(\mathcal{M}_{C^*})$ is a torsion class (closed under extensions) has been studied in [4], [5], [8], [10], and, recently, in [2]. In this note we continue the study of this problem and relate it to a finiteness condition on $\mathcal{F}_C$, that every $I \in \mathcal{F}_C$ be finitely generated. Coalgebras satisfying this property are called right $\mathcal{F}$-noetherian. It was proved in [8] that if C is right $\mathcal{F}$-noetherian, then $Rat(\mathcal{M}_{C^*})$ is a torsion class. In this paper we find an hypothesis under which the converse holds. This is a consequence of our main theorem on $\mathcal{F}$-noetherian coalgebras that allows to lift the property of being $\mathcal{F}$-noetherian through the terms of the coradical filtration. It states that C is right $\mathcal{F}$-noetherian if and only if each term C_n of the coradical filtration is so, or equivalently, C_1 is right $\mathcal{F}$-noetherian and $\mathcal{F}_C$ is closed under products. From this theorem it follows that for C_1 being right $\mathcal{F}$-noetherian, $Rat(\mathcal{M}_{C^*})$ is a torsion class if and only if C is right $\mathcal{F}$-noetherian. The hypothesis of C_1 being right $\mathcal{F}$-noetherian is investigated in several cases. When C is almost connected, C_1 is right $\mathcal{F}$-noetherian if and only if C_1 is finite dimensional. If C is pointed, C_1 is right $\mathcal{F}$-noetherian if and only if for every group-like element $x \in G(C)$ the set $\{dim(P_{x,y}(C)) : y \in G(C)\}$ is bounded. Finally, as another consequence of the main theorem we provide new examples of right $\mathcal{F}$-noetherian coalgebras. These are constructed by imposing the terms of the coradical filtration to be left semiperfect.

2 Preliminaries

Let us to present several facts on torsion theory and coalgebras that we will need in the sequel.

Torsion theory (see [11, Chapter VI]): Let R be a ring and $\mathcal{M}_R$ the category of right R-modules. A class $\mathcal{C} \subset \mathcal{M}_R$ is called an *hereditary pretorsion class* if it is closed under subobjects, quotients, and direct sums. If, in addition, $\mathcal{C}$ is closed under extensions, then it is called a *torsion class*. A functor $r : \mathcal{M}_R \to \mathcal{M}_R$ is called a *left exact preradical* if it is a subfunctor of the identity functor of $\mathcal{M}_R$, it is left exact, and $r \circ r = r$. Such a functor is called a *radical* if $r(M/r(M)) = \{0\}$ for all $M \in \mathcal{M}_R$. A *right linear topology* on R is a family $\mathcal{T}$ of right ideals of R satisfying:

T1. If $I \in \mathcal{T}$ and $I \subset J$, then $J \in \mathcal{T}$.
T2. If $I, J \in \mathcal{T}$, then $I \cap J \in \mathcal{T}$.
T3. For any $I \in \mathcal{T}$ and $a \in R$, the right ideal $(I : a) \in \mathcal{T}$.

A *Gabriel topology* is a right linear topology $\mathcal{T}$ satisfying the additional axiom:

T4. If I is a right ideal of R and there is $J \in \mathcal{T}$ such that $(I : b) \in \mathcal{T}$ for all $b \in J$, then $I \in \mathcal{T}$.

There is a bijective correspondence between:

1) Right linear topologies on R.
2) Hereditary pretorsion classes in $\mathcal{M}_R$.
3) Left exact preradicals in $\mathcal{M}_R$.

Given a linear topology $\mathcal{T}$ the associated hereditary pretorsion class is $\mathcal{C} = \{M \in \mathcal{M}_R : Ann(m) \in \mathcal{T} \text{ for all } m \in M\}$. The associated left exact preradical r is defined as follows: for any $M \in \mathcal{M}_R$, $r(M) = \{m \in M : Ann(m) \in \mathcal{T}\}$. Conversely, if $\mathcal{C}$ is an hereditary pretorsion class, the corresponding linear topology $\mathcal{T}$ consists of all right ideals I of R for which $R/I \in \mathcal{C}$. The associated left exact preradical is defined in the following way: for any $M \in \mathcal{M}_R$, $r(M)$ is the sum of all submodules of M belonging to $\mathcal{C}$. This correspondence becomes a bijective correspondence between Gabriel topologies, torsion classes, and radicals.

Coalgebras and comodules: Throughout all vector spaces, algebras, coalgebras, $\otimes$, etc are over a fixed ground field k. For general facts on coalgebras and comodules we refer to [1], [7] and [12]. For a coalgebra C its dual algebra C^* is a topological vector space with the weak-* topology. The *closed* subspaces of C^* are the annihilators $W^{\perp(C^*)}$ of subspaces W of C. A subspace U of C^* is called *cofinite* if C^*/U is of finite dimension. A right ideal J of C^* is *closed and cofinite* if and only there is a finite dimensional right coideal W of C such that $J = W^{\perp(C^*)}$.

The category ${}^C\mathcal{M}$ of left C-comodules is isomorphic to the full subcategory $Rat(\mathcal{M}_{C^*})$ of $\mathcal{M}_{C^*}$ consisting of all rational right C^*-modules. Let $M \in \mathcal{M}_{C^*}$ and $m \in M$, we say that m is a *rational element* if there is $\rho_m = \sum_i c_i \otimes m_i \in C \otimes M$ such that

$$m \cdot c^* = \sum_{i=1}^{n} \langle c^*, c_i \rangle m_i \quad \forall c^* \in C^*.$$

The set consisting of all rational elements of M, denoted by $Rat(M)$, is a C^*-submodule of M. When $M = Rat(M)$, M is called *rational.* The assignment $Rat_C(-) : \mathcal{M}_{C^*} \to \mathcal{M}_{C^*}, M \mapsto Rat(M)$, called the *rational functor*, is a left exact preradical. The hereditary pretorsion class associated to this preradical is the subcategory $Rat(\mathcal{M}_{C^*})$ of all rational left C^*-modules. The linear topology

$\mathcal{F}_C$ corresponding to this class is the family of all closed cofinite right ideals of C^*. It follows from the Fundamental Theorem on Coalgebras that $\mathcal{F}_C$ is a symmetric linear topology. This means that for every $J \in \mathcal{F}_C$ there is a two-sided ideal K of C^* such that $K \in \mathcal{F}_C$ and $K \subset J$.

3 When is the rational functor a radical?

Definition 3.1 *A coalgebra C is said to have a right torsion rat functor if it satisfies one of the following equivalent conditions:*

i) $Rat(\mathcal{M}_{C^*})$ is closed under extensions.

ii) The rational functor is a radical.

iii) $\mathcal{F}_C$ is a Gabriel topology.

It was proved in [5, Proposition 4, Theorem 6] that these coalgebras enjoy the following properties:

Proposition 3.2 *Having a right torsion rat functor is closed under subcoalgebras and arbitrary direct sums.*

We give a necessary condition to have a torsion rat functor. We recall from [6] that a coalgebra C is *locally finite* if $D \wedge_C D$ is finite dimensional for any finite dimensional subcoalgebra D where $\wedge_C$ denotes the wedge product over C. See [12, Section 9.0] for the definition of the wedge product and its properties.

Lemma 3.3 *Let C be a coalgebra such that $\mathcal{F}_C$ is closed under products. Then C is locally finite.*

Proof: Let D be a finite dimensional subcoalgebra of C. By hypothesis, there is a finite dimensional subspace W of C such that $D^{\perp(C^*)}D^{\perp(C^*)} = W^{\perp(C^*)}$. Now, $D \wedge_C D = (D^{\perp(C^*)}D^{\perp(C^*)})^{\perp(C)} = W^{\perp(C^*)\perp(C)} = W$. Hence $D \wedge_C D$ is of finite dimension. ∎

The converse of Lemma 3.3 is not true. A counterexample may be found in [9, Example 3.4]. Since any Gabriel topology is closed under products ([11, Lemma 5.3]) we have:

Corollary 3.4 *Any coalgebra having a right torsion rat functor is locally finite.*

A sufficient condition to have a torsion rat functor was given in [8, page 521]:

Proposition 3.5 *Let C be a coalgebra such that that every $J \in \mathcal{F}_C$ is finitely generated. Then C has a right torsion rat functor.*

Definition 3.6 *A coalgebra C satisfying the hypothesis of Proposition 3.5 is called right $\mathcal{F}$-noetherian.*

Examples 3.7

i) Let A be an algebra such that every cofinite right ideal of A is finitely generated. It follows from [6, Theorem 3.3] that the finite dual A^o is right $\mathcal{F}$-noetherian. In particular, the dual of a finitely generated algebra is right and left $\mathcal{F}$-noetherian.

ii) Recall from [4] that a coalgebra C is called *left semiperfect* if the injective hull of any simple left C-comodule is finite dimensional. It was proved in [2, Theorem 2.12] that any left semiperfect coalgebra is right $\mathcal{F}$-noetherian.

iii) Subcoalgebras and arbitrary direct sums of right $\mathcal{F}$-noetherian are too, see [10, Corollary 4.9], [2, Proposition 2.8].

Our next step is to find out some structural properties of right $\mathcal{F}$-noetherian coalgebras. Recall from [13, Example 1.2] that a right C-comodule M is said to be *finitely cogenerated* if there is an injective C-comodule map $f : M \to C^{(n)}$ for some $n \in I\!N$. For any $M \in \mathcal{M}^C$, let $E(M)$ denote its injective hull.

Proposition 3.8 *Let I be a right coideal of C. The following assertions are equivalent:*

i) C/I is finitely cogenerated.

ii) $E(I)/I$ is finitely cogenerated.

iii) $I^{\perp(C^*)}$ is a finitely generated right ideal of C^*.

Proof: *i) $\Leftrightarrow$ ii)* It is just to take into account that $C \cong E(I) \oplus T$ for some subcomodule T of C and $C/I \cong E(I)/I \oplus T$, see [3, 1.5g].

i) $\Rightarrow$ iii) Notice that $(C/I)^* \cong I^{\perp(C^*)}$ as right C^*-modules. Since C/I is finitely cogenerated, there is an injective C-comodule map $f : C/I \to C^{(n)}$ for some $n \in I\!N$. The dual map $f^* : C^{*(n)} \to (C/I)^* \cong I^{\perp(C^*)}$ is a surjective C^*-module map. Thus $I^{\perp(C^*)}$ is finitely generated.

iii) $\Rightarrow$ *i)* We can express $C^{(n)} = W \otimes C$ with W a space of dimension n. Let $\{w_i\}_{i=1}^n$ be a basis of W and $\{w_i^*\}_{i=1}^n$ be its dual basis in W^*. By hypothesis, there is a surjective C^*-module map $f : W^* \otimes C^* \to I^{\perp(C^*)}$. Let $c_i^* = f(w_i^* \otimes \varepsilon)$ for all $i = 1, ..., n$ (ε is the counit of C). We define $g : C \to C^{(n)}$ by $g(c) = \sum_{i=1}^n w_i \otimes (\sum_{(c)} \langle c_i^*, c_{(1)} \rangle c_{(2)})$ for all $c \in C$. It is easy to check that $g^* = f$. Hence g is a C-comodule map. Moreover, $I^{\perp(C^*)} = Im(g^*) = Ker(g)^{\perp(C^*)}$. Then $I = Ker(g)$. Therefore C/I is finitely cogenerated. ∎

In view of the preceding result the closed right ideals of C^* are finitely generated if the quotients of C, as a right comodule, are finitely cogenerated. For a locally finite coalgebra this latter property may be characterized by studying the socle of such a quotients. It is known that there is a bijective correspondence between simple subcoalgebras of C and isomorphism classes of simple right C-comodules, see [1, Theorem 3.1.4]. Let $\mathcal{S}$ denote the set of simple subcoalgebras of C. For each $D \in \mathcal{S}$, let S_D be the corresponding simple right comodule. It is also known that the socle of C as a right comodule coincides with the coradical of C, C_0, and it decomposes as $C_0 = soc(C) \cong \oplus_{D \in \mathcal{S}} S_D^{(n_D)}$ where the $n_D's$ are natural numbers, see [3, 1.3.2]. For any right C-comodule M, $soc(M) \cong M \square_C C_0$. The isotypic component of S_D in M is given by $\rho^{-1}(M \square_C D)$ with $\rho : M \to M \otimes C$ being the structure map of M.

The following technical lemma is very useful to describe the simple comodules appearing in $soc(C/I)$ for a right coideal I of C.

Lemma 3.9 *Let I be a right coideal of C and E be a subcoalgebra of C. Then $(I \wedge_C E)/I \cong (C/I) \square_C E$ as right E-comodules.*

Proof: It is not difficult to verify that the map

$$\Phi : I \wedge_C E \to (C/I) \square_C E, \ c \mapsto \sum_{(c)} (c_{(1)} + I) \otimes c_{(2)}$$

is a surjective E-comodule map whose kernel is I. ∎

Lemma 3.10 *Let C be a locally finite coalgebra and I a right coideal of finite dimension. Then C/I is finitely cogenerated if and only if there is $\gamma \in \mathbb{N}$ such that $dim((I \wedge_C D)/I) \leq \gamma n_D dim(S_D)$ for every $D \in \mathcal{S}$.*

Proof: Using Lemma 3.9 we have that

$$soc(C/I) \cong (C/I) \square_C C_0 \cong \oplus_{D \in \mathcal{S}} (C/I) \square_C D \cong \oplus_{D \in \mathcal{S}} (I \wedge_C D)/I.$$

Each $(I \wedge_C D)/I \cong S_D^{(m_D)}$ and m_D is finite because C is locally finite. Notice that C/I is finitely cogenerated if and only if $soc(C/I)$ is finitely cogenerated as C_0-comodule. This happens if and only if there is $\gamma \in I\!N$ such that $m_D \leq \gamma n_D$ for all $D \in \mathcal{S}$. Equivalently, $dim((I \wedge_C D)/I) = m_D dim(S_D) \leq \gamma n_D dim(S_D)$. ∎

This description of the socle using the wedge yields a method to lift the property of being right $\mathcal{F}$-noetherian through the terms of the coradical filtration.

Theorem 3.11 *Let $\{C_n\}_{n \in I\!N}$ be the coradical filtration of C. The following assertions are equivalent:*

i) C is right $\mathcal{F}$-noetherian.

ii) C_n is right $\mathcal{F}$-noetherian for all $n \in I\!N$.

iii) C_1 is right $\mathcal{F}$-noetherian and $\mathcal{F}_C$ is closed under products.

Proof: Let first prove that if C is right $\mathcal{F}$-noetherian, then $\mathcal{F}_C$ is closed under products. Let $J, K \in \mathcal{F}_C$. By hypothesis J, K are finitely generated. Let H be a two-sided ideal such that $H \in \mathcal{F}_C$ and $H \subseteq J$. Again by hypothesis, H is finitely generated as a right ideal. By [6, Lemma 1.1.1], KH is finitely generated and cofinite. Since it is finitely generated, it is closed by [6, Proposition 1.3.1 b]. Hence $KH \in \mathcal{F}_C$. From $KH \subset KJ$, it follows that $KJ \in \mathcal{F}_C$.

i) ⇒ ii) and *i) ⇒ iii)* Being right $\mathcal{F}$-noetherian is closed under subcoalgebras.

ii) ⇒ i) Since C_n is right $\mathcal{F}$-noetherian, $\mathcal{F}_C$ is closed under products. From Lemma 3.3, C_n is locally finite for all $n \in I\!N$. In particular, C_1 is locally finite. By [6, Proposition 2.4.5], C is locally finite.

Let I be a finite dimensional right coideal of C. In view of Proposition 3.8 it suffices to prove that C/I is finitely cogenerated. There is $m \in I\!N$ such that $I \subset C_m$. By hypothesis and Proposition 3.8, C_{m+1}/I is finitely cogenerated. Applying Lemma 3.10 there is $\gamma \in I\!N$ such that $dim(I \wedge_{C_{m+1}} D)/I) \leq \gamma n_D dim(S_D)$ for every $D \in \mathcal{S}$. On the other hand, $I \wedge_C D \subset C_m \wedge_C C_0 = C_{m+1}$. By [6, 2.3.4], $I \wedge_C D = I \wedge_{C_{m+1}} D$. Then $dim((I \wedge_C D)/I) \leq \gamma n_D dim(S_D)$ for each $D \in \mathcal{S}$. Lemma 3.10 implies that C/I is finitely cogenerated.

iii) ⇒ i) We first check that every closed and cofinite maximal ideal is finitely generated as a right ideal. Let M be such an ideal and D be a simple subcoalgebra of C such that $M = D^{\perp(C^*)}$. Arguing as in *ii)* $\Rightarrow$ *i)* we obtain that C/D is finitely cogenerated. Thus M is finitely generated.

Let $J \in \mathcal{F}_C$ and K be a two-sided ideal such that $K \in \mathcal{F}_C$ and $K \subset J$. Let $P_1, ..., P_n$ be maximal closed and cofinite two-sided ideals containing J (there are only finitely many because J is cofinite). Set $N = \cap_{i=1}^{n} P_i$, then $Rad(C^*/K) = N/K$. Hence there is $n \in I\!N$ such that $N^n \subseteq K$. Let $H = P_1 \cdot ... \cdot P_n$, then $H^n \subseteq N^n \subseteq K \subset J$. The ideal H is closed and cofinite by hypothesis, and finitely generated because the $P_i's$ are so. Therefore J is finitely generated. ∎

As an immediate consequence:

Corollary 3.12 *Let C be a coalgebra such that C_1 is right $\mathcal{F}$-noetherian. The following assertions are equivalent:*

i) C has a right torsion rat functor.

ii) $\mathcal{F}_C$ is closed under products.

iii) C is right $\mathcal{F}$-noetherian.

We analyse in some cases the hypothesis of C_1 being right $\mathcal{F}$-noetherian. Recall that a coalgebra C is called *almost connected* if C_0 is finite dimensional.

Proposition 3.13 *Let C be an almost connected coalgebra. The following assertions are equivalent:*

i) C_1 is right $\mathcal{F}$-noetherian.

ii) C is locally finite.

iii) C_n is finite dimensional for all $n \in I\!N$.

Proof: *i)* $\Rightarrow$ *ii)* If C_1 is right $\mathcal{F}$-noetherian, then C_1 is locally finite. By [6, Theorem 2.4.5], C is locally finite.

ii) $\Rightarrow$ *iii)* Since C is locally finite, $C_n = C_0 \wedge_C C_{n-1}$ is finite dimensional.

iii) $\Rightarrow$ *i)* It is clear since C_1^* is finite dimensional. ∎

Corollary 3.14 *Let C be an almost connected coalgebra. The following assertions are equivalent:*

i) C has a right torsion rat functor.

ii) $\mathcal{F}_C$ is closed under products.

iii) C is locally finite.

iv) C is right $\mathcal{F}$-noetherian.

Proof: It is sufficient to prove *iii)* $\Rightarrow$ *iv)*. Let I be a finite dimensional right coideal of C. There is $n \in I\!N$ such that $I \subseteq C_n$. For any $D \in \mathcal{S}$, $I \wedge_C D \subseteq C_{n+1}$ which is finite dimensional. Setting $r = dim(C_{n+1}/I)$, we have that $dim((I \wedge_C D)/I) \leq rn_D dim(S_D)$ for all $D \in \mathcal{S}$. From Lemma 3.10 and Proposition 3.8, $I^{\perp(C^*)}$ is finitely generated. ∎

Remark 3.15 1.- Corollary 3.14 is a generalization of [4, Corollary 21] where C_0 was assumed 1-dimensional. Notice that Corollary 3.14 is equivalent to [10, Theorem 4.6] and [2, Theorem 2.10].

2.- The hypothesis of C being almost connected in Corollary 3.14 may be replaced by C being a direct sum of almost connected coalgebras. This holds from the fact that a direct sum of coalgebras is right $\mathcal{F}$-noetherian if and only if each term is so, see [10, Corollary 4.9], [2, Proposition 2.8]. This includes the cocommutative case.

3.- Note that being locally finite does not depend of the right or left side. Thus the statements of Corollary 3.14 are equivalent to their left versions.

In the pointed case the bound of Lemma 3.10 takes a clearer form. Let C be a pointed coalgebra and $G(C)$ its set of group-like elements. For $x, y \in G(C)$, let $P_{x,y}(C)$ denote the space of (x, y)-primitive elements.

Proposition 3.16 *Let C be a pointed coalgebra. Then C_1 is right $\mathcal{F}$-noetherian if and only if for each $x \in G(C)$ the set $\{dim(P_{x,y}(C)) : y \in G(C)\}$ is bounded.*

Proof: Assume that C_1 is right $\mathcal{F}$-noetherian. Then C_1 is locally finite. Let $x \in G(C)$ and $M = (kx)^{\perp(C_1^*)}$. By hypothesis, M is finitely generated. Proposition 3.8 yields that C_1/kx is finitely cogenerated. For each $y \in G(C)$ the isotypic component of ky in C_1/kx is $(C_1/kx)\square_C ky \cong ky^{(n_{x,y})}$ with $n_{x,y} = dim(P_{x,y}(C))$. By hypothesis and Lemma 3.10, the set $\{n_{x,y} : y \in G(C)\}$ is bounded.

Conversely, let D be a finite dimensional subcoalgebra of C_1. We will show that C/D is finitely cogenerated as a right comodule. For any $g, h \in G(C)$ let $P'_{g,h}(C)$ be a subspace of $P_{g,h}(C)$ such that $P_{g,h}(C) = k(g-h) \oplus P'_{g,h}(C)$. By the Taft-Wilson Lemma ([7, Theorem 5.4.1]), $C_1 = kG(C) \oplus (\oplus_{g,h\in G(C)} P'_{g,h}(C))$. We can write $D = (\oplus_{g\in F} kg) \oplus (\oplus_{g,h\in F} P'_{g,h}(D))$ with F being a finite subset of $G(C)$. For each $g \in F$ set $D_g = kg \oplus (\oplus_{h\in F} P'_{g,h}(D))$. Then $D = \oplus_{g\in F} D_g$ as right C-comodules. In order to prove that D is finitely cogenerated, it suffices to prove that D_g is so. The injective hull of D_g, $E(D_g) = kg \oplus (\oplus_{h\in G(C)} P'_{g,h}(C))$. Now, $E(D_g)/D_g \cong \oplus_{h\in G(C)-F} (kh)^{(m_{g,h})}$ with $m_{g,h} = dim(P'_{g,h}(C)/P'_{g,h}(D))$. From the hypothesis and Proposition 3.8 we deduce that D_g is finitely cogenerated. ∎

Corollary 3.17 *Let C be a pointed coalgebra satisfying that for each $x \in G(C)$ the set $\{dim(P_{x,y}(C)) : y \in G(C)\}$ is bounded. The following assertions are equivalent:*

i) C has a right torsion rat functor.

ii) $\mathcal{F}_C$ is closed under products.

iii) C is right $\mathcal{F}$-noetherian.

We finish this paper by constructing new examples of right $\mathcal{F}$-noetherian coalgebras. Combining Examples 3.7 ii) and Theorem 3.11 we have:

Corollary 3.18 *Let C be a coalgebra. If C_n is left semiperfect for all $n \in \mathbb{N}$, then C is right $\mathcal{F}$-noetherian.*

Example 3.19 For a quiver Γ, the path coalgebra $k\Gamma$ is the k-vector space generated by the paths in Γ with comultiplication Δ and counit ε defined by

$$\Delta(\gamma) = \sum_{\alpha\beta=\gamma} \alpha \otimes \beta, \qquad \varepsilon(\gamma) = \begin{cases} 0 & if\ \ |\gamma| > 0 \\ 1 & if\ \ |\gamma| = 0, \end{cases}$$

where α, β, γ are paths, $\alpha\beta$ is the concatenation of paths, and $|\cdot|$ denotes the length of a path. Assume that for every vertex $v \in \Gamma$ and any $n \in \mathbb{N}$ there is a finite number of paths of length less or equal than n ending at v. This condition assures that $(k\Gamma)_n$ is left semiperfect for all $n \in \mathbb{N}$. Then, the path coalgebra $k\Gamma$ is right $\mathcal{F}$-noetherian.

References

[1] E. Abe, *Hopf Algebras*. Cambridge University Press, 1977.

[2] J. Cuadra; C. Năstăsescu; F. Van Oystaeyen, *Graded Almost Noetherian Rings and Applications to Coalgebras*. To appear in J. Algebra.

[3] J.A. Green, *Locally Finite Representations*. J. Algebra **41** (1976), 137-171.

[4] B. I-Peng Lin, *Semiperfect Coalgebras.* J. Algebra **49** (1977), 357-373.

[5] B. I-Peng Lin, *Products of Torsion Theories and Applications to Coalgebras*. Osaka J. Math. **12** (1975), 433-439.

[6] R.G. Heyneman; D.E. Radford, *Reflexivity and Coalgebras of Finite Type.* J. Algebra **28** (1974), 215-246.

[7] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, CBMS Regional Conf. Ser. Math. **82**, Amer. Math. Soc., 1993.

[8] D.E. Radford, *Coreflexive Coalgebras*. J. Algebra **26** (1973), 512-535.

[9] D.E. Radford, *On the Structure of Ideals of the Dual Algebra of a Coalgebra.* Trans. Amer. Math. Soc. **198** (1974), 123-137.

[10] T. Shudo, *A Note on Coalgebras and Rational Modules*. Hiroshima Math. J. **6** (1976), 297-304.

[11] B. Stentröm, *Rings of Quotients*. Springer-Verlag, 1975.

[12] M. E. Sweedler, *Hopf Algebras*. Benjamin, 1969.

[13] M. Takeuchi, *Morita Theorems for Categories of Comodules.* J. Fac. Sci. Univ. Tokyo **24** (1977), 629-644.