Galois Corings and a Jacobson-Bourbaki type Correspondence.

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Abstract

The Jacobson-Bourbaki Theorem for division rings was formulated in terms of corings by Sweedler in [14]. Finiteness conditions hypotheses are not required in this new approach. In this paper we extend Sweedler's result to simple artinian rings using a particular class of corings, comatrix corings. A Jacobson-Bourbaki like correspondence for simple artinian rings is then obtained by duality.

Introduction

One of the key pieces in the Galois theory of fields and more generally of division rings is the Jacobson-Bourbaki Theorem, see [11, Chapter 7, Sections 2, 3] and [10, Section 8.2]. Let E be a division ring with prime field k. Consider the injective ring homomorphism $r : E \to \text{End}(_k E), e \mapsto r_e$ where $r_e(e') = e'e$ for all $e' \in E$ (the multiplication in $\text{End}(_k E)$ is the opposite of the composition). The Jacobson-Bourbaki Theorem states that there is a bijective correspondence between the set of division subrings D of E such that $_D E$ is finite dimensional and the set of subrings S of $\text{End}(_k E)$ such that $Im(r) \subseteq S$ and S_E is finite dimensional. The ring $\text{End}(_k E)$ is indeed an E-ring and the condition $Im(r) \subseteq S$ can be rephrased as S being an E-subring of $\text{End}(_k E)$. This correspondence is hidden behind the veil of the Galois connection in a Galois extension of fields or more generally of division rings, see [11] and [4, 5].

Using the dual structure of E-ring, the structure of E-coring, in [14] Sweedler gave a dual result to the Jacobson-Bourbaki Theorem. The advantage of using this dual structure is that the finiteness conditions needed in the Jacobson-Bourbaki Theorem can be dropped.

The finiteness conditions come implicit in the structure of coring through the fact that an element is mapped into a finite sum of elements via the comultiplication map. Sweedler's result asserts that there is a bijective correspondence between division subrings of E and quotient corings of the E-coring $E \otimes_k E$. The Jacobson-Bourbaki Theorem can be obtained from Sweedler's result by duality and this process makes clear why the finiteness conditions are needed.

The goal of this paper is to extend Sweedler's result from division rings to simple artinian rings replacing the Sweedler coring $E \otimes_k E$ by a more general type of coring, the comatrix coring introduced in [6] to describe the structure of cosemisimple corings. Let Σ be a finitely generated and projective right module over a ring A and let B be a simple artinian subring of $End(\Sigma_A)$. Let $\mathfrak{C}$ denotes the comatrix A-coring $\Sigma^* \otimes_B \Sigma$ constructed on the bimodule Σ and with coefficients B, see (1). Our main theorem (Theorem 2.4) states that there is a bijective correspondence between the set of all simple artinian subrings $B \subseteq C \subseteq End(\Sigma_A)$ and the set of all coideals J of $\mathfrak{C}$ such that the quotient coring $\mathfrak{C}/J$ is simple cosemisimple. If in addition Σ_A is simple, then any quotient coring of $\mathfrak{C}$ is simple cosemisimple, thus obtaining a bijective correspondence between intermediate division subrings of $B \subseteq End(\Sigma_A)$ and coideals of $\mathfrak{C}$ (see Remark 2.5). Sweedler's result, together with some additional information on conjugated subextensions, is then obtained as a consequence by taking A a division ring and $\Sigma = A$ (Corollary 2.6). An example illustrating the bijective correspondence is worked out. This closes Section 2, which thus contains our main results.

Section 1 recalls the most fundamental results on comatrix corings, Galois corings, and cosemisimple corings needed in the sequel. We also include a homological characterization of Galois corings (Theorem 1.5), which gives as a consequence that if the canonical map is surjective for a quasi-projective comodule with a generating condition, then the coring is Galois (Corollary 1.6). This corollary is used in the proof of the main result of Section 3, namely, Theorem 3.2, which states a Jacobson-Bourbaki correspondence for simple artinian subextensions of a ring extension. This correspondence is dual to the stated in Section 2. We complete the paper with an Appendix that contains a complete classification of the simple cosemisimple $\mathbb{C}/\mathbb{R}$–corings.

We next fix notation and present some basic definitions. In the sequel A, B denote associative and unitary algebras over a commutative ring K. By $\otimes_A$ we denote the tensor product over A. The category of right A-modules is denoted by $\mathcal{M}_A$. Bimodules are assumed to be centralized by K. An A-coring (or A/K–coring, when K is not obvious from the context) is a triple $(\mathfrak{C}, \Delta, \epsilon)$ where $\mathfrak{C}$ is an A-bimodule and $\Delta : \mathfrak{C} \to \mathfrak{C} \otimes_A \mathfrak{C}$ (comultiplication) and $\epsilon : \mathfrak{C} \to A$ (counit) are A-bimodule maps such that $(id_{\mathfrak{C}} \otimes_A \Delta)\Delta = (\Delta \otimes_A id_{\mathfrak{C}})\Delta$ and $(\epsilon \otimes_A id_{\mathfrak{C}})\Delta = (id_{\mathfrak{C}} \otimes_A \epsilon)\Delta = id_{\mathfrak{C}}$. In a categorical language, a coring is just a coalgebra in the monoidal category of A-bimodules with the tensor product $\otimes_A$ as a product. For $c \in \mathfrak{C}$ we will write $\Delta(c) = \sum c_{(1)} \otimes_A c_{(2)}$. The left dual $^*\mathfrak{C} = Hom(_A\mathfrak{C}, _AA)$ of the coring $\mathfrak{C}$ is an A^{op}-ring (A^{op} denotes the opposite ring of A) with the product $(f * g)(c) = \sum f(c_{(1)}g(c_{(2)}))$ for all $f, g \in {}^*\mathfrak{C}$ and $c \in \mathfrak{C}$. Similarly, the right dual $\mathfrak{C}^*$ of $\mathfrak{C}$ is an A^{op}-ring with the product $(f * g)(c) = \sum g(f(c_{(1)})c_{(2)})$ for $f, g \in \mathfrak{C}^*$ and $c \in \mathfrak{C}$.

A right $\mathfrak{C}$-comodule is a right A-module together with an A-module map $\rho_M : M \rightarrow M \otimes_A \mathfrak{C}$ such that $(id_M \otimes_A \Delta)\rho_M = (\rho_M \otimes id_\mathfrak{C})\rho_M$ and $(id_M \otimes \epsilon)\rho_M = id_M$. A $\mathfrak{C}$-comodule map between two right $\mathfrak{C}$-comodules M and N is an A-module map $f : M \rightarrow N$ such that $(f \otimes_A id_\mathfrak{C})\rho_M = \rho_N f$. By $Hom_\mathfrak{C}(M, N)$ we will denote the K-module of all $\mathfrak{C}$-comodule maps between M and N. The category whose objects are right $\mathfrak{C}$-comodules and whose morphisms are $\mathfrak{C}$-comodule maps is denoted by $\mathcal{M}^\mathfrak{C}$. It is an additive K–linear category and if $_A\mathfrak{C}$ is flat, then it is a Grothendieck category. The product of every endomorphism ring of an object in an additive category is by default the composition. We adopt, however, the following convention in the case of modules: the product of the endomorphism ring $End(M_A)$ of a right A–module M is the composition, although by $End(_A N)$ we will denote the opposite ring of the endomorphism ring of a left A–module N, being then its product the opposite of the composition.

1 Comatrix corings, Galois comodules and cosemisimple corings

Let Σ be a $B - A$–bimodule, and assume that Σ_A is finitely generated and projective with a finite dual basis $\{(e_i^*, e_i)\} \subseteq \Sigma^* \times \Sigma$. We can consider a coring structure [6, Proposition 2.1] over the A–bimodule $\Sigma^* \otimes_B \Sigma$ with comultiplication and counit defined respectively by

$$\Delta(\phi \otimes_B x) = \sum_i \phi \otimes_B e_i \otimes_A e_i^* \otimes_B x, \qquad \epsilon(\phi \otimes_B x) = \phi(x). \tag{1}$$

The comultiplication is independent of the choice of the dual basis. This coring will be called the A-comatrix coring on Σ with coefficients in B. The A-module Σ becomes a right $\Sigma^* \otimes_B \Sigma$–comodule with coaction

$$\varrho_\Sigma(x) = \sum_i e_i \otimes_A e_i^* \otimes_B x.$$

Assume Σ to be the underlying A–module of a right comodule over some A–coring $\mathfrak{C}$, with structure map $\rho_\Sigma : \Sigma \rightarrow \Sigma \otimes_A \mathfrak{C}$. In this case, with $T = End(\Sigma_\mathfrak{C})$, we have from [6, Proposition 2.7] that the map $\mathsf{can} : \Sigma^* \otimes_T \Sigma \rightarrow \mathfrak{C}$ defined by

$$\mathsf{can}(\phi \otimes_T x) = \sum \phi(x_0)x_1 \qquad (\rho_\Sigma(x) = \sum x_0 \otimes_A x_1)$$

is a homomorphism of A–corings. This canonical map allowed to extend [6, Definition 3.4] the notion of a Galois coring without assuming the existence of group-like elements. When the role of the comodule Σ is stressed, the terminology of Galois comodules introduced in [3] is more convenient. Probably, the best solution here is to mention both the coring and the comodule.

Definition 1.1. The pair $(\mathfrak{C}, \Sigma)$ is said to be *Galois* if can is an isomorphism. In such a case, we say that $\mathfrak{C}$ is a *Galois coring* and Σ is termed a *Galois comodule*. The extension $T \subseteq End(\Sigma_\mathfrak{C})$ is called a $(\mathfrak{C}, \Sigma)$-*Galois extension*.

The notion of a noncommutative G–Galois extension, may be recovered from this definition, see [6, Example 2.9]. In this case, the corresponding Galois coring has a group-like element. We give an example of a noncommutative Galois extension for a coring without group-like elements.

Example 1.2. Let $\mathbb{C}$ and $\mathbb{H}$ denote the complex number field and the Hamilton's quaternions algebra respectively. Consider the right $\mathbb{C}$-vector space $\mathfrak{T}$ with basis $\{c, s\}$. This vector space becomes a $\mathbb{C}$-bimodule with left action $ic = c$ and $is = -s$ and a $\mathbb{C}$-coring with comultiplication and counity defined by

$$\Delta(c) = c \otimes c - s \otimes s, \quad \epsilon(c) = 1,$$
$$\Delta(s) = c \otimes s + s \otimes c, \quad \epsilon(s) = 0.$$

Analogously to the coalgebra case this coring is called *the trigonometric coring*. This coring has no group-like elements. Let Σ be a right $\mathbb{C}$-vector space with basis $\{v_1, v_2\}$. The map $\rho : \Sigma \to \Sigma \otimes \mathfrak{T}$ defined by

$$v_1 \mapsto v_1 \otimes c + v_2 \otimes s, \ v_2 \mapsto v_2 \otimes c - v_1 \otimes s$$

makes Σ into a right $\mathfrak{T}$-comodule. It is not difficult to check that $(\mathfrak{T}, \Sigma)$ is a Galois $\mathbb{C}$-coring. The corresponding Galois extension is the well-known embedding of $\mathbb{H}$ into $M_2(\mathbb{C})$:

$$i \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad j \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Our first objective is to enrich [3, 18.26] with a new characterization of Galois comodules. We need two previous observations. The first one is that if $\mathfrak{C}$ is flat as a left A–module then, using that the forgetful functor $U : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$ is faithful and exact, the following lemma can be proved (see [9]).

Lemma 1.3. *If $\mathfrak{C}$ is flat as a left A–module, then a right $\mathfrak{C}$–comodule M is finitely generated in the Grothendieck category $\mathcal{M}^{\mathfrak{C}}$ if and only if M is finitely generated as a right A–module.*

We have a pair of functors

$$\mathcal{M}_T \underset{\mathrm{Hom}_{\mathfrak{C}}(\Sigma, -)}{\overset{-\otimes_T \Sigma}{\rightleftarrows}} \mathcal{M}^{\mathfrak{C}} \tag{2}$$

where $- \otimes_T \Sigma$ is left adjoint to $\mathrm{Hom}_{\mathfrak{C}}(\Sigma, -)$. If $\chi : \mathrm{Hom}_{\mathfrak{C}}(\Sigma, -) \otimes_T \Sigma \to id_{\mathcal{M}^{\mathfrak{C}}}$ is the counit of this adjunction, then the canonical map can be expressed [6, Lemma 3.1] as the composite

$$\mathsf{can} : \Sigma^* \otimes_T \Sigma \xrightarrow{(-\otimes_A id_{\mathfrak{C}}) \circ \rho_\Sigma \otimes_T \Sigma} \mathrm{Hom}_{\mathfrak{C}}(\Sigma, \mathfrak{C}) \otimes_T \Sigma \xrightarrow{\chi_{\mathfrak{C}}} \mathfrak{C} \tag{3}$$

Observe that $\chi_{\mathfrak{C}}$ is an isomorphism if and only if can is so. Our second observation is the following lemma.

Lemma 1.4. *Let $\eta : id_{\mathcal{M}_T} \to \mathrm{Hom}_{\mathfrak{C}}(\Sigma, - \otimes_T \Sigma)$ be the unit of the adjunction (2). Consider the map* $\mathsf{can}_* : \mathrm{Hom}_{\mathfrak{C}}(\Sigma, \Sigma^* \otimes_T \Sigma) \to \mathrm{Hom}_{\mathfrak{C}}(\Sigma, \mathfrak{C})$, $f \mapsto \mathsf{can} \circ f$. *Then the following composition is the identity map on* Σ^*

$$\Sigma^* \xrightarrow{\;\;\eta_{\Sigma^*}\;\;} \mathrm{Hom}_{\mathfrak{C}}(\Sigma, \Sigma^* \otimes_T \Sigma) \xrightarrow{\;\;\mathsf{can}_*\;\;} \mathrm{Hom}_{\mathfrak{C}}(\Sigma, \mathfrak{C}) \cong \Sigma^*$$

Proof. If we apply the displayed composite map to $\phi \in \Sigma^*$, then we obtain the map from Σ to A given by $x \mapsto \sum \epsilon_{\mathfrak{C}}(\phi(x_0)x_1)$, which is nothing but ϕ since

$$\sum \epsilon_{\mathfrak{C}}(\phi(x_0)x_1) = \sum \phi(x_0)\epsilon_{\mathfrak{C}}(x_1) = \sum \phi(x_0\epsilon_{\mathfrak{C}}(x_1)) = \phi(x)$$

$\square$

We are now in position to state our homological characterization of Galois comodules.

Theorem 1.5. *Let Σ be a right comodule over an A–coring $\mathfrak{C}$ and assume that Σ is finitely generated and projective as a right A-module. If $_A\mathfrak{C}$ is flat, then $(\mathfrak{C}, \Sigma)$ is Galois if and only if there exists an exact sequence $\Sigma^{(J)} \to \Sigma^{(I)} \to \mathfrak{C} \to 0$ in $\mathcal{M}^{\mathfrak{C}}$ such that the sequence*

$$\mathrm{Hom}_{\mathfrak{C}}(\Sigma, \Sigma^{(J)}) \longrightarrow \mathrm{Hom}_{\mathfrak{C}}(\Sigma, \Sigma^{(I)}) \longrightarrow \mathrm{Hom}_{\mathfrak{C}}(\Sigma, \mathfrak{C}) \longrightarrow 0$$

is exact.

Proof. Consider $T^{(J)} \to T^{(I)} \to \Sigma^* \to 0$ a free presentation of the left T–module Σ^*. By tensorizing with $_T\Sigma$ we obtain an exact sequence $\Sigma^{(J)} \to \Sigma^{(I)} \to \Sigma^* \otimes_T \Sigma \to 0$. We have then the following commutative diagram in $\mathcal{M}_T$

$$
\begin{array}{ccccccccc}
T^{(J)} & \longrightarrow & T^{(I)} & \longrightarrow & \Sigma^* & \longrightarrow & 0 & & (4)\\
\big\downarrow{\scriptstyle \eta_{T^{(J)}}} & & \big\downarrow{\scriptstyle \eta_{T^{(I)}}} & & \big\downarrow{\scriptstyle \eta_{\Sigma^*}} & & & & \\
\mathrm{Hom}_{\mathfrak{C}}(\Sigma, \Sigma^{(J)}) & \longrightarrow & \mathrm{Hom}_{\mathfrak{C}}(\Sigma, \Sigma^{(I)}) & \longrightarrow & \mathrm{Hom}_{\mathfrak{C}}(\Sigma, \Sigma^* \otimes_T \Sigma) & \longrightarrow & 0 & &
\end{array}
$$

Now, $\eta_{T^{(J)}}$ and $\eta_{T^{(I)}}$ are isomorphisms because Σ is finitely generated in the Grothendieck category $\mathcal{M}^{\mathfrak{C}}$ (see [9]). By Lemma 1.4, η_{Σ^*} is an isomorphism if and only if can_* is bijective. So if we assume that $(\mathfrak{C}, \Sigma)$ is Galois, then η_{Σ^*} is an isomorphism and, from (4), the following sequence is exact

$$\mathrm{Hom}_{\mathfrak{C}}(\Sigma, \Sigma^{(J)}) \longrightarrow \mathrm{Hom}_{\mathfrak{C}}(\Sigma, \Sigma^{(I)}) \longrightarrow \mathrm{Hom}_{\mathfrak{C}}(\Sigma, \Sigma^* \otimes_T \Sigma) \longrightarrow 0$$

Observe that the isomorphism of A–corings $\mathsf{can} : \Sigma^* \otimes_T \Sigma \cong \mathfrak{C}$ is also an isomorphism of right $\mathfrak{C}$–comodules. This finishes the proof of the necessary condition. For the sufficiency, consider the commutative diagram in $\mathcal{M}^{\mathfrak{C}}$ with exact rows

$$
\begin{array}{ccccccccc}
\Sigma^{(J)} & \longrightarrow & \Sigma^{(I)} & \longrightarrow & \mathfrak{C} & \longrightarrow & 0 \\
\big\uparrow{\scriptstyle \chi_{\Sigma^{(J)}}} & & \big\uparrow{\scriptstyle \chi_{\Sigma^{(I)}}} & & \big\uparrow{\scriptstyle \chi_{\mathfrak{C}}} & & \\
\mathrm{Hom}_{\mathfrak{C}}(\Sigma, \Sigma^{(J)}) \otimes_T \Sigma & \longrightarrow & \mathrm{Hom}_{\mathfrak{C}}(\Sigma, \Sigma^{(I)}) \otimes_T \Sigma & \longrightarrow & \mathrm{Hom}_{\mathfrak{C}}(\Sigma, \mathfrak{C}) \otimes_T \Sigma & \longrightarrow & 0
\end{array}
$$

In this diagram, $\chi_{\Sigma^{(J)}}$ and $\chi_{\Sigma^{(I)}}$ are isomorphisms because Σ is finitely generated in $\mathcal{M}^{\mathfrak{C}}$. We have then that $\chi_{\mathfrak{C}}$ is an isomorphism and, thus, $(\mathfrak{C}, \Sigma)$ is Galois. $\square$

T. Brzeziński has shown in [2] that a simple comodule with `can` surjective is Galois. As a consequence of Theorem 1.5 we derive a generalization of Brzeziński's result. Following the definition given for modules in [1], we say the comodule Σ is *quasi-projective* if for every exact sequence $\Sigma \to N \to 0$ in $\mathcal{M}^{\mathfrak{C}}$ then the sequence of abelian groups $\mathrm{Hom}_{\mathfrak{C}}(\Sigma, \Sigma) \to \mathrm{Hom}_{\mathfrak{C}}(\Sigma, N) \to 0$ is exact. Since, by Lemma 1.3, Σ is finitely generated in $\mathcal{M}^{\mathfrak{C}}$, a straightforward adaptation of [1, Proposition 16.2.(2)] to Grothendieck categories gives that $\mathrm{Hom}_{\mathfrak{C}}(\Sigma, -)$ will already preserve exact sequences of the form $\Sigma^{(I)} \to N \to 0$. The following corollary is then easily deduced from Theorem 1.5, making use once more of the "*AB5*" condition.

Corollary 1.6. *Assume that Σ is quasi-projective in $\mathcal{M}^{\mathfrak{C}}$ and generates every subcomodule of any finite direct sum of copies of Σ (e.g., Σ is a semisimple comodule). If `can` is surjective, then $(\mathfrak{C}, \Sigma)$ is Galois.*

Now, let us recall from [7, 6] the structure of cosemisimple corings, which is tightly related to the coring version of the Generalized Descent Theorem formulated in [6, Theorem 3.10]. A coring is said to be *cosemisimple* if it satisfies one of the equivalent conditions of the following theorem.

Theorem 1.7. *[7, Theorem 3.1] The following assertions for an A-coring $\mathfrak{C}$ are equivalent:*
- (i) *Every left $\mathfrak{C}$-comodule is semisimple and ${}^{\mathfrak{C}}\mathcal{M}$ is abelian.*
- (ii) *Every right $\mathfrak{C}$-comodule is semisimple and $\mathcal{M}^{\mathfrak{C}}$ is abelian.*
- (iii) *${}_{\mathfrak{C}}\mathfrak{C}$ is semisimple and $\mathfrak{C}_A$ is flat.*
- (iv) *$\mathfrak{C}_{\mathfrak{C}}$ is semisimple and ${}_A\mathfrak{C}$ is flat.*

A coring is called *simple* if it has no non trivial subbicomodules. It was proved in [7, Theorem 3.7] that any cosemisimple coring decomposes in a unique way as a direct sum of simple cosemisimple corings. An example of simple cosemisimple coring is the comatrix A-coring $\Sigma^* \otimes_B \Sigma$, where Σ_A is finitely generated and projective and $B \subseteq End(\Sigma_A)$ is simple artinian. The following result shows that indeed all simple cosemisimple corings can be obtained in this way.

Proposition 1.8. *[6, Proposition 4.2] Let $\mathfrak{C}$ be a simple cosemisimple A-coring and $\Sigma_{\mathfrak{C}}$ a finitely generated right $\mathfrak{C}$-comodule. Then $T = End(\Sigma_{\mathfrak{C}})$ is simple artinian, Σ_A is finitely generated projective and the canonical map $\mathsf{can} : \Sigma^* \otimes_T \Sigma \to \mathfrak{C}$ is an isomorphism.*

A more precise description of simple cosemisimple corings is given by the following structure theorem. It may be viewed as a generalization of the Artin-Wedderburn Theorem.

Theorem 1.9. *[6, Theorem 4.3] An A-coring $\mathfrak{C}$ is simple cosemisimple if and only if there is a finitely generated projective right A-module Σ and a division subring $D \subseteq End(\Sigma_A)$ such that $\mathfrak{C} \cong \Sigma^* \otimes_D \Sigma$ as A-corings.*

In such a case, if Γ is another finitely generated and projective right A-module and $E \subseteq End(\Gamma_A)$ is a division subring, then $\mathfrak{C} \cong \Gamma^* \otimes_E \Gamma$ if and only if there is an isomorphism of right A-modules $g : \Sigma \to \Gamma$ such that $gDg^{-1} = E$.

In view of this structure theorem, for a field extension A/k, the classification of simple cosemisimple A-corings centralized by k is reduced to the classification of finite dimensional division algebras over k and to the study of how these division algebras embed in matrix algebras over A. The first problem leads to the Brauer group theory of a field and the second one can be treated with the help of the Skolem-Noether Theorem. The complete classification for the field extension $\mathbb{C}/\mathbb{R}$ is obtained in the Appendix.

2 The Galois connection from a coring point of view

Let Σ_A be a finitely generated and projective right A-module and denote by $S = \operatorname{End}(\Sigma_A)$ its endomorphism ring. Let $B \subseteq S$ be a subring, and consider the comatrix A–coring $\mathfrak{C} = \Sigma^* \otimes_B \Sigma$. A typical situation is to consider $B = k$, a field, and $S = M_n(k)$, the ring of square matrices of order n over k. Let $\mathsf{Subext}(S/B)$ denote the set of all ring subextensions $B \subseteq C \subseteq S$, and $\mathsf{Coideals}(\mathfrak{C})$ be the set of all coideals of $\mathfrak{C}$. Consider the maps

$$\mathcal{J} : \mathsf{Subext}(S/B) \; \underset{\longleftarrow}{\overset{\longrightarrow}{\rule{1.5cm}{0pt}}} \; \mathsf{Coideals}(\mathfrak{C}) : \mathcal{R} \tag{5}$$

defined as follows. For each subextension $C \in \mathsf{Subext}(S/B)$ we have a canonical homomorphism of A–corings $\mathfrak{C} = \Sigma^* \otimes_B \Sigma \to \Sigma^* \otimes_C \Sigma$, whose kernel $\mathcal{J}(C)$ is a coideal of $\mathfrak{C}$. Conversely, given a coideal J of $\mathfrak{C}$, then, by [6, Proposition 2.5], $B \subseteq \operatorname{End}(\Sigma_{\mathfrak{C}}) \subseteq \operatorname{End}(\Sigma_{\mathfrak{C}/J}) \subseteq S$, so that $\mathcal{R}(J) = \operatorname{End}(\Sigma_{\mathfrak{C}/J})$ is a subextension of $B \subseteq S$. Both $\mathcal{J}$ and $\mathcal{R}$ are inclusion preserving maps. The following proposition, which generalizes [13, Proposition 6.1], collects some more of their relevant general properties.

Proposition 2.1. *The maps defined in* (5) *enjoy the following properties.*

(1) $\mathcal{R}\mathcal{J}(C) \supseteq C$ *for every* $C \in \mathsf{Subext}(S/B)$.

(2) $\mathcal{J}\mathcal{R}(J) \subseteq J$ *for every* $J \in \mathsf{Coideals}(\mathfrak{C})$.

(3) $\mathcal{R}\mathcal{J}(C) = C$ *if and only if* $\operatorname{End}(\Sigma_{\Sigma^* \otimes_C \Sigma}) = C$.

(4) $\mathcal{J}\mathcal{R}(J) = J$ *if and only if* $(\mathfrak{C}/J, \Sigma)$ *is Galois.*

(5) The maps $\mathcal{J}$ *and* $\mathcal{R}$ *establish a bijection between the set consisting of ring subextensions* $B \subseteq C \subseteq S$ *such that* $\operatorname{End}(\Sigma_{\Sigma^* \otimes_C \Sigma}) = C$ *and the set of coideals* J *of* $\mathfrak{C}$ *such that* $(\mathfrak{C}/J, \Sigma)$ *is Galois.*

Proof. A direct computation gives that $\mathcal{R}\mathcal{J}(C) = \operatorname{End}(\Sigma_{\Sigma^* \otimes_C \Sigma})$ for every $C \in \mathsf{Subext}(S/B)$. Thus, *(1)* follows from [6, Proposition 2.5]. This gives also *(3)*. Now, for a given coideal $J \in \mathsf{Coideals}(\mathfrak{C})$, we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{J}\mathcal{R}(J) & \longrightarrow & \Sigma^* \otimes_B \Sigma & \longrightarrow & \Sigma^* \otimes_{\operatorname{End}(\Sigma_{\mathfrak{C}/J})} \Sigma & \longrightarrow & 0 \\
& & \downarrow & & \| & & \downarrow{\scriptstyle \mathsf{can}_{\mathfrak{C}/J}} & & \\
0 & \longrightarrow & J & \longrightarrow & \mathfrak{C} & \longrightarrow & \mathfrak{C}/J & \longrightarrow & 0
\end{array} \tag{6}$$

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which implies *(2)* and *(4)*. Finally, let us prove *(5)*: for any ring subextension $C \in$ Subext(S/B), the A-coring $\Sigma^* \otimes_C \Sigma$ is Galois by [6, Lemma 3.9]. Hence $\mathfrak{C}/\mathcal{J}(C)$ is Galois. Conversely, for any $J \in$ Coideals$(\mathfrak{C})$ we have End$(\Sigma_{\mathfrak{C}}) \subseteq$ End$(\Sigma_{\mathfrak{C}/J})$. If $\mathfrak{C}/J$ is Galois, then the canonical map $\mathsf{can}_{\mathfrak{C}/J}$ is an isomorphism and, therefore, End$(\Sigma_{\Sigma^* \otimes_{\mathrm{End}(\Sigma_{\mathfrak{C}/J})} \Sigma}) =$ End$(\Sigma_{\mathfrak{C}/J})$. Statement *(5)* follows now from *(3)* and *(4)*. □

Remark 2.2. If $_A(\Sigma^* \otimes_C \Sigma)$ is locally projective (see e.g. [3, 42.10] for this notion), then $C = \mathrm{End}(\Sigma_{\Sigma^* \otimes_C \Sigma})$ if and only if $_C\Sigma$ is faithfully balanced, i.e., C is isomorphic to the biendomophisms ring of $_C\Sigma$ under the natural map. This is because for $_A(\Sigma^* \otimes_C \Sigma)$ locally projective, End$(\Sigma_{\Sigma^* \otimes_C \Sigma}) = \mathrm{End}(_{*(\Sigma^* \otimes_C \Sigma)}\Sigma)$, see [3, 19.3]. Then, by [6, Proposition 2.1], $^*(\Sigma^* \otimes_C \Sigma) \cong \mathrm{End}(_C\Sigma)^{op}$ canonically and, therefore,

$$\mathrm{End}(\Sigma_{\Sigma^* \otimes_C \Sigma}) \;\; = \mathrm{End}(_{*(\Sigma^* \otimes_C \Sigma)}\Sigma) = \mathrm{End}(_{\mathrm{End}(_C\Sigma)^{op}}\Sigma) = \mathrm{End}(\Sigma_{\mathrm{End}(_C\Sigma)}).$$

Remark 2.3. We have from [6, Proposition 2.5] that

$$\mathrm{End}(\Sigma_{\Sigma^* \otimes_C \Sigma}) = \{f \in \mathrm{End}(\Sigma_A) \mid f \otimes_C x = 1 \otimes_C f(x), \text{ for every } x \in \Sigma\} := \overline{C}.$$

Since $\Sigma^* \otimes_C \Sigma$ is Galois, we get that $\overline{\overline{C}} = \overline{C}$. Proposition 2.1 gives a bijective correspondence between coideals J of $\mathfrak{C}$ such that $\mathfrak{C}/J$ is Galois and subextensions C such that $C = \overline{C}$.

We are now ready to state a generalization of Sweedler's predual to Jacobson-Bourbaki Theorem [14, Theorem 2.1]. We will say that $C, C' \in$ Subext(S/B) are *conjugated* in S if there is an unit $g \in S$ such that $C' = gCg^{-1}$.

Theorem 2.4. *Let Σ be a finitely generated projective right A-module and let $S = \mathrm{End}(\Sigma_A)$. Let B be a subring of S and consider the comatrix A-coring $\mathfrak{C} = \Sigma^* \otimes_B \Sigma$. Denote by $\mathcal{S}$ the set of all simple artinian subrings $B \subseteq C \subseteq S$ and let $\mathcal{T}$ denote the set of all coideals J of $\mathfrak{C}$ such that $\mathfrak{C}/J$ is simple cosemisimple. Then the maps*

$$\mathcal{R}(-) : \mathcal{T} \to \mathcal{S}, \;\; J \mapsto \mathrm{End}(\Sigma_{\mathfrak{C}/J})$$
$$\mathcal{J}(-) : \mathcal{S} \to \mathcal{T}, \;\; C \mapsto Ker(\mathfrak{C} \twoheadrightarrow \Sigma^* \otimes_C \Sigma)$$

are inverse to each other. If, in addition, A is a (noncommutative) local ring, then two intermediate simple artinian subrings C and C' are conjugated in S if and only if $\mathfrak{C}/\mathcal{J}(C)$ and $\mathfrak{C}/\mathcal{J}(C')$ are isomorphic as A-corings.

Proof. For $C \in \mathcal{S}$, the A-coring $\Sigma^* \otimes_C \Sigma$ is a simple cosemisimple A-coring in virtue of [6, Proposition 4.2]. Hence $\mathfrak{C}/\mathcal{J}(C) \cong \Sigma^* \otimes_C \Sigma$ is simple cosemisimple and so $\mathcal{J}(C) \in \mathcal{T}$. From [6, Proposition 2.5], $C \subseteq \mathrm{End}(\Sigma_{\Sigma^* \otimes_C \Sigma})$. Since $_C\Sigma$ is faithfully flat, [6, Theorem 3.10] yields $C = \mathrm{End}(\Sigma_{\Sigma^* \otimes_C \Sigma})$. By Theorem 2.1, $\mathcal{R}\mathcal{J}(C) = C$.

Assume that $J \in \mathcal{T}$, then $\mathfrak{C}/J$ is simple cosemisimple. By [6, Theorem 4.1], $\mathfrak{C}/J$ is flat as a left A-module which implies, by [9, Lemma 3.1], that Σ is finitely generated as a right $\mathfrak{C}/J$-comodule. Hence End$(\Sigma_{\mathfrak{C}/J})$ is a simple artinian ring. So $\mathcal{R}(J) \in \mathcal{S}$. Furthermore, $\mathfrak{C}/J$ is Galois by [6, Proposition 4.2]. By Theorem 2.1, $\mathcal{J}\mathcal{R}(J) = J$.

For the second assertion, let C, C' be intermediate simple artinian subrings and let $g \in S$ be invertible such that $gCg^{-1} = C'$. We check that $\Sigma^* \otimes_C \Sigma$ is isomorphic to $\Sigma^* \otimes_{C'} \Sigma$. The set $\{e_i^* g, g(e_i)\}_{i=1}^n$ is a dual basis for Σ and, by [6, Remark 2.2] the coring structure on $\Sigma^* \otimes_{C'} \Sigma$ defined by this dual basis is the same as the one defined by $\{e_i^*, e_i\}_{i=1}^n$. Consider the A-bimodule map,

$$\psi : \Sigma^* \times \Sigma \to \Sigma^* \otimes_{C'} \Sigma, \ (\varphi, x) \mapsto \varphi g^{-1} \otimes_{C'} g(x).$$

Let $c \in C$ and $c' = gcg^{-1} \in C'$. Then,

$$\begin{aligned}
\psi(\varphi c, x) &= \varphi c g^{-1} \otimes_{C'} g(x) = \varphi g^{-1} c' \otimes_{C'} g(x) = \varphi g^{-1} \otimes_{C'} c' g(x) \\
&= \varphi g^{-1} \otimes_{C'} gc(x) = \psi(\varphi, cx).
\end{aligned}$$

Hence ψ defines a unique A-bimodule map $\bar{\psi} : \Sigma^* \otimes_C \Sigma \to \Sigma^* \otimes_{C'} \Sigma$. It is routine to verify that ψ is an isomorphism of A-corings. Conversely, let $\chi : \Sigma^* \otimes_C \Sigma \to \Sigma^* \otimes_{C'} \Sigma$ be an isomorphism of A-corings. Let S, S' be the unique, up to isomorphism, simple right $\Sigma^* \otimes_C \Sigma$-comodule and $\Sigma^* \otimes_{C'} \Sigma$-comodule, respectively. Then $\Sigma \cong S^{(m)}$ as a $\Sigma^* \otimes_C \Sigma$-comodule and $\Sigma \cong S'^{(n)}$ as a $\Sigma^* \otimes_{C'} \Sigma$-comodule for some $m, n \in \mathbb{N}$. Let S^χ denote S viewed as a right $\Sigma^* \otimes_{C'} \Sigma$-comodule via χ. Then $S^\chi \cong S'$ as a $\Sigma^* \otimes_{C'} \Sigma$-comodule. In particular, they are isomorphic as right A-modules. Since A is local, $S^\chi \cong S' \cong A^{(l)}$ as right A-modules for a certain $l \in \mathbb{N}$. Let Σ^χ denote Σ when considered as a right $\Sigma^* \otimes_{C'} \Sigma$-comodule. Then $\Sigma^\chi \cong (S^\chi)^{(m)} \cong A^{(lm)}$ as a right A-module. On the other hand, $\Sigma \cong A^{(ln)}$ as a right A-module. Since the underlying right A-module of Σ^χ and Σ is the same, $m = n$ and hence $\Sigma^\chi \cong \Sigma$ as a right $\Sigma^* \otimes_{C'} \Sigma$-comodule. Denote this isomorphism by g. Then $\mathrm{End}(\Sigma_{\Sigma^* \otimes_{C'} \Sigma}) = \mathrm{End}(\Sigma^\chi_{\Sigma^* \otimes_{C'} \Sigma})$ via $d \mapsto g^{-1} dg$. As $\mathrm{End}(\Sigma^\chi_{\Sigma^* \otimes_{C'} \Sigma}) = \mathrm{End}(\Sigma_{\Sigma^* \otimes_C \Sigma})$, $C = \mathrm{End}(\Sigma_{\Sigma^* \otimes_C \Sigma})$ and $C' = \mathrm{End}(\Sigma_{\Sigma^* \otimes_{C'} \Sigma})$, the assertion holds. □

Remark 2.5. With hypothesis as in Theorem 2.4, if we assume in addition that Σ_A is simple, then any quotient coring of $\mathfrak{C} = \Sigma^* \otimes_B \Sigma$ is simple cosemisimple: since Σ_A is simple, Σ is simple as a right $\mathfrak{C}/J$-comodule for any coideal J of $\mathfrak{C}$. It is easy to see that the canonical map $\mathsf{can} : \Sigma^* \otimes_{\mathrm{End}(\Sigma_{\mathfrak{C}/J})} \Sigma \to \mathfrak{C}/J$ is surjective. Using that $\Sigma_{\mathfrak{C}/J}$ is simple, by Corollary 1.6, can is an isomorphism. As $\mathrm{End}(\Sigma_{\mathfrak{C}/J})$ is a division ring, $\mathfrak{C}/J$ is simple cosemisimple. Thus we have a bijective correspondence between intermediate division subrings of $B \subseteq \mathrm{End}(\Sigma_A)$ and coideals of $\mathfrak{C}$. Observe that no assumptions are made on A. As a consequence we can derive Sweedler's predual to the Jacobson-Bourbaki Theorem, with the additional information concerning conjugated subrings.

Corollary 2.6. *[14, Theorem 2.1] Let $D \subseteq E$ be division rings. Set $\mathfrak{C} = E \otimes_D E$ and let $g = 1 \otimes_D 1$ be the distinguished group-like element. For a coideal J of $\mathfrak{C}$ let $\pi_J : \mathfrak{C} \to \mathfrak{C}/J$ denote the canonical projection. Then, the maps*

$$\begin{aligned}
\mathcal{R}(-) &: \mathcal{T} \to \mathcal{S}, \ J \mapsto \{e \in E : e\pi_J(g) = \pi_J(g)e\} \\
\mathcal{J}(-) &: \mathcal{S} \to \mathcal{T}, \ C \mapsto Ker(\mathfrak{C} \twoheadrightarrow \Sigma^* \otimes_C \Sigma)
\end{aligned}$$

establish a bijective correspondence between the set $\mathcal{S}$ of intermediate division rings $D \subseteq C \subseteq E$ and the set $\mathcal{T}$ of coideals J of $\mathfrak{C}$. Moreover, two intermediate division rings C and C' are conjugated in S if and only if $\mathfrak{C}/\mathcal{J}(C) \cong \mathfrak{C}/\mathcal{J}(C')$.

Proof. It only remains to prove that $\text{End}(E_{\mathfrak{C}/J}) = \{e \in E : e\pi_J(g) = \pi_J(g)e\}$ but this is easily checked. $\square$

Remark 2.7. If Σ_A is not simple, then the quotient corings of $\Sigma^* \otimes_B \Sigma$ need not in general to be simple cosemisimple. Let k be a field, $\Sigma = k^{(2)}$ and $T = T_2(k) \subseteq \text{End}(_k\Sigma) = M_2(k)$ the upper triangular matrix algebra. Then $\Sigma^* \otimes_T \Sigma$ is a non simple cosemisimple quotient coalgebra of $\Sigma^* \otimes_k \Sigma$. This example also serves to show that factor corings of Galois corings are not Galois. The module Σ_T is isomorphic to the indecomposable projective eT, where $e \in M_2(k)$ is the elementary matrix with 1 in the $(1,1)$-entry and zero elsewhere. Thus, $\text{End}(\Sigma_T) \cong eTe \cong k$, and the canonical map (15) gives here a surjective k-coalgebra homomorphism $\textsf{can} : \Sigma^* \otimes_k \Sigma \to T^*$ which, obviously, cannot be bijective. Thus, if we take $\mathfrak{C} = \Sigma^* \otimes_k \Sigma$, and $\mathfrak{D} = T^*$, then the factor coalgebra $(\mathfrak{D}, \Sigma)$ of the Galois coalgebra $(\mathfrak{C}, \Sigma)$ is not Galois. On the other hand, observe that Σ_{T^*} is projective but does not generate all its submodules, which sheds some light on the conditions involved in Theorem 1.5.

Example 2.8. We next illustrate the Galois connection established in Theorem 2.4 by a concrete example. Assume that k has an n-th primitive root of unity ω. For α, β non zero elements in k let $A_\omega(\alpha, \beta)$ denote the associative k-algebra generated by two elements x, y subject to the relations $x^n = \alpha, y^n = \beta$ and $yx = \omega xy$. Details on the properties of this algebra to be used in the sequel may be consulted in [12, Chapter 15]. The algebra $A_\omega(\alpha, \beta)$ is a central simple k-algebra. For our purposes we will assume that the subalgebras $C(\alpha) = k\langle x : x^n = \alpha\rangle$ and $C(\beta) = k\langle y : y^n = \beta\rangle$ are fields. Let Σ be an n-dimensional $C(\alpha)$-vector space with basis $B = \{v_1, ..., v_n\}$ and consider a dual basis $B^* = \{v_1^*, ..., v_n^*\}$ in Σ^*. The algebra $A_\omega(\alpha, \beta)$ can be embedded in $M_n(C(\alpha))$ by assigning

$$x \mapsto X = xe_{1,1} + \omega xe_{2,2} + ... + \omega^{n-1}xe_{n,n}, \qquad y \mapsto Y = e_{1,2} + ... + e_{n-1,n} + \beta e_{n,1},$$

where $e_{i,j}$ denotes the elementary matrix in $M_n(C(\alpha))$ with 1 in the (i,j)-entry and zero elsewhere. The action of X and Y on the bases B and B^* is:

$$X \cdot v_j = \omega^{j-1} v_j x \qquad\qquad v_j^* \cdot X = \omega^{j-1} x v_j^*$$
$$Y \cdot v_j = \begin{cases} \beta v_n & \text{if } j = 1 \\ v_{j-1} & \text{if } j > 1 \end{cases} \qquad\qquad v_j^* \cdot Y = \begin{cases} \beta v_1^* & \text{if } j = n \\ v_{j+1}^* & \text{if } j < n \end{cases} \qquad (7)$$

If either α or β is equal to 1, then the algebra $A_\omega(\alpha, \beta)$ is isomorphic to $M_n(k)$.

We will next describe the coideals of the $C(\alpha)$–coring $\Sigma^* \otimes_k \Sigma$ corresponding to the intermediate extensions of $k \subset M_n(C(\alpha))$ given in the following diagram:
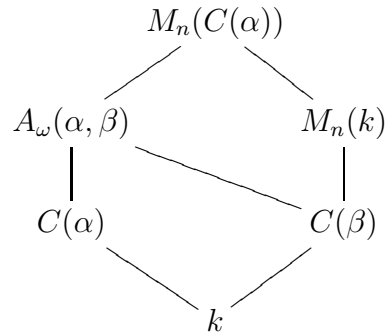
For $l, m, i = 1, ..., n$ set $z_{l,m}^i = \alpha^{-1} x^{n-i} v_l^* \otimes_k v_m x^i$. Observe that the set $\{z_{l,m}^i : l, m, i = 1, ..., n\}$ is a basis of $\Sigma^* \otimes_k \Sigma$ as a right $C(\alpha)$-vector space. We have that $x z_{l,m}^i = z_{l,m}^{i-1} x$ for $1 \leq i \leq n$ with the convention $z_{l,m}^0 = z_{l,m}^n$. The comultiplication and counit of $\Sigma^* \otimes_k \Sigma$ reads:

$$
\begin{aligned}
\Delta(z_{l,m}^i) &= \sum_{j=1}^n (\alpha^{-1} x^{n-i} v_l^* \otimes_k v_j) \otimes_{C(\alpha)} (v_j^* \otimes_k v_m x^i) \\
&= \sum_{j=1}^n (\alpha^{-1} x^{n-i} v_l^* \otimes_k v_j x^i) \otimes_{C(\alpha)} (\alpha^{-1} x^{n-i} v_j^* \otimes_k v_m x^i) \\
&= \sum_{j=1}^n z_{l,j}^i \otimes_{C(\alpha)} z_{j,m}^i, \\
\epsilon(z_{l,m}^i) &= \delta_{l,m}.
\end{aligned}
\tag{8}
$$

Trivial extensions: The trivial extensions $k \subseteq k \subset M_n(C(\alpha))$ and $k \subset M_n(C(\alpha)) \subseteq M_n(C(\alpha))$ correspond to the coideals $\{0\}$ and $Ker(\epsilon)$ respectively.

Extension $k \subset C(\alpha) \subset M_n(C(\alpha))$: We embed $C(\alpha)$ into $M_n(C(\alpha))$ by mapping x to X and consider the $C(\alpha)$-coring $\Sigma^* \otimes_{C(\alpha)} \Sigma$. The action of X on B and B^* gives the following relations in $\Sigma^* \otimes_{C(\alpha)} \Sigma$:

$$
\alpha^{-1} x^{n-i} v_l^* \otimes_{C(\alpha)} v_m x^i = v_l^* \otimes_{C(\alpha)} v_m.
\tag{9}
$$

The set $\{v_l^* \otimes_{C(\alpha)} v_m : l, m = 1, ..., n\}$ is a basis of $\Sigma^* \otimes_{C(\alpha)} \Sigma$ as a right $C(\alpha)$-vector space. Set $c_{l,m} = v_l^* \otimes_{C(\alpha)} v_m$. The bimodule structure on this coring is given by $x c_{l,m} = \omega^{m-l} c_{l,m} x$. The comultiplication and counit in $\Sigma^* \otimes_{C(\alpha)} \Sigma$ are defined by:

$$
\Delta(c_{l,m}) = \sum_{j=1}^n c_{l,j} \otimes_{C(\alpha)} c_{j,m}, \qquad \epsilon(c_{l,m}) = \delta_{l,m}.
$$

The coideal $J_{C(\alpha)}$ of $\Sigma^* \otimes_k \Sigma$ corresponding to this extension is the right subspace generated by the set

$$
\{z_{l,m}^n - \omega^{i(m-l)} z_{l,m}^i : l, m, i = 1, ..., n-1\}.
$$

Observe that if we embed diagonally $C(\alpha)$ into $M_n(C(\alpha))$, then Σ is centralized by $C(\alpha)$ and hence $\Sigma^* \otimes_{C(\alpha)} \Sigma$ is indeed a coalgebra, the comatrix coalgebra over $C(\alpha)$ of order n. Hence the diagonal embedding of $C(\alpha)$ is not conjugated with the preceding one.

Extension $k \subset C(\beta) \subset M_n(C(\alpha))$: We embed $C(\beta)$ into $M_n(C(\alpha))$ by mapping y to Y and consider the $C(\alpha)$-coring $\Sigma^* \otimes_{C(\beta)} \Sigma$. Taking into account the action of Y on B and B^*, the following relations in $\Sigma^* \otimes_{C(\beta)} \Sigma$ are obtained:

$$
v_i^* \otimes_{C(\beta)} v_j = \begin{cases} \beta^{-1} v_{n-(j-i)+1}^* \otimes_{C(\beta)} v_1 & \text{if } i < j \\ v_{i-j+1}^* \otimes_{C(\beta)} v_1 & \text{if } i \geq j \end{cases}
\tag{10}
$$

A basis of $\Sigma^* \otimes_{C(\beta)} \Sigma$ as a left $C(\alpha)$-vector space is:

$$
\{\alpha^{-1} x^{n-i} v_l^* \otimes_{C(\beta)} v_1 x^i : l, i = 1, ..., n\}.
$$

Setting $c_l^i = \alpha^{-1}x^{n-i}v_l^* \otimes_{C(\beta)} v_1 x^i$, the left action of $C(\alpha)$ on $\Sigma^* \otimes_{C(\beta)} \Sigma$ reads as $xc_l^i = c_l^{i-1}x$ with the convention $c_l^0 = c_l^n$. The comultiplication and counit of $\Sigma^* \otimes_{C(\beta)} \Sigma$ is given by:

$$
\begin{aligned}
\Delta(c_l^i) &= \sum_{j=1}^n (\alpha^{-1}x^{n-i}v_l^* \otimes_{C(\beta)} v_j) \otimes_{C(\alpha)} (v_j^* \otimes_{C(\beta)} v_1 x^i) \\
&= \sum_{j=1}^n (\alpha^{-1}x^{n-i}v_l^* \otimes_{C(\beta)} v_j x^i) \otimes_{C(\alpha)} (\alpha^{-1}x^{n-i}v_j^* \otimes_{C(\beta)} v_1 x^i) \\
&= \sum_{j=1}^l (\alpha^{-1}x^{n-i}v_l^* \otimes_{C(\beta)} v_j x^i) \otimes_{C(\alpha)} (\alpha^{-1}x^{n-i}v_j^* \otimes_{C(\beta)} v_1 x^i) \\
&\quad + \sum_{j=l+1}^n (\alpha^{-1}x^{n-i}v_l^* \otimes_{C(\beta)} v_j x^i) \otimes_{C(\alpha)} (\alpha^{-1}x^{n-i}v_j^* \otimes_{C(\beta)} v_1 x^i) \\
&= \sum_{j=1}^l (\alpha^{-1}x^{n-i}v_{l-j+1}^* \otimes_{C(\beta)} v_1 x^i) \otimes_{C(\alpha)} (\alpha^{-1}x^{n-i}v_j^* \otimes_{C(\beta)} v_1 x^i) \\
&\quad + \beta^{-1}\sum_{j=l+1}^n (\alpha^{-1}x^{n-i}v_{n-(j-l)+1}^* \otimes_{C(\beta)} v_1 x^i) \otimes_{C(\alpha)} (\alpha^{-1}x^{n-i}v_j^* \otimes_{C(\beta)} v_1 x^i) \\
&= \sum_{j=1}^l c_j^i \otimes_{C(\alpha)} c_{l-j+1}^i + \beta^{-1}\sum_{j=l+1}^n c_j^i \otimes_{C(\alpha)} c_{n-(j-l)+1}^i, \\
\epsilon(c_l^i) &= \delta_{l,1}.
\end{aligned}
$$

The coideal $J_{C(\beta)}$ of $\Sigma^* \otimes_k \Sigma$ corresponding to $C(\beta)$ is the right subspace generated by the following set:

$$
\{z_{l,m}^i - \beta^{-1}z_{n-(m-l)+1,1}^i : l, m, i = 1, ..., n; l < m\} \cup \{z_{l,m}^i - z_{l-m+1,1}^i : l, m, i = 1, ..., n; l \geq m\}.
$$

Extension $k \subset A_\omega(\alpha, \beta) \subset M_n(C(\alpha))$: Consider $A_\omega(\alpha, \beta)$ as embedded into $M_n(C(\alpha))$ by mapping x to X and y to Y. We next describe the $C(\alpha)$-coring $\Sigma^* \otimes_{A_\omega(\alpha,\beta)} \Sigma$. Since $C(\alpha)$ and $C(\beta)$ are contained in $A_\omega(\alpha, \beta)$, similar relations to (9) and (10) are obtained. The set $\{v_i \otimes_{A_\omega(\alpha,\beta)} v_1 : i = 1, ..., n\}$ is a basis of $\Sigma^* \otimes_{A_\omega(\alpha,\beta)} \Sigma$ as a right $C(\alpha)$-vector space. Set $c_i = v_i \otimes_{A_\omega(\alpha,\beta)} v_1$. The bimodule structure of $\Sigma^* \otimes_{A_\omega(\alpha,\beta)} \Sigma$ is $xc_i = \omega^{-i+1}c_i x$. The comultiplication and the counit of $\Sigma^* \otimes_{A_\omega(\alpha,\beta)} \Sigma$ are:

$$
\begin{aligned}
\Delta(c_i) &= \sum_{l=1}^n (v_i^* \otimes_{A_\omega(\alpha,\beta)} v_l) \otimes_{C(\alpha)} (v_l^* \otimes_{A_\omega(\alpha,\beta)} v_1) \\
&= \sum_{l=1}^i (v_{i-l+1}^* \otimes_{A_\omega(\alpha,\beta)} v_1) \otimes_{C(\alpha)} (v_l^* \otimes_{A_\omega(\alpha,\beta)} v_1) \\
&\quad + \beta^{-1}\sum_{l=i+1}^n (v_{n-(l-i)+1}^* \otimes_{A_\omega(\alpha,\beta)} v_1) \otimes_{C(\alpha)} (v_l^* \otimes_{A_\omega(\alpha,\beta)} v_1), \\
&= \sum_{l=1}^i c_l \otimes_{C(\alpha)} c_{i-l+1} + \beta^{-1}\sum_{l=i+1}^n c_l \otimes_{C(\alpha)} c_{n-(l-i)+1}, \\
\epsilon(c_i) &= \delta_{i,1}.
\end{aligned}
$$

The coideal $J_{A_\omega(\alpha,\beta)}$ of $\Sigma^* \otimes_k \Sigma$ that corresponds to this intermediate extension is the right subspace generated by the set:

$$
\{z_{l,m}^n - \omega^{i(m-l)}z_{l,m}^i : l, m, i = 1, ..., n\} \cup \{z_{l,m}^i - \beta^{-1}z_{n-(m-l)+1,1}^i : l, m, i = 1, ..., n;
$$
$$
l < m\} \cup \{z_{l,m}^i - z_{l-m+1,1}^i : l, m, i = 1, ..., n; l \geq m\}.
$$

Extension $k \subset M_n(k) \subset M_n(C(\alpha))$: Let

$$
X = e_{1,1} + \omega e_{2,2} + ... + \omega^{n-1}e_{n,n}, \qquad Y = e_{1,2} + ... + e_{n-1,n} + e_{n,1}.
$$

Then $X^n = 1, Y^n = 1$ and $YX = \omega XY$ and the k-algebra generated by X and Y is $M_n(k)$. The action of X and Y on the basis B and B^* is obtained from (7) for $\beta = 1$. From these actions we get the relations:

$$
\begin{aligned}
v_1^* \otimes_{M_n(k)} v_1 &= v_n^* \cdot Y \otimes_{M_n(k)} v_1 = v_n^* \otimes_{M_n(k)} Y \cdot v_1 = v_n^* \otimes_{M_n(k)} v_n \\
&= v_{n-1}^* \cdot Y \otimes_{M_n(k)} v_n = v_{n-1}^* \otimes_{M_n(k)} Y \cdot v_n = v_{n-1}^* \otimes_{M_n(k)} v_{n-1} \\
&= .... \\
&= v_2^* \otimes_{M_n(k)} v_2. \\
v_i^* \otimes_{M_n(k)} v_j &= \omega^{-j+1} v_i^* \otimes_{M_n(k)} X \cdot v_j = \omega^{-j+1} v_i^* \cdot X \otimes_{M_n(k)} v_j \\
&= \omega^{i-j} v_i^* \otimes_{M_n(k)} v_j.
\end{aligned}
$$

Then $v_i^* \otimes_{M_n(k)} v_j = 0$ for $i \neq j$. Set $c_i = \alpha^{-1} x^{n-i} v_1^* \otimes_{M_n(k)} v_1 x^{n-i}$ for $i = 1, ..., n$. Then the set $\{c_i : i = 1, ..., n\}$ is a basis of $\Sigma^* \otimes_{M_n(k)} \Sigma$ as a right $C(\alpha)$-vector space. The bimodule structure of this coring is given by $xc_i = c_{i-1}x$ with the convention $c_0 = c_n$. The comultiplication and counit of $\Sigma^* \otimes_{M_n(k)} \Sigma$ takes the form:

$$
\begin{aligned}
\Delta(c_i) &= \sum_{j=1}^n (\alpha x^{n-i} v_1^* \otimes_{M_n(k)} v_j) \otimes_{C(\alpha)} (v_j^* \otimes_{M_n(k)} v_1) x^i \\
&= (\alpha x^{n-i} v_1^* \otimes_{M_n(k)} v_1) \otimes_{C(\alpha)} (v_1^* \otimes_{M_n(k)} v_1) x^i \\
&= (\alpha x^{n-i} v_1^* \otimes_{M_n(k)} v_1 x^i) \otimes_{C(\alpha)} (\alpha^{-1} x^{n-i} v_1^* \otimes_{M_n(k)} v_1) x^i \\
&= c_i \otimes_{C(\alpha)} c_i, \\
\epsilon(c_i) &= 1.
\end{aligned}
$$

This coring can also be obtained as the Sweedler coring associated to the extension $k \subset C(\alpha)$. The coideal $J_{M_n(k)}$ of $\Sigma^* \otimes_k \Sigma$ corresponding to this extension is the right subspace spanned by the set

$$
\{z_{l,m}^i : l, m, i = 1, ..., n; l \neq m\} \cup \{z_{1,1}^i - z_{l,l}^i : l, i = 1, ..., n\}.
$$

3   Duality

Let $f : \mathfrak{C} \to \mathfrak{D}, g : \mathfrak{C} \to \mathfrak{E}$ be surjective homomorphisms of A–corings. Then $Kerf = Kerg$ if and only if there exists an isomorphism of A–corings $\mathfrak{D} \cong \mathfrak{E}$ making commute the diagram

(11)

Thus, every coideal J of $\mathfrak{C}$ determines a class of surjective homomorphisms $\mathfrak{C} \to \mathfrak{D}$ having J as their common kernel or, alternatively, the morphisms in each class are connected by commutative triangles as in (11).

From a formal point of view, corings over A are dual to A–rings, being these last understood to be morphisms of rings $A \to U$. The definition of a homomorphism of

A–rings is obvious, and we will conceive an A–*subring* of a given A–ring $A \to E$ as an isomorphism class of injective homomorphisms of A–rings $U \to E$. Obviously, every A–subring of E may be represented by an inclusion $U \subseteq E$. We can thus consider the set $\mathsf{Subrings}(E)$ of A–subrings of E.

One of the possible concrete dual correspondences from A–corings to A–rings goes as follows: if $\mathfrak{C}$ is an A–coring, then $^*\epsilon_{\mathfrak{C}} : A \to {}^*\mathfrak{C}^{op}$ is an A–ring (see [14, Proposition 3.2]), and under this mapping, homomorphisms of A–corings give homomorphisms of A–rings. In particular, if $J \in \mathsf{Coideals}(\mathfrak{C})$ and $\mathfrak{C} \to \mathfrak{D} = \mathfrak{C}/J$ is the corresponding canonical projection, then we have an injective homomorphism of A–rings $^*\mathfrak{D}^{op} \to {}^*\mathfrak{C}^{op}$ (see again [14, Proposition 3.2]). If $\mathfrak{C} = \Sigma^* \otimes_B \Sigma$ is a comatrix A–coring, then, by [6, Proposition 2.1], we have an injective homomorphism of A–rings $^*\mathfrak{D}^{op} \to {}^*\mathfrak{C}^{op} \cong \mathrm{End}(_B\Sigma)$. The corresponding A–subring of $\mathrm{End}(_B\Sigma)$ will be denoted by $\mathcal{R}'(J)$. Conversely, given an A–subring $U \to \mathrm{End}(_B\Sigma)$, then $\mathrm{End}(\Sigma_U)$ is independent on the representative U of the A–subring. We define the coideal $\mathcal{J}'(U)$ of $\Sigma^* \otimes_B \Sigma$ as the kernel of the homomorphism of A–corings $\Sigma^* \otimes_B \Sigma \to \Sigma^* \otimes_{\mathrm{End}(\Sigma_U)} \Sigma$ induced by the ring homomorphism $B \to \mathrm{End}(\Sigma_U)$.

These considerations lead to a Galois connection for $E = \mathrm{End}(_B\Sigma)$:

$$\mathcal{J}' : \mathsf{Subrings}(E) \;\overset{\longrightarrow}{\underset{\longleftarrow}{}}\; \mathsf{Coideals}(\mathfrak{C}) : \mathcal{R}' \tag{12}$$

The following proposition collects some of its relevant properties. A right comodule Σ over an A–coring $\mathfrak{D}$ is said to be *loyal* if the canonical map

$$M^* \otimes_{\mathrm{End}(M_{\mathfrak{D}})} M \longrightarrow M^* \otimes_{\mathrm{End}(_{*\mathfrak{D}}M)} M$$

induced by the inclusion $\mathrm{End}(\Sigma_{\mathfrak{D}}) \subseteq \mathrm{End}(_{*\mathfrak{D}}\Sigma)$ is a bijection. By [3, 19.2,19.3], if $_A\mathfrak{D}$ is locally projective, then every right $\mathfrak{D}$–comodule is loyal.

Proposition 3.1. *The mappings defined in* (12) *enjoy the following properties.*

(1) $\mathcal{R}'\mathcal{J}'(U) = \mathrm{End}(_{\mathrm{End}(\Sigma_U)}\Sigma)$ *for every* $U \in \mathsf{Subrings}(E)$ *and, thus,* $U \subseteq \mathcal{R}'\mathcal{J}'(U)$.

(2) $J \supseteq \mathcal{J}'\mathcal{R}'(J)$ *for every* $J \in \mathsf{Coideals}(\Sigma^* \otimes_B \Sigma)$ *such that* $\Sigma_{\mathfrak{C}/J}$ *is loyal.*

(3) The maps $\mathcal{J}'$ *and* $\mathcal{R}'$ *establish a bijection between the set of* A–*subrings* U *of* $\mathrm{End}(_B\Sigma)$ *such that* Σ_U *is faithful and balanced, and the set of coideals* J *of* $\mathfrak{C}$ *such that* $(\mathfrak{C}/J, \Sigma)$ *is Galois and* $\Sigma_{\mathfrak{C}/J}$ *is loyal.*

Proof. (1) Given an A–subring $U \subseteq \mathrm{End}(\Sigma_B)$, $\mathcal{J}'(U)$ is defined as the kernel of the surjective homomorphism of A–corings $\Sigma^* \otimes_B \Sigma \to \Sigma^* \otimes_{\mathrm{End}(\Sigma_U)} \Sigma$. The commutative diagram of injective homomorphisms of A–rings

$$
\begin{array}{ccc}
^*\big(\Sigma^* \otimes_{\mathrm{End}(\Sigma_U)} \Sigma\big)^{op} & \longrightarrow & ^*\big(\Sigma^* \otimes_B \Sigma\big)^{op} \\
\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\
\mathrm{End}(_{\mathrm{End}(\Sigma_U)}\Sigma) & \longrightarrow & \mathrm{End}(_B\Sigma)
\end{array}
$$

deduced from [6, Proposition 2.1], shows that $\mathcal{R}'\mathcal{J}'(U) = \mathrm{End}(_{\mathrm{End}(\Sigma_U)}\Sigma)$.

(2) With $p : \mathfrak{C} \to \mathfrak{D} = \mathfrak{C}/J$ the canonical projection, consider the commutative diagram

$$\Sigma^* \otimes_B \Sigma \xrightarrow{\quad p \quad} \mathfrak{D} \ ,$$

with arrows f and can to $\Sigma^* \otimes_{\mathrm{End}(\Sigma_{\mathfrak{D}})} \Sigma$
$$\tag{13}$$

where f is induced by the morphism $B \to \mathrm{End}(\Sigma_{\mathfrak{D}})$. Since $\Sigma_{\mathfrak{D}}$ is loyal, we get from the diagram that $J \supseteq \ker(f) = \mathcal{J}'\mathcal{R}'(J)$.

(3) Let $U \subseteq E = \mathrm{End}(_B\Sigma)$ be an A–subring. By definition, $\mathcal{J}'(U)$ is such that $\mathfrak{C}/\mathcal{J}'(U) \cong \Sigma^* \otimes_{\mathrm{End}(\Sigma_U)} \Sigma$. By [6, Lemma 3.9], $(\Sigma^* \otimes_{\mathrm{End}(\Sigma_U)} \Sigma, \Sigma)$ is Galois. This means that the canonical map

$$\mathsf{can} : \Sigma^* \otimes_{\mathrm{End}(\Sigma_{\Sigma^* \otimes_{\mathrm{End}(\Sigma_U)} \Sigma})} \Sigma \longrightarrow \Sigma^* \otimes_{\mathrm{End}(\Sigma_U)} \Sigma$$

is an isomorphism. But this map is nothing but the one induced by the ring extension

$$\mathrm{End}(\Sigma_{\Sigma^* \otimes_{\mathrm{End}(\Sigma_U)}\Sigma}) \subseteq \mathrm{End}(_*(\Sigma^* \otimes_{\mathrm{End}(\Sigma_U)}\Sigma)\Sigma) = \mathrm{End}(\Sigma_{\mathrm{End}(\Sigma_{\mathrm{End}(\Sigma_U)})}) = \mathrm{End}(\Sigma_U),$$

which proves that Σ is a loyal right $\Sigma^* \otimes_{\mathrm{End}(\Sigma_U)} \Sigma$–comodule. Part *(1)* gives obviously $\mathcal{R}'\mathcal{J}'(U) = U$. Conversely, let J be a coideal of $\mathfrak{C}$ such that $(\mathfrak{C}/J, \Sigma)$ is Galois and $\Sigma_{\mathfrak{C}/J}$ is Galois. Put $\mathfrak{D} = \mathfrak{C}/J$. From the triangle (13), and the fact that $\Sigma_{\mathfrak{D}}$ is loyal, we compute the A–subring $\mathcal{R}'(J)$ of $\mathrm{End}(_R\Sigma)$ as

$$^*(\mathfrak{D})^{op} \cong {}^*(\Sigma^* \otimes_{\mathrm{End}(\Sigma_{\mathfrak{D}})} \Sigma)^{op} \cong {}^*(\Sigma^* \otimes_{\mathrm{End}(\Sigma_{*\mathfrak{D}^{op}})} \Sigma)^{op} \cong \mathrm{End}(_{\mathrm{End}(\Sigma_{*\mathfrak{D}^{op}})}\Sigma) \tag{14}$$

This means that $\mathcal{R}'(J)$, defined as $^*\mathfrak{D}^{op}$, is such that $\Sigma_{\mathcal{R}'(J)}$ is faithful and balanced. The diagram (13) gives in addition that $\mathcal{J}'\mathcal{R}'(J) = J$. $\qquad\square$

We are now in position to prove our version for simple artinian rings of the Jacobson-Bourbaki theorem.

Theorem 3.2. *Let Σ be a finitely generated projective right module over a Quasi-Frobenius ring A and let $S = \mathrm{End}(\Sigma_A)$. Let B be a subring of S. Denote by $\mathcal{I}$ the set of intermediate simple artinian subrings $B \subseteq C \subseteq S$ such that $_C\Sigma$ is finitely generated. Let $\mathcal{D}$ denote the set of all A-subrings U of $\mathrm{End}(_B\Sigma)$ such that U_A is finitely generated and projective, and Σ_U is semisimple and isotypic. The maps*

$$\mathcal{R}^*(-) : \mathcal{D} \to \mathcal{I}, \ U \mapsto \mathrm{End}(\Sigma_U),$$
$$\mathcal{J}^*(-) : \mathcal{I} \to \mathcal{D}, \ C \mapsto \mathrm{End}(_C\Sigma).$$

establish a bijective correspondence. This correspondence is dual to that of Theorem 2.4. Moreover, if A is in addition local, then two intermediate simple artinian rings C and C' are conjugated if and only if $\mathrm{End}(_CA)$ and $\mathrm{End}(_{C'}A)$ are isomorphic as A–rings.

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Proof. If C is simple artinian and $_C\Sigma$ is finitely generated, then $U = \text{End}(_C\Sigma)$ is a simple artinian A–ring. Thus, Σ_U is semisimple isotypic. The comatrix A–coring $\Sigma^* \otimes_C \Sigma$ is then finitely generated and projective as a left A–module. The ring isomorphism $U^{op} \cong {}^*(\Sigma^* \otimes_C \Sigma)$ given in [6, Proposition 2.1] is an isomorphism of A–bimodules. In particular, U_A becomes a finitely generated module. Obviously, $_C\Sigma$ is faithful and balanced. Conversely, assume that Σ_U is semisimple isotypic for an A–subring $U \subseteq \text{End}(_B\Sigma)$ such that U_A is finitely generated. Then $C = \text{End}(\Sigma_U)$ is a simple artinian subring of S, because Σ_U is clearly finitely generated. To prove that Σ_U is faithful and balanced, consider that the comatrix A–coring structure on $U^* \otimes_U U$ induces, via the isomorphism $U^* \otimes_U U \cong U^*$ an A–coring structure on this last A–bimodule [6, Example 2.4]. If $\{u_\alpha^*, u_\alpha\}_{\alpha=1}^n$ is a finite dual basis for U_A, the comultiplication and counit are given explicitly by

$$\Delta : U^* \to U^* \otimes_A U^*, \quad \varphi \mapsto \sum_\alpha \varphi u_\alpha \otimes_A u_\alpha^*,$$

$$\epsilon : U^* \to A, \quad \varphi \mapsto \varphi(1)$$

The canonical map $U \to {}^*(U^*)$ is then an anti-isomorphism of rings. We may then identify the categories of $\mathcal{M}^{U^*}$ and $_{U^{op}}\mathcal{M} = \mathcal{M}_U$ (see e.g. [3, 19.6]). In fact, an explicit isomorphism of categories is given as follows. For each element m in a right U–module M, the equality $mu = \sum_\alpha mu_\alpha u_\alpha^*(u)$ for $u \in U$ says that $\{mu_\alpha, u_\alpha^*\}$ is a set of right rational parameters in the sense of [7]. Thus [7, Corollary 4.7] gives the isomorphism of categories $\mathcal{M}_U = \mathcal{M}^{U^*}$, where the right U^*–comodule structure on M_U is given by $\rho_M(m) = \sum_\alpha mu_\alpha \otimes_A u_\alpha^*$. In particular, the (finitely generated) simple isotypic right U–module may be considered as a right U^*–comodule, and we have a canonical map given by

$$\text{can} : \Sigma^* \otimes_C \Sigma \to U^*, \quad \varphi \otimes_C x \mapsto \sum_\alpha \varphi(xu_\alpha)u_\alpha^* \tag{15}$$

Now, for every $0 \neq u \in U$, let $x \in \Sigma$ such that $xu \neq 0$. Then, for $\varphi \in \Sigma^*$ with $\varphi(xu) \neq 0$, we have

$$\text{can}(\varphi \otimes_C x)(u) = \sum_\alpha \varphi(xu_\alpha)u_\alpha^*(u) = \sum_\alpha \varphi(xu_\alpha u_\alpha^*(u)) = \varphi(xu) \neq 0,$$

which implies, being A Quasi-Frobenius and U_A finitely generated, that can is surjective. Finally, since Σ_{U^*} is semisimple, we get from Corollary 1.6 that can an isomorphism. Therefore, we have an isomorphism

$$U^{op} \cong {}^*(U^*) \xrightarrow{\ \text{can}^*\ } \text{End}(_C\Sigma)^{op}$$

which turns out to be, by using [6, equation (3)], the canonical homomorphism $U \to \text{End}(_C\Sigma)$. In this way, Σ_U is faithful and balanced and $_C\Sigma$ must be finitely generated. $\square$

Remark 3.3. If the base ring A is simple artinian then Theorem 3.2 takes a simpler form. Thus, for a simple artinian ring B, the maps

$$\mathcal{R}^*(-) : \mathcal{D} \to \mathcal{I}, \ U \mapsto End(\Sigma_U),$$
$$\mathcal{J}^*(-) : \mathcal{I} \to \mathcal{D}, \ C \mapsto End(_C\Sigma).$$

establish a bijective correspondence between the set $\mathcal{D}$ of simple artinian A-subrings U of $End(_B\Sigma)$ such that U_A is finitely generated and the set $\mathcal{I}$ of intermediate simple artinian subrings $B \subseteq C \subseteq S$ such that $_CS$ is finitely generated.

If $\Sigma = A$, and $B \subseteq A$ is a subring, then for every A–subring $U \subseteq End(_BA)$, we have that $\mathcal{R}^*(U) = End(U_A) = \{c \in A : u(ca) = cu(a), u \in U, a \in A\}$. If, moreover, A is a division ring, then we deduce from Theorem 3.2 the following version of the Jacobson-Bourbaki theorem (c.f. [14, §4]).

Corollary 3.4. *Let $B \subseteq A$ be division rings. The maps $\mathcal{R}^*$ and $\mathcal{J}^*$ establish a bijective correspondence between the intermediate division rings $B \subseteq C \subseteq A$ such that $_CA$ is finite dimensional and the A–subrings $U \subseteq End(_BA)$ such that U_A is finite dimensional. Moreover, two intermediate division subrings C and C' are conjugated if and only if $End(_CA)$ and $End(_{C'}A)$ are isomorphic A–rings.*

Proof. The pertinent remark here is that A_U is always a simple module (since A_A is simple). ☐

Appendix

Throughout this section all corings are $\mathbb{C}/\mathbb{R}$-corings and all $\mathbb{C}$-bimodules are assumed to be centralized by the field of real numbers $\mathbb{R}$.

We know by the Structure Theorem of simple and cosemisimple corings that any such a $\mathbb{C}$-coring $\mathfrak{C}$ is of the form $\Sigma^* \otimes_D \Sigma$ where Σ is a finite dimensional complex vector space and D is a $\mathbb{R}$-division algebra embedded in $End(\Sigma_\mathbb{C}) \cong M_n(\mathbb{C})$ where $n = dim(\Sigma_\mathbb{C})$. Furthermore, two corings $\Sigma^* \otimes_D \Sigma$ and $\Sigma^* \otimes_E \Sigma$ are isomorphic if and only if there is an invertible $u \in End(\Sigma_\mathbb{C})$ such that $uEu^{-1} = D$, i.e., E and D are conjugated in $End(\Sigma_\mathbb{C})$.

By Fröbenius Theorem, $D = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. We study the possible ways of embedding these division algebras in $M_n(\mathbb{C})$. As a consequence of the proof of the Skolem-Noether Theorem ([10, Theorem 4.9]) the non conjugated ways of embedding D in $M_n(\mathbb{C})$ are in bijective correspondence with the simple left $D \otimes_\mathbb{R} M_n(\mathbb{C})$-modules.

1. Case $D = \mathbb{R}$. Since $M_n(\mathbb{C})$ is an $\mathbb{R}$-algebra, the only way of embedding $\mathbb{R}$ in $M_n(\mathbb{C})$ is the obvious one. The comultiplication and counit of the coring $\Sigma^* \otimes_\mathbb{R} \Sigma$ is described in (8). If $n = 1$, then this coring is isomorphic to the coring $[\mathbb{Z}/2]\mathbb{C}$ associated to the canonical $\mathbb{Z}/2$–grading on $\mathbb{C}$. As a right $\mathbb{C}$–vector space, $[\mathbb{Z}/2]\mathbb{C}$ is free over the basis $\mathbb{Z}/2 = \{[0], [1]\}$. Its left $\mathbb{C}$-vector space structure is determined by the rules $i[0] = [1]i$, $i[1] = [0]i$. In this coring, $[0]$ and $[1]$ are group-like elements. For $n > 1$, the coring $\Sigma^* \otimes_\mathbb{R} \Sigma$ is isomorphic with the tensor product coring (see [8, Proposition 1.5]) $[\mathbb{Z}/2]\mathbb{C} \otimes_\mathbb{R} M^c(\mathbb{R}, n)$, where $M^c(\mathbb{R}, n)$ is the comatrix $\mathbb{R}$–coalgebra. An explicit isomorphism sends $z^0_{m,l}$ onto $[0] \otimes_\mathbb{R} x_{m,l}$ and $z^1_{m,l}$ onto $[1] \otimes_\mathbb{R} x_{m,l}$, where $x_{m,l}$ denotes the matrix with 1 in the component (l, m) and 0 elsewhere.

2. Case $D = \mathbb{C}$. Since $\mathbb{C} \otimes_\mathbb{R} M_n(\mathbb{C}) \cong M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$, there are two ways (for $n > 0$) of embedding $\mathbb{C}$ into $M_n(\mathbb{C})$. Let $e_{p,q}$ denote the elementary matrix in $M_n(\mathbb{C})$ with 1 in the (p, q)-entry and zero elsewhere. The two non conjugate embeddings are represented by the

one sending i to $i(\sum_{l=1}^{n} e_{l,l})$ and the one sending i to $\bar{i} = \sum_{l=1}^{n} e_{l,n-l+1}$. In the first case, Σ is centralized by $\mathbb{C}$ and therefore $\Sigma^* \otimes_{\mathbb{C}} \Sigma$ is a $\mathbb{C}$-coalgebra. Thus $\Sigma^* \otimes_{\mathbb{C}} \Sigma$ is isomorphic to the comatrix coalgebra of order n. Let us study the second case. The action of $\bar{i}$ on Σ and Σ^* is:

$$\bar{i} \cdot v_j = v_{n-j+1}i, \qquad v_j^* \cdot \bar{i} = iv_{n-j+1}^*.$$

Then the bimodule structure on $\Sigma^* \otimes_{\mathbb{C}} \Sigma$ is given by:

$$\begin{aligned}
i(v_p^* \otimes_{\mathbb{C}} v_q) &= iv_p^* \otimes_{\mathbb{C}} v_q = v_{n-p+1}^* \cdot \bar{i} \otimes_{\mathbb{C}} v_q = v_{n-p+1}^* \otimes_{\mathbb{C}} \bar{i} \cdot v_q \\
&= v_{n-p+1}^* \otimes_{\mathbb{C}} v_{n-q+1}i = (v_{n-p+1}^* \otimes_{\mathbb{C}} v_{n-q+1})i.
\end{aligned}$$

The comultiplication and counit of $\Sigma^* \otimes_{\mathbb{C}} \Sigma$ is defined by:

$$\Delta(v_p^* \otimes v_q) = \sum_{l=1}^{n}(v_p^* \otimes v_l) \otimes (v_l^* \otimes v_q), \qquad \epsilon(v_p^* \otimes v_q) = \delta_{p,q}.$$

This coring can be described as the right $\mathbb{C}$-vector space $\mathfrak{C} = \oplus_{p,q=1}^{n} v_{p,q}\mathbb{C}$ with bimodule structure $iv_{p,q} = v_{n-p+1,n-q+1}i$ and with comultiplication and counit given by:

$$\Delta(v_{p,q}) = \sum_{l=1}^{n} v_{p,l} \otimes_{\mathbb{C}} v_{l,q}, \qquad \epsilon(v_{p,q}) = \delta_{p,q}.$$

3. Case $D = \mathbb{H}$. Since $\mathbb{H}$ is a central simple $\mathbb{R}$-algebra and $M_n(\mathbb{C})$ is simple, $\mathbb{H} \otimes_{\mathbb{R}} M_n(\mathbb{C})$ is simple. Hence all embeddings of $\mathbb{H}$ in $M_n(\mathbb{C})$ are conjugate. Let us observe that if $\mathbb{H}$ embeds in $M_n(\mathbb{C})$, then n is even. By the Double Centralizer Theorem we would have $M_n(\mathbb{C}) \cong \mathbb{H} \otimes_{\mathbb{R}} C(\mathbb{H})$, where $C(\mathbb{H})$ denotes the centralizer of $\mathbb{H}$ in $M_n(\mathbb{C})$. Comparing real dimensions, 4 divides $2n^2$. Let $\bar{i}, \bar{j}$ be the generators of $\mathbb{H}$. Consider the following embedding of $\mathbb{H}$ in $M_2(\mathbb{C})$:

$$\bar{i} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \bar{j} \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Since n is even, the above embedding gives an embedding of $\mathbb{H}$ in $M_n(\mathbb{C})$ by placing each block repeatedly in the main diagonal. Any other embedding of $\mathbb{H}$ in $M_n(\mathbb{C})$ is conjugated to this one.

We next describe the coring $\Sigma^* \otimes_{\mathbb{H}} \Sigma$. We first assume that $n = 2$. Observe that this case is a particular case of our example in the second section by taking $k = \mathbb{R}$ and $\alpha = \beta = -1$. The bimodule structure on $\Sigma^* \otimes_{\mathbb{H}} \Sigma$ is the following:

$$i(v_1^* \otimes v_1) = (v_1^* \otimes v_1)i, \qquad i(v_2^* \otimes v_1) = -(v_2^* \otimes v_1)i.$$

The comultiplication and counit of $\Sigma^* \otimes_{\mathbb{H}} \Sigma$ read as:

$$\begin{aligned}
\Delta(v_1^* \otimes v_1) &= (v_1^* \otimes v_1) \otimes (v_1^* \otimes v_1) - (v_2^* \otimes v_1) \otimes (v_2^* \otimes v_1), & \epsilon(v_1^* \otimes v_1) &= 1, \\
\Delta(v_2^* \otimes v_1) &= (v_2^* \otimes v_1) \otimes (v_1^* \otimes v_1) + (v_1^* \otimes v_1) \otimes (v_2^* \otimes v_1), & \epsilon(v_2^* \otimes v_1) &= 0.
\end{aligned}$$

This coring is precisely the *trigonometric coring*. We now discuss the general case $dim(\Sigma_{\mathbb{C}}) = n = 2m$. Let us recall that $\mathbb{H}$ is embedded in $M_n(\mathbb{C})$ in the following way:

$$\bar{i} \mapsto \sum_{i=1}^{m} e_{2l-1,2l} - \sum_{l=1}^{m} e_{2l,2l-1}, \qquad \bar{j} \mapsto i(\sum_{l=1}^{n} (-1)^{l+1} e_{l,l}).$$

Through this embedding the action of $\mathbb{H}$ on Σ and Σ^* is:

$$\bar{i} \cdot v_q = \begin{cases} v_{q-1} & \text{if q is even} \\ -v_{q+1} & \text{if q is odd} \end{cases} \qquad \bar{j} \cdot v_q = (-1)^{q+1} v_q i$$

$$v_q^* \cdot \bar{i} = \begin{cases} -v_{q-1}^* & \text{if q is even} \\ v_{q+1}^* & \text{if q is odd} \end{cases} \qquad v_q^* \cdot \bar{j} = (-1)^{q+1} i v_q^*$$

These actions give rise to the following relations in $\Sigma^* \otimes_{\mathbb{H}} \Sigma$. Let $p, q \in \{1, 2, ..., m\}$. Then:

$$v_{2p}^* \otimes v_{2q} = -v_{2p}^* \otimes \bar{i} \cdot v_{2q-1} = -v_{2p}^* \cdot \bar{i} \otimes v_{2q-1} = v_{2p-1}^* \otimes v_{2q-1},$$
$$v_{2p-1}^* \otimes v_{2q} = -v_{2p-1}^* \otimes \bar{i} \cdot v_{2q} = v_{2p-1}^* \cdot \bar{i} \otimes v_{2q-1} = -v_{2p}^* \otimes v_{2q-1}.$$

The set $\{v_{2p} \otimes v_l | p = 1, ..., m; l = 1, ..., n\}$ is a basis of $\Sigma^* \otimes_{\mathbb{H}} \Sigma$ as a right $\mathbb{C}$-vector space. The left $\mathbb{C}$-action on this coring is:

$$i(v_{2p}^* \otimes v_{2q}) = iv_{2p}^* \otimes v_{2q} = (-1)^{2p+1} v_{2p}^* \cdot \bar{j} \otimes v_{2q} = (-1)^{2p+1} v_{2p}^* \otimes \bar{j} \cdot v_{2q}$$
$$= v_{2p}^* \otimes v_{2q} i = (v_{2p}^* \otimes v_{2q}) i,$$

$$i(v_{2p}^* \otimes v_{2q-1}) = iv_{2p}^* \otimes v_{2q-1} = (-1)^{2p+1} v_{2p}^* \cdot \bar{j} \otimes v_{2q-1} = (-1)^{2p+1} v_{2p}^* \otimes \bar{j} \cdot v_{2q-1}$$
$$= -v_{2p}^* \otimes v_{2q} i = -(v_{2p}^* \otimes v_{2q-1}) i.$$

The comultiplication and counit of $\Sigma^* \otimes_{\mathbb{H}} \Sigma$ is:

$$\Delta(v_{2p} \otimes v_{2q}) = \sum_{l=1}^{m} (v_{2p}^* \otimes v_{2l}) \otimes_{\mathbb{C}} (v_{2l}^* \otimes v_{2q}) + \sum_{l=1}^{m} (v_{2p}^* \otimes v_{2l-1}) \otimes_{\mathbb{C}} (v_{2l-1}^* \otimes v_{2q})$$
$$= \sum_{l=1}^{m} (v_{2p}^* \otimes v_{2l}) \otimes_{\mathbb{C}} (v_{2l}^* \otimes v_{2q}) - \sum_{l=1}^{m} (v_{2p}^* \otimes v_{2l-1}) \otimes_{\mathbb{C}} (v_{2l}^* \otimes v_{2q-1})$$
$$\epsilon(v_{2p} \otimes v_{2q}) = \delta_{p,q}$$
$$\Delta(v_{2p} \otimes v_{2q-1}) = \sum_{l=1}^{m} (v_{2p}^* \otimes v_{2l}) \otimes_{\mathbb{C}} (v_{2l}^* \otimes v_{2q-1}) + \sum_{l=1}^{m} (v_{2p}^* \otimes v_{2l-1}) \otimes_{\mathbb{C}} (v_{2l-1}^* \otimes v_{2q-1})$$
$$= \sum_{l=1}^{m} (v_{2p}^* \otimes v_{2l}) \otimes_{\mathbb{C}} (v_{2l}^* \otimes v_{2q}) + \sum_{l=1}^{m} (v_{2p}^* \otimes v_{2l-1}) \otimes_{\mathbb{C}} (v_{2l}^* \otimes v_{2q})$$
$$\epsilon(v_{2p} \otimes v_{2q-1}) = 0.$$

Let $\mathfrak{T} = \mathbb{C}c \oplus \mathbb{C}s$ denote the trigonometric coring and let $M^c(\mathbb{R}, m)$ be the $\mathbb{R}$-comatrix coalgebra of order m. Then $\mathfrak{T} \otimes_{\mathbb{R}} M^c(\mathbb{R}, m)$ becomes a $\mathbb{C}$-coring in the natural way [8]. It may be verified that the map from $\mathfrak{T} \otimes_{\mathbb{R}} M^c(\mathbb{R}, m)$ to $\Sigma^* \otimes_{\mathbb{H}} \Sigma$ defined by

$$c \otimes_{\mathbb{R}} x_{pq} \mapsto v_{2p}^* \otimes v_{2q}, \qquad s \otimes_{\mathbb{R}} x_{pq} \mapsto v_{2p}^* \otimes v_{2q-1},$$

is an isomorphism of corings.

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