Convocatorias 2014
Proyectos de I+D "EXCELENCIA" y Proyectos de I+D+I "RETOS INVESTIGACIÓN"
Dirección General de Investigación Científica y Técnica
Subdirección General de Proyectos de Investigación

AVISO IMPORTANTE

En virtud del artículo 11 de la convocatoria **NO SE ACEPTARÁN NI SERÁN SUBSANABLES MEMORIAS CIENTÍFICO-TÉCNICAS** que no se presenten en este formato.

Lea detenidamente las instrucciones que figuran al final de este documento para rellenar correctamente la memoria científico-técnica.

Parte A: RESUMEN DE LA PROPUESTA/SUMMARY OF THE PROPOSAL

INVESTIGADOR PRINCIPAL 1 (Nombre y apellidos):

Andrei Martínez Finkelshtein

INVESTIGADOR PRINCIPAL 2 (Nombre y apellidos):

Juan José Moreno Balcázar

TÍTULO DEL PROYECTO: Aproximación y ortogonalidad: de la teoría a las aplicaciones

ACRÓNIMO: APORTA

RESUMEN Máximo 3500 caracteres (incluyendo espacios en blanco):

Este proyecto combina objetivos ambiciosos en investigación básica sobre polinomios ortogonales, funciones especiales y sus propiedades analíticas y estructurales, con las aplicaciones de estos conocimientos en otras ramas de las matemáticas (procesos estocásticos, combinatoria, análisis numérico), física (física estadística, sistemas integrables, mecánica cuántica, caminos aleatorios cuánticos, computación cuántica) y tecnología (procesado de señales y herramientas de diagnóstico en oftalmología, con aplicaciones clínicas). Algunos de los problemas que se consideran son:

1. Ulteriores contribuciones a la teoría general de polinomios ortogonales (PO) y aproximación racional, incluyendo el desarrollo del análisis asintótico de Riemann-Hilbert y sus aplicaciones al estudio de fenómenos críticos relacionados con funciones especiales no lineales o con singularidades que aparecen en la descripción de la estructura electrónica del grafeno. Análisis asintótico de polinomios con ortogonalidad no estandar (Sobolev, Hermite-Padé, Wronskianos) y nuevos métodos para el estudio de PO sobre la circunferencia unidad.

2. Desarrollo de modelos electrostáticos de ceros de varias clases de polinomios y análisis de problemas extremales en la teoría del potencial logarítmico. En particular, puntos de silla de funcionales de la energía en el plano, relacionados con varios asuntos de la teoría geométrica de funciones y de los fenómenos no lineales, tales como el crecimiento laplaciano.

3. Aplicaciones de herramientas recientemente desarrolladas al estudio de los procesos de difusión multipartícula y las matrices aleatorias, sus posibles transiciones de fase, así como profundización en los procesos de Markov clásicos desde puntos de vista novedosos.

4. Desarrollo del enfoque basado en PO a los caminos aletorios cuánticos (CAC), iniciado por los miembros de este equipo, en particular mediante funciones de Schur como herramienta para estudiar la dinámica y las fases topológicas en CAC, con aplicaciones a la computación cuántica.

5. Extensión de ciertas conexiones conocidas entre problemas biespectrales, sistemas integrables, transformaciones de Darboux y procesamiento de señales a contextos más amplios de matrices Jacobi por bloques, matrices de CMV y otros.

6. Búsqueda de algoritmos más eficientes para la medición objetiva de las características del ojo y su diagnóstico, como la reconstrucción de la forma de la córnea, el cálculo de funciones ópticas del ojo utilizando las medidas de aberraciones del frente de onda, así como el análisis de estas aberraciones utilizando PSF (Point Spread Function) correspondientes a varios parámetros de desenfoque.

Los resultados esperados pueden generar transferencia de conocimiento, principalmente en la Actividad 6. Esta expectativa se basa en la experiencia previa del equipo (con una patente nacional concedida) y en la colaboración con el "Instituto de Investigaciones Neurobiológicas Neurobia", que es una entidad interesada en los resultados del proyecto.

PALABRAS CLAVE: polinomios ortogonales, funciones especiales, teoría asintótica, procesos estocásticos clásicos y cuánticos, imagen médica

TITLE OF THE PROJECT: Approximation and orthogonality: from theory to applications

ACRONYM: APORTA

SUMMARY Maximum 3500 characters (including spaces):

This project combines both ambitious goals in fundamental research on orthogonal polynomials, special functions and their analytic and structural properties, with the applications of this knowledge in other branches of mathematics (stochastic processes, combinatorics, numerical analysis), physics (statistical physics, integrable systems, quantum mechanics, quantum random walks, quantum computation), and technology (signal processing and diagnostic tools in ophthalmology, with applications in clinical practice). Some of the problems to be tackled are:

1. Further contributions to general orthogonal polynomials (OP) and rational approximation, including the development of the Riemann-Hilbert asymptotic analysis and its applications to the study of critical phenomena related to non-linear special functions or singularities appearing in the description of the electronic structure of graphene; asymptotic analysis of polynomials of non-standard orthogonality (Sobolev, Hermite-Padé, Wronskians), and new approaches to OP on the unit circle.

2. Development of electrostatic models for zeros of several classes of polynomials and analysis of different extremal problems in logarithmic potential theory, in particular, saddle points of energy functionals on the plane, connected to several object from the geometric function theory and non-linear phenomena such as laplacian growth.

3. Applications of newly developed tools to the study of multiparticle diffusion processes and random matrices, their possible phase transitions, as well as new insights into classical Markov processes.

4. Further development of the OP approach to Quantum Random Walks (QRW), started by members of this team, in particular via Schur functions as a tool to study the dynamics and topological phases in QRW, with applications to quantum computing.

5. Extension of known connections between bispectral problems, integrable systems, Darboux transformations and signal processing to broader contexts of block-Jacobi matrices, CMV matrices, and others.

6. Search for more efficient algorithms for objective measurements of the eye characteristics and diagnostics, such as the cornea shape reconstruction, calculation of optical functions of the eye from the measured wavefront aberrations, as well as analysis of these aberrations from the known PSF (Point Spread Function) corresponding to several defocus parameters.

Expected results may generate transfer of knowledge mainly from the outcome of Activity 6. This expectation is based both on the previous experience of the team (with already one national patent granted) and on the ongoing collaboration with the "Instituto de Investigaciones Neurobiológicas Neurobia", which is an entity interested in the results of this project.

KEY WORDS: orthogonal polynomials, special functions, asymptotic theory, classical and quantum random processes, medical imaging

Parte B: INFORMACIÓN ESPECÍFICA DEL EQUIPO

B.1. RELACIÓN DE LAS PERSONAS NO DOCTORES QUE COMPONEN EL EQUIPO DE TRABAJO (se recuerda que los doctores del equipo de trabajo y los componentes del equipo de investigación no se solicitan aquí porque deberán incluirse en la aplicación informática de solicitud). Repita la siguiente secuencia tantas veces como precise.

1. Nombre y apellidos: Ana Belén Castaño Fernández
   Titulación: Licenciatura en Matemáticas e Ingeniería Técnica en Informática de Sistemas
   Tipo de contrato: en formación
   Duración del contrato: temporal (hasta 1/9/2017)

B.2. FINANCIACIÓN PÚBLICA Y PRIVADA (PROYECTOS Y/O CONTRATOS DE I+D+I) DEL EQUIPO DE INVESTIGACIÓN (repita la secuencia tantas veces como se precise hasta un máximo de 10 proyectos y/o contratos).

1. Investigador del equipo de investigación que participa en el proyecto/contrato (nombre y apellidos): Andrei Martínez-Finkelshtein, Juan José Moreno Balcázar, Alejandro Zarzo Altarejos.
   Referencia del proyecto: Proyecto de Excelencia P11-FQM-7276
   Título: "Teoría de la aproximación, funciones especiales y modelos matemáticos: de la teoría a las aplicaciones oftalmológicas"
   Investigador principal (nombre y apellidos): Andrei Martínez-Finkelshtein
   Entidad financiadora: Junta de Andalucía
   Duración: 30/04/2013-01/09/2017
   Financiación recibida (en euros): 239.478,30
   Relación con el proyecto que se presenta: muy relacionado
   Estado del proyecto o contrato: concedido

2. Investigador del equipo de investigación que participa en el proyecto/contrato (nombre y apellidos): Andrei Martínez-Finkelshtein, Juan José Moreno Balcázar, Alejandro Zarzo Altarejos, Pedro Martínez-González, Luis F. Velázquez Campoy, María José Cantero Medina, Leandro Moral Ledesma.
   Referencia del proyecto: Proyecto MTM2011-28952-C02-01
   Título: "Funciones especiales y teoría de aproximación: aplicaciones en ciencia y tecnología"
   Investigador principal (nombre y apellidos): Andrei Martínez-Finkelshtein
   Entidad financiadora: Ministerio de Ciencia e Innovación (MICINN)
   Duración: 01/01/2012-31/12/2014 (prorrogado hasta 01/01/2015).
   Financiación recibida (en euros): 74.000
   Relación con el proyecto que se presenta: muy relacionado
   Estado del proyecto o contrato: concedido

3. Investigador del equipo de investigación que participa en el proyecto/contrato (nombre y apellidos): Andrei Martínez-Finkelshtein.
   Referencia del proyecto: Proyecto FIS PI10/01843 de Acción Estratégica en Salud de España
   Título: "Desarrollo y Evaluación de un Sistema de Monitorización del efecto de diferentes técnicas quirúrgicas en queratocono"
   Investigador principal (nombre y apellidos): Andrei Martínez-Finkelshtein
   Entidad financiadora: Instituto de Salud Carlos III
   Duración: 01/01/2011-31/12/2013.
   Financiación recibida (en euros): 30.000
   Relación con el proyecto que se presenta: algo relacionado
   Estado del proyecto o contrato: concedido

4. Investigador del equipo de investigación que participa en el proyecto/contrato (nombre y apellidos): Andrei Martínez Finkelshtein, Juan José Moreno Balcázar.
   Referencia del proyecto: Proyecto de Excelencia P09-FQM-4643
   Título: “Ortogonalidad, aproximación y complejidad cuántica: teoría y aplicaciones científicas y tecnológicas”
   Investigador principal (nombre y apellidos): Antonio Durán Guardeño
   Entidad financiadora: Junta de Andalucía
   Duración: 02/02/2010 -31/12/2013.
   Financiación recibida (en euros): 293.939,68
   Relación con el proyecto que se presenta: muy relacionado
   Estado del proyecto o contrato: concedido

5. Investigador del equipo de investigación que participa en el proyecto/contrato (nombre y apellidos): Andrei Martínez Finkelshtein, Juan José Moreno Balcázar, Alejandro Zarzo Altarejos, Pedro Martínez González, Luis F. Velázquez Campoy, María José Cantero Medina, Leandro Moral Ledesma.
   Referencia del proyecto: Proyecto MTM2008-06689-C02-01
   Título: “Polinomios Ortogonales: Teoría y Aplicaciones en las Matemáticas, la Física y la Tecnología”
   Investigador principal (nombre y apellidos): Andrei Martínez-Finkelshtein
   Entidad financiadora: Ministerio de Ciencia e Innovación (MICINN)
   Duración: 01/01/2009 -31/12/2011.
   Financiación recibida (en euros): 108.900,00
   Relación con el proyecto que se presenta: muy relacionado
   Estado del proyecto o contrato: concedido

6. Investigador del equipo de investigación que participa en el proyecto/contrato (nombre y apellidos): Andrei Martínez Finkelshtein, Juan José Moreno Balcázar.
   Referencia del proyecto: Proyecto de Excelencia P06-FQM-01735
   Título: “Ortogonalidad, no-linealidad y teoría de la información: interacciones y aplicaciones físicas, clínicas y nanotecnológicas”
   Investigador principal (nombre y apellidos): Andrei Martínez-Finkelshtein
   Entidad financiadora: Junta de Andalucía
   Duración: 01/06/2007 -10/10/2010.
   Financiación recibida (en euros): 152.800,00
   Relación con el proyecto que se presenta: muy relacionado
   Estado del proyecto o contrato: concedido

7. Investigador del equipo de investigación que participa en el proyecto/contrato (nombre y apellidos): Andrei Martínez Finkelshtein, Juan José Moreno Balcázar, Alejandro Zarzo Altarejos.
   Referencia del proyecto: Proyecto de Excelencia FQM 481
   Título: “Funciones especiales, entropías cuánticas y aplicaciones bio- y nanotecnológicas”
   Investigador principal (nombre y apellidos): Jesús Sánchez-Dehesa
   Entidad financiadora: Junta de Andalucía
   Duración: 01/02/2006 -30/06/2009.
   Financiación recibida (en euros): 85.000,00
   Relación con el proyecto que se presenta: muy relacionado
   Estado del proyecto o contrato: concedido

8. Investigador del equipo de investigación que participa en el proyecto/contrato (nombre y apellidos): Andrei Martínez-Finkelshtein, Juan José Moreno Balcázar, Alejandro Zarzo Altarejos, Pedro Martínez-González, Luis F. Velázquez Campoy, María José Cantero Medina, Leandro Moral Ledesma.

Referencia del proyecto: Proyecto MTM2005-08648-C02-01
Título: "Funciones especiales y Polinomios Ortogonales. Aproximación Constructiva y Aplicaciones"
Investigador principal (nombre y apellidos): Andrei Martínez-Finkelshtein
Entidad financiadora: Ministerio de Educación y Ciencia
Duración: 15/10/2005 - 14/10/2008.
Financiación recibida (en euros): 65.450,00
Relación con el proyecto que se presenta: muy relacionado
Estado del proyecto o contrato: concedido

Parte C: DOCUMENTO CIENTÍFICO

C.1. PROPUESTA CIENTÍFICA

C.1.1. Background

(Remark: all the bibliographic references appearing in this proposal starting with "t" correspond to publications of the team members; those starting with a capital letter are external to the team).

Orthogonal polynomials (OP), born as an independent theory in the XIX century, are traditionally considered "classical analysis". Despite of this, this theory has been living a spectacular development starting from the 80-ies (a MathSciNet search for "orthogonal polynomials" renders more than 8000 hits, approximately 6700 appeared after 1980). The modern OP theory extends far beyond the classical families. Nowadays "non-standard" orthogonality conditions are of great interest. For instance, a burst of activity in rational approximation 30 years ago brought the need to study OP with an orthogonality measure depending on the degree of the polynomial (varying orthogonality) and corresponding to non-hermitian bilinear functionals. Number theory problems, simultaneous approximation, or stochastic processes lead to multiple OP (MOP), where the orthogonality conditions are distributed among several measures. Recurrence relations with many terms, problems in quantum mechanics or generalizations of Markov processes yield OP with matrix coefficients (matrix OP). The need to fit a function controlling oscillations of it approximants motivates the Sobolev OP. And the description of spectral properties of Hamiltonians of quantum mechanical systems generated a burst of activity on exceptional OP.

The OP lie on the crossroad of many branches of mathematics, which stimulates the implementation of new ideas coming from other areas. Several of those branches that have supplied techniques to and have benefited from the theory of OP during the last decades are:

- Potential theory, electrostatic equilibrium, and geometric function theory;
- Boundary value problems and the Riemann-Hilbert techniques;
- Integrable systems and inverse scattering;
- Random walks, random matrices and growth models;
- Spectral theory of operators;
- Harmonic analysis and Schur functions theory.

We will describe briefly some of these ideas that serve as a background for this project.

The **logarithmic potential** theory [NS91, ST97], with its roots in mathematical physics, is a crucial tool for OP and rational approximation, contributing for example with the notion of equilibrium or extremal measure. This measure is intrinsically related with the zeros of norm-minimizing polynomials, among others. This basic concept has been extended and adapted to tackle increasingly sophisticated problems, being an object of intense study the equilibrium in an *external field*, the equilibrium with *constraints* or the *vector equilibrium* problems, and their combinations. In all cases we deal with minimizing an energy functional. But even these models are not sufficient; an emergent line of research is based on the observation that in many situations we must go beyond local minima of the energy and consider its saddle points, which leads to the *critical measures*. In the case of a continuous energy, the resulting measure is the key component of the asymptotic expression of some kind of OP, such as those with a varying weight (external field), discrete orthogonality (upper constraint), multiple (vector equilibrium) or non-hermitian orthogonality (critical measures). Analogous problems for a discrete energy are connected with an important class of linear ODE (the generalized Heun equations [R95]), with interest for integrable systems (Gaudin model [HW95]). The study of critical measures is closely related to classical problems of geometric function theory, in particular, to the structure of trajectories of quadratic differentials on a Riemann surface. The analysis of critical measures can be key in open problems in the asymptotic theory, see e.g. [tMFR11] for further details.

Other classes of "non-standard" OP are still waiting to benefit fully from these methods and demand new ideas for their study. Among them we may highlight the Sobolev OP, since Spanish groups have played a leading role in their development. Their main feature is that they exhibit orthogonality with respect to an inner product involving derivatives. Although their original motivation was simultaneous approximation of a function and its derivatives, they became related to spectral methods for boundary value problems of differential equations [H05]. Within the theory of general OP, **Sobolev polynomials** occupy a special place, their analytic theory is largely unknown beyond some established facts (see e.g. [tAMBPR09], [tAMBPR11], [tMF00]); however, more than 100 papers on this subject have appeared in the last 15 years, a number that continues growing. Somewhat related are the polynomials defined as Wronskians of standard OP. Again, their analytic properties are basically unknown (see e.g. [FHV12]), but they are crucial for the construction of exceptional OP [D14a], [D14b], [D14c].

A system, described by non linear differential or difference equations is, generally speaking, virtually "intractable", unless it is integrable. Although there is no unique definition of integrability, all these systems have in common the existence of nonlinear phenomena, such as solitons, with increasing applications in technology. Integrable systems, being a meeting point for different branches of knowledge, constitute a source of inter-disciplinary interaction. Among the paradigms are the Toda lattices arising in the discretization of the KdV equation, the first example of an integrable system with solitonic solutions. A formulation of Toda lattices in terms of Lax pairs involving Jacobi matrices is essential for understanding their integrability, but also for establishing a fruitful connection with the theory of selfadjoint operators and OP. This relationship among OP, integrable systems [De99a, De99b] and inverse scattering [GC79] had its second essential moment in the works of several mathematical physicists. Fokas, Its and Kitaev [FIK92] found a characterization of families of OP in terms of matrix **Riemann-Hilbert** (**RH**) problems, opening new horizons whose dimensions we are starting to appreciate only now. Based on this characterization, Deift and Zhou [DZ93, De99a] put forward a powerful method of asymptotic analysis of OP (nonlinear steepest descent method) that renders results of maximum precision, and is being developed further (in particular, by members of this team). These techniques have allowed to obtain results that were considered beyond of reach just a few years ago, such as the study of critical phenomena in transition between asymptotic regimes, where the behavior of OP and their associated parameters is described by means of **Painlevé transcendents** and other nonlinear special functions (another connection with integrable systems). A key ingredient are the so-called "g-functions", connected with the critical and equilibrium measures described above. Since the original version of the RH method for OP is intimately connected with the analyticity of the orthogonality weights, there have been attempts to extend it to discrete orthogonality [BKMM07] and to non-analytic weights. Furthermore, these techniques could possibly tackle the problem of the detailed asymptotics of OP with respect to a weight with logarithmic singularities. This problem has direct connections with the study of the electronic structure of **graphene**, see [H72], [FHV12].

Jacobi matrices, canonical representatives of self-adjoint operators, are the cornerstone of the **spectral approach** to OP on the real line (**OPRL**). A generalized version of this connection links block-Jacobi matrices to matrix OPRL. A similar spectral approach to OP on the unit circle (**OPUC**), connected analogously with unitary operators, is of much more recent vintage. The starting point was the discovery in 2003 by members of this team [tCMV03] of the unitary analogue of Jacobi matrices, the **CMV matrices** (acronym of the last names of the authors), which became Trojan horses for the introduction of spectral methods in OPUC theory. The extremely active participation of B. Simon (Caltech) and collaborators in this spectral approach [Si05], [Si10], have fostered this part of the theory. Since its foundations, the advances in OPUC go hand in hand with developments in harmonic analysis. **Schur functions**, central in this intertwining as an alternative to the spectral approach, are protagonists of a recent revolution in OPUC theory, almost contemporary to that of CMV matrices, due to S. Khrushchev. Its cornerstone is a factorization property of Schur functions called Khrushchev's formula. The combined use of this formula and analytic techniques originated a number of deep results and innovative techniques for OPUC [Kh01], [Kh02] nowadays known under the name of Khrushchev theory (see [Si05]).

However, the intense development of OP theory in the last years has not been motivated by the new methods only, but also by recent and spectacular applications of this theory in different branches of science. We will only mention briefly a few of them, related to some lines of research of this project.

Some of the most remarkable applications of OP to stochastic processes have their origin in two papers of Karlin and McGregor [KM59a], [KM59b]. On one hand, reversible and irreducible **Markov processes**, such as random walks, can be described in terms of self-adjoint operators, which provide a natural connection to OPRL. This relation is even closer for birth-death processes, modeled by tridiagonal matrices which constitute a direct link to

OPRL [KM59a]. One of our members, F.A. Grünbaum, has a number of recent contributions in this topic [tGrIg08], [tGr08], [tGr09a], [tGr09b], [tGr10]. On the other hand, [KM59b] uncovered an interpretation of certain determinants in terms of joint probability of diffusion of non-colliding particles ("non-colliding point processes"). This gave birth to a wide and intensively studied class of **determinantal point processes**, one of whose paradigms is the eigenvalue distribution of random matrices. Random matrix models [M04], [De99a] are ubiquitous in mathematics and modern physics due to their high capacity of describing very diverse phenomena in quantum gravity [FIK92], solid state physics, statistical mechanics, statistics, combinatorics [BDJ99], tiling and random graphs [J02], polynuclear growth [J03], percolation, mathematical biology [BS06] and even public transportation modeling [BBDS06] (the list is far from being complete). The intensity of research and the rate of new results and ideas is endorsed by the publication of at least 12 monographs on this topic in the last years [ABDF11], [AGZ09], [BL14], [BS09], [BFL14], [B09], [DG09], [K14], [Fo10], [Gh09], [Gui09], [T12]. The impressive progress in the solution of random matrix models (from the so-called "universality" to the asymptotics of critical phenomena) has been based on the development of the RH approach, large deviation results, methods of integrable systems and equilibrium measures. Random matrix models with interactions and normal random models are important steps in future developments and their applications.

Random walks have a quantum version, the **quantum random walks** (**QRW**), nowadays a hot topic in quantum information. They appeared in the early nineties [ADZ93] by the hand of Wolf Prize Y. Aharonov and coworkers. The widespread use of random walks in randomized algorithms motivated the use of QRW in quantum computation, discovering that they provide quantum algorithms with an exponential speedup over any classical one [Am03], [Ke03], [Ke06], [Ch09]. QRW have been experimentally reproduced in different ways, and there are strong evidences of their role in natural processes such as the highly efficient energy transfer during the photosynthesis [En07], [Mo08]. Regarding the mathematical setting, a connection with OPUC was very recently discovered by members of our team [tCGMV10], leading to a new method of analysis of QRW which goes further than standard ones (see [tCGMV10], [tCGMV12a], [tCGMV12b], [tGV12] and compare with [Ko08, SKJ09]). The name CGMV method, coined by N. Konno and E. Segawa [KS11], [KS14], refers to its authors, among them those involved in the discovery of CMV matrices. This is not a coincidence since CMV matrices constitute a meeting point for QRW and OPUC: the conservation of probability in QRW is encoded in the unitarity of the evolution operator, so the canonical representatives of the unitaries provide the simplest QRW. This parallels the relation between birth-death processes and OPRL via Jacobi matrices. Another recent result of our team and collaborators adds value to OPUC in quantum physics: Schur functions codify the return properties of discrete time quantum systems [tGVWW13], [tBGVW14] revealing their geometric-topological meaning. This suggests the use of Schur functions in a subject with deep impact in quantum information engineering: the study of topological phases of quantum matter [Ki12], [TAD14], [Ki08], [Ch13]. A promising way to overcome the stability problems in building a quantum computer is the so-called **topological quantum computation** [Ki03], [N-S08], [Ki09], which takes advantage of the stability of topological restrictions. This idea already explains the recently discovered topological insulators and superconductors [Ki08], [Be13], [HK14], platform for the next generation of electronics and potential building blocks of future quantum computers. The robustness of their properties is due to the stability of phases characterized by topological invariants ("topological phases"). These invariants are sometimes computed by splitting the quantum system, which uncovers the interest of Khrushchev splitting formula in the study of topological phases.

Concerning spectral theory, OPRL are related to the fruitful **bispectral problem** [tGr01]. The classical example was found by Slepian, Landau and Pollack [Sl83], from Bell Labs, regarding the **time-band-limiting problem** originated from Shanon's seminal work in communication theory. This motivated the use of bispectral techniques in tomography by a member of our team, F.A. Grünbaum [tGr82], [tGr83], whose results have been used for signal processing in geophysics. The unexpected connection of the bispectral problem with signal theory is as remarkable as its surprising links with integrable systems [tDG86]. These miracles need a deeper understanding and deserve further research. OPRL enter this game because they provide a rich variety of bispectral instances [tGH94], [tGH96], [tGH97a], [tGH97b], [tGH99], [tGHH99], [tGY02], [tGr07]. Nevertheless, integrable systems and bispectral situations are rara avis. Luckily enough, **Darboux transformations** have proved to be a powerful tool to generate hierarchies of such strange phenomena, starting from simple instances. Actually, Darboux transformations lead to the complete classification of certain bispectral problems, a question in which F.A. Grünbaum has key contributions [tDG86], [tGH94], [tGH96], [tGH97b], [tGHH99], [tGY02].

Approximation theory and OP find other technological applications. The **reconstruction** and representation of **surfaces** are central for graphical computation, medical imaging and other fields. For example, the Hartmann-Shack sensors and the Placido disk-based corneal topographers, used in ophthalmological practice, try to determine the wavefront errors of the eye and the properties of the anterior face of the cornea by analyzing either the shift of the light points on the target or the reflected image of a concentric ring pattern, respectively. In all cases, a reconstruction step is unavoidable. A standard approach consists in expressing the corresponding function in an appropriate basis, and the most popular one in optometry and vision is that of Zernike polynomials [At04] that form a complete set of polynomials in two variables orthogonal with respect to the disk area. For many years this was the only option, according to a standard of the Optical Society of America (OSA). The development of more sophisticated tools and higher resolutions made apparent that this choice is not always optimal [Da06], [tMFDCZA09]; several alternatives are being considered (in particular, by members of this team). Computerization of medical technology in ophthalmology and vision research allows also the introduction of much more advanced techniques and mathematical tools, such as Fourier and wavelet analysis, pattern recognition, signal processing and others. Methods from approximation theory and OP are especially promising in this context.

State of the art

Due to the restriction in space we have to outline, very schematically, the latest developments in the fields exposed in the “Background” section, giving preference to those areas where the members of our team have played a prominent role.

- The toolbox of RH techniques, potential theory, spectral theory, complex analysis and special functions has allowed to establish new fine asymptotic results, either of great generality, or in special asymptotic regimes revealing critical phenomena. These phenomena are usually described in terms of the “special functions of the XXI Century”, namely the Painlevé transcendents. On the other hand, motivated by a universality conjecture in random matrix theory, impressively general results on the asymptotics of reproducing kernels for OP have been established by Lubinsky, Simon and others. These results have a close relation with the equal spacing of zeros (“clock behavior”, term coined by B. Simon), a property whose violation turns out to be especially interesting [tFMFS10], [tFMFS11].
- The exciting connections of OP with random processes, both classical and quantum, has brought a variety of ground breaking results. Classical continuous stochastic processes (diffusion processes) that benefited clearly from this approach are the so-called biorthogonal ensembles and non-colliding point processes. These include different types of random matrices, as well as diffusion processes of many particles, that constitute an efficient modeling tool in physics and technology. In special situations (such as coupled random matrices or with an external source, confluent Brownian motion and others) the correlation kernel, which describes the local statistics of these processes, can be written in terms of multiple OP. Until very recently these problems were considered intractable, but new developments of RH techniques allow expecting advances in this area. In particular, it is of great interest the study of **“universality”** (independence of the asymptotic properties of the model from the concrete associated p.d.f.), as well as critical phenomena of transition and phase change. While one-matrix models describe random surfaces and pure gravity, interacting random matrix models describe models of statistical physics on random surfaces, and quantum gravity with interaction. The development of methods for interacting random matrix models and random matrix models with external source requires far going extensions of the matrix RH approach and the theory of vector equilibrium measures on the complex plane. The appropriate RH problem is an $m\times m$ matrix-valued problem where $m\geq 3$, and the steepest descent analysis of such problems is highly nontrivial.
- Another field is related to normal, or non-Hermitian, matrix models. There is an enormous interest in studying this ensemble, which appear in numerous applications including the theory of the Hele-Shaw flow, quantum chromodynamics, neural networks, combinatorics and string theory. A very fast growing number of mostly physical works is devoted to this

model. However, a rigorous mathematical study of non-Hermitian matrix models is at its very initial stage. No rigorous results on higher order asymptotics of the partition function and associated OP, local correlations of the eigenvalues, and, most importantly, universality issues, are available so far (except for some important but still particular cases of potentials).
- QRW exhibit a large variety of quantum effects. For instance, the walker can be in a superposition of states, making QRW capable of performing different tasks simultaneously (quantum parallelism), so that they can spread out much quicker than random walks (ballistic transport), which is the reason for the great speedup of quantum algorithms. On the other hand, interference between multiple-scattering paths can completely trap the walker inside a disordered medium (Anderson localization [An58]). Localization acts oppositely to parallelism, thus the study of localization-delocalization regimes is crucial. The team has several contributions to this problem using the CGMV method [tCGMV10], [tCGMV12a], [tCGMV12b], [tGV12], and is currently working on QRW with extreme ballistic diffusion.
- The study of return properties of QRW via Schur functions has led us to uncover a dynamical interpretation of Khrushchev formula: it reflects a splitting of a quantum system into subsytems whose return properties control those of the original system [tCGVWW]. This formula can be used for practical computations [tGVWW13], [tBGVW14], but has also profound consequences in the qualitative behavior of QRW which are under study at present. In a general quantum setting, matrix Schur functions take the place of the scalar ones [BGVW14], so one needs to allow for matrix OPUC (MOPUC) in the CGMV method. A drawback is the absence of a Khrushchev formula for MOPUC. Luckily enough, our team, in collaboration with the group on Quantum Information led by R.F. Werner, has shown that a new diagrammatic proof of Khrushchev formula works for MOPUC [tCGVWW]. This result opens this research line into multiple promising directions: extension of Khrushchev theory to MOPUC and OPRL, further development of diagrammatic methods for OP, Khrushchev type splitting techniques for Markov processes, qualitative return properties of classical and quantum random walks, Schur function approach to the classification of topological phases.
- Darboux transformations for Jacobi matrices and their implications in OPRL theory are well known [tGH94], [tGH96], [tGH97b], [tGHH99], [tGY02], but they have not been developed yet in the context of CMV matrices and OPUC. This extension is a project in progress within our research group [tCMMV]. Darboux transformations for CMV share many features with those of the Jacobi case, like the relation with spectral transformations, but also exhibit new phenomena such as the appearance of spurious solutions whose interpretation deserves further research. Regarding the matrix OP setting, the contribution [tGr11] by our member F.A. Grünbaum has recently opened up a research line on Darboux transformations for block-Jacobi matrices and matrix OPRL (MOPRL). As in the standard Jacobi case, we expect links of these two new types of Darboux transformations with integrable systems, bispectral problems and signal processing.
- Other known results for OP can be pursued in innovative directions. An example is the generating functions technique, so fruitful for OPRL. Surprisingly, so far, it has not been extended to OPUC, where much less instances are known. A first step in filling this gap is given in [tCIa] by our member M.J. Cantero in collaboration with A. Iserles. This work shows that generating functions can be so useful for OPUC as for OPRL, but also opens interesting connections with the so-called pantograph equations [Is93]. Another instance of revisiting an old result is the case of Heine integral representation of Hankel determinants. A work in progress by A. Iserles and coworkers [ADHI] illustrates its efficiency for large parameter asymptotics of OPRL. M.J. Cantero and A. Iserles are currently involved in the extension of these ideas to OPUC [CIb], which needs a Heine integral representation for Toeplitz determinants. This has important ramifications to highly oscillatory quadrature, random matrix models, integrable sytems and high energy physics [BCM12], [CJJM13].
- Medical imaging, in particular, in ophthalmology and vision science, is advancing so fast that describing the most recent achievements is a formidable task. Narrowing the field to diagnostic tools of the human corneal diseases, we can mention both the introduction of new technology, such as the Optical Coherence Tomography (OCT), and the revision of standard mathematical algorithms used in more traditional devices, such as corneal topographers. The latter is fostered also by the development and cheapening of computers, allowing for more sophisticated methods implemented and useful in the clinical practice. Beyond the classical Zernike-based reconstruction, ideas like Fourier transform, wavelets, convex optimization, artificial intelligence, filtering and pattern recognition become pervasive. OP and

approximation theory have still much to do in these applications. It is interesting, for instance, to study the efficiency of multivariate Chebyshev [GSJ14] or Sobolev OP in this context. In fact, generalizations of Sobolev OP to several variables have found already applications in the analysis of polishing tools used in manufacturing of optical surfaces

Related research teams

This research group has intense and stable connections within **Spain**, with a traditional solid school on OP and approximation theory, represented by very active groups (working from different and complementary perspectives), at the *University Carlos III of Madrid* (led by F. Marcellán and G. López Lagomasino, Activities 1, 2, 5), at *University of Zaragoza* (led by M. Alfaro, Activities 1, 2), from the *University of Seville* (led by A. Durán, Activities 1, 5), *Vigo University* (E. Godoy and A. Cachafeiro, Activities 1, 2), or *University of La Laguna* (R. Orive, Activities 1, 5). Related with the topic of this project is also the team at *Public University of Navarra*, led by J.L. López, working on asymptotic analysis of special functions (Activity 1). We should emphasize the tight contacts with *Institute Carlos I of Theoretical and Computational Physics* of the Granada University (led by P. Garrido), being the PI its adjunct researchers. In topics of ophthalmology and optometry (Activity 6), we have stable contacts with researchers from *VISSUM Ophthalmological Corporation* and *University Miguel Hernández*, from Elche (J. Alió) and with *Instituto de Investigaciones Neurobiológicas Neurobia* (G. Castro). Scientific contacts have been established with groups led by R. Montés-Micó (*U. Valencia*), R. Navarro (*CSIC Zaragoza*), and N. López-Gil (*U. Murcia*).
This research team has also stable relationship with many groups **abroad**, representing virtually all continents. There exists a very strong connection with the **Russian** and **Ukrainian** mathematical schools, represented by researchers from *Moscow State University* (V. Sorokin and V. Buyarov, with several research visits in Almeria), *Institutes Steklov of Mathematics* (*S. Suetin*) and *Keldysh of Applied Mathematics* (A. Aptekarev, also with several research visits to Almería and Granada), as well as the *Institute of Low Temperatures* in Kharkov (L. Golinsky, with a research stay in Almería in 2010). In **Europe**, the most stable collaboration has taken place with the *Katholieke Universiteit Leuven* in Belgium (A.B.J. Kuijlaars and W. Van Asche), *Université Lille I* (B. Beckermann), *Université d'Aix-Marseille* (F. Wielonsky), *Bordeaux* (S. Kupin) and *INRIA* in Sophia-Antipolis (L. Barachart) in France, Universities of *Coimbra* (A. Branquinho) and *Aveiro* (A. Foulquié and I. Caçao) in Portugal, *University of Cambridge*, in UK (A. Iserles). An important group in optometry and vision science, maintaining contacts with us, is located at the *Institute of Biomedical Engineering and Instrumentation* in Wroclaw, Poland, led by R. Iskander and W. Okrasinski, and who visited the University of Almería recently; another team we want to expand our contacts with is located at the *Queensland University of Technology*, **Australia** (M.J. Collins and D. Alonso-Caneiro). There is also a stable collaboration with researchers from **Brasil**, in particular, with D. Dimitrov, S. Ranga, C.F. Bracciali, and E.X.L. de Andrade, from *UNESP*; this collaboration is supported by a "Special Visiting Researcher" Fellowship under the Brazilian Scientific Mobility Program "Ciências sem Fronteiras" (2013–2015), recently won by the first PI. Another new collaboration is with researchers from **China**, at the *Shanghai Jiao Tong University* (M. Tiaglov), where the first PI recently was awarded with a Fellowship under the Recruitment Program of High-end Foreign Experts of the China State Foreign Expert Bureau (2014–2016). We have traditional collaboration with many groups in the **United States** (with regular long term visits of researchers in both directions). In particular, the research groups with the most stable collaboration with our team have been those from the *Vanderbilt University* (E.B. Saff), *University of Arizona* (K.T.-R. McLaughlin), *University of South Florida* (E.A. Rakhmanov), *Georgia Institute of Technology* (D. Lubinsky, P. Iliev and J. Geronimo). It has been crucial for our team the collaboration with colleagues from the *University of California Berkeley* (several visits and research stays of members of the team there, actually the intense collaboration with F.A. Grünbaum was the reason for his inclusion in the present project), *Institute for Advanced Study* (J. Bourgain) and *California Institute of Technology* (B. Simon, with 4 long term visits of A. Martínez-Finkelshtein to Caltech in the last 4 years). Finally, the research in QRW has benefited greatly from the scientific contacts in **Germany** and **Japan**, in particular, with R.F. Werner and its group on Quantum Information from *Leibniz Universität Hannover*, A.H. Werner from *Freie Universität Berlin*, N. Konno from *Yokohama National University, and* E. Segawa, from *Tohoku University*.

All these collaborations are backed up either by research exchange and visits (carried out or in immediate plans) of scholars and students, or by joint publications and works of members of the team.

Main achievements of the previous project

This project is a natural continuation of the Coordinated Project MTM2005-08648-C02 from the Ministry of Education and Science (October 15, 2005 - October 14, 2008), and of two Coordinated Projects from the Ministry of Science and Innovation: MTM2008-06689-C02, (January 1, 2009 - December 31, 2011), and MTM2011-28952-C02 (January 1, 2012 - December 31, 2014, extended until December 31, 2015).

Among the most recent achievements (in particular, within the framework of the projects mentioned above) are:

- Development of RH asymptotic techniques for weights with singularities [tFMFS10, tFMFS11] or non-standard orthogonality [tMFRS12, AMFMT14].
- Description of the global structure of trajectories of rational quadratic differentials on the Riemann sphere [AMFMT14].
- Introduction of critical measures on the plane, their connections to quadratic differentials [tMFR11], and variational techniques to study their supports [tMFRS11, BMFS12].
- Advances in quadrature formulas on the unit circle [tBC09].
- Introduction and study of the notion of normality of derivative behavior for OPUC [tMFS11].
- Study of spectral transformations on the unit circle [tCMV11, tCMV12].
- Interconnections between Darboux transformations for Jacobi matrices, integrable systems, bispectral situations and time-band-limiting problems in signal processing [tGr94], [tGH94], [tGH96], [tGH97a], [tGH97b], [tGH99], [tGHH99], [tGr01], [tGY02], [tGY04].
- Analytic theory of Sobolev OP, especially on unbounded sets [tAMBPR11, tAMBPR15] or for discrete inner products [tMMMB13], and their zero interlacing [tCGMB13].
- Asymptotic study of the local and global statistics of the spectrum of diffusion processes of many non-colliding particles, especially in the critical regime [KMFW11].
- Convergence properties of Padé and Hermite-Padé approximants [tMFRS12, tMFRS13].
- Applications of OP, CMV matrices and Schur functions to QRW [tCGMV10], [tCGMV12a], [tCGMV12b], [tGV12], [tGVWW13], [tBGVW14].
- Extensions of Khrushchev formula [tCGVWW].
- Pantograph equations and generating functions for OP on the unit circle [tCIa].
- Multiscale adaptive algorithm for reconstruction of the shape of the cornea from the keratoscopic measurements [tMFRCA11].
- Development of a simple set of indices for screening subclinical keratoconus with Placido-based corneal keratoscopes [tRMFCPA11, tRMFCB13].
- Development of new procedures for the computation of the through-focus characteristics of a human eye [tRMFI14, tMFRI].

Selected publications of the team related with the project

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C.1.2. General goals

We identify 6 general goals, that correspond to 6 main activities listed in Section C.1.4:

1. Further contributions to general orthogonal polynomials (OP) and rational approximation, including the development of the Riemann-Hilbert asymptotic analysis and its applications to the study of critical phenomena related to non-linear special functions or singularities appearing in the description of the electronic structure of graphene; asymptotic analysis of polynomials of non-standard orthogonality (Sobolev, Hermite-Padé, Wronskians), and new approaches to OP on the unit circle (OPUC).
2. Development of electrostatic models for zeros of several classes of polynomials and analysis of different extremal problems in logarithmic potential theory, in particular, saddle points of energy functionals on the plane, connected to several object from the geometric function theory and non-linear phenomena such as laplacian growth.
3. Applications of newly developed tools to the study of multiparticle diffusion processes and random matrices, their possible phase transitions, as well as new insights into classical Markov processes.
4. Further development of the OP approach to Quantum Random Walks (QRW), started by members of this team, in particular via Schur functions as a tool to study the dynamics and topological phases in QRW, with applications to quantum computing.
5. Extension of known connections between bispectral problems, integrable systems, Darboux transformations and signal processing to broader contexts of block-Jacobi matrices, CMV matrices, and others.
6. Search for more efficient algorithms for objective measurements of the eye characteristics and diagnostics, such as the cornea shape reconstruction, calculation of optical functions of the eye from the measured wavefront aberrations, as well as analysis of these aberrations from the known PSF for several defocus parameters.

C.1.3. Specific goals and activities

They are enumerated in correspondence with the 6 general goals listed above (and correspond also to 6 main Activities of this Project, appearing in Section C.1.4).

<u>Goal 1</u>:

- Further development of the nonlinear steepest descent method for higher-order matrices, required in the analysis of multiple orthogonality and random matrix models, in connection with critical phenomena (phase transitions and Painlevé trascendents).
- Asymptotics of OP with respect to logarithmic singularities and application to the study of the electronic structure of graphene.
- Algebraic and asymptotic properties of Sobolev OP (Mehler-Heine formulas, asymptotics of varying Sobolev OP), with respect to continuous, discrete and plane measures.
- Wronskians of families of polynomials: zero localization and asymptotics.
- Asymptotics of Padé, Hermite-Padé and general rational approximation to analytic functions, in particular, those with algebraic singularities.
- Matrix Khrushchev formulas and Khrushchev theory for matrix OPUC and OPRL.
- Further development and uses of diagrammatic techniques for OP.
- Further development of generating functions methods for OPUC and non-standard OP, and their connections with functional equations.
- Development of Heine type integral representation for OPUC and its applications to OPUC theory, highly oscillatory quadrature on the unit circle and random matrix models.

<u>Goal 2:</u>

- Study of critical measures on the plane and their connection with geometric function theory (quadratic differentials on a Riemann surface and problem of moduli). Extension of the notion of the S-property to vector equilibria.
- Electrostatic models for zeros of Heine-Stieltjes and multiple OP. Further asymptotics of polynomial solutions of ODE of Heun type, and relation with the monodromy problem.
- Study of the dependence of the support of the equilibrium and critical measures on the intrinsic parameters of the system, description of the mechanisms of phase transitions, and connections wiith differenty notions of balayage.

Goal 3:
- Study of stochastic models (random matrices, non-colliding diffusion and other determinantal processes) associated to multiple OP, their asymptotics and universality.
- Asymptotics of normal matrix models and relation to the laplacian growth and other diffusion processes on the plane.
- Development of splitting techniques for Markov processes, based on Khrushchev type formulas for generating functions of first time return probabilities.
Goal 4:
- General relations between limit laws for QRW and spectral measures. In particular, analysis of singular continuous measures and measures giving strongly ballistic diffusion.
- Return properties of interacting QRW. Applications to situations of experimental interest.
- CGMV approach to QRW on graphs.
- Use of Schur functions techniques and Khrushchev formulas for the classification of topological phases in QRW. Rigorous proof of bulk-edge correspondence.
Goal 5:
- Interconnections between Darboux transformations for block-Jacobi matrices, integrable systems, bispectral situations and the time-band-limiting problem in signal processing.
- Darboux for CMV matrices, integrable systems, bispectral problems and signal theory.
Goal 6:
- Further study and improvement of the reconstruction techniques of wavefronts and corneal topography, alternative or complementary to Zernike polynomials fitting.
- Study of applicability of methods of compressed sensing and sparse signal reconstruction in the setting of corneal topography.
- Simplification of ray-tracing procedures for a real-time modeling of the internal structure of an in-vivo eyes.
- Study of applicability of phase-recovery techniques in the reconstruction of wavefront aberrations, as a possible alternative to the Hartman-Shack aberrometry.

C.1.4. Methodology
For methodological purposes the project is divided in several Activities and Tasks, as follows:

Activity 1: general orthogonal polynomials (OP) and rational approximation
The **asymptotic analysis** of OP plays a central role in the analytic theory of OP. We plan to continue the development of the matrix **Riemann-Hilbert** (RH) analysis, which leads to structure relations and asymptotics of OP via Deift-Zhou's method. Especially important and non-trivial is the analysis of the $m \times m$ matrix-valued problems with $m \geq 3$, characterizing Hermite-Padé or multiple OP (MOP), whose solution requires ingredients from logarithmic potential theory and Riemann surfaces. It reveals also deep connections of some critical regime asymptotics of OP and phase transitions in random matrix models with non-linear special functions, such as **Painlevé trascendents**. Another singular behavior is related to orthogonality weights with logarithmic singularities; by finding a detailed asymptotics of the recurrent coefficients that constitute a discretization of the corresponding Hamiltonian, we hope to gain some insight into the electronic structure of the **graphene**.
Non-standard orthogonality on the plane is connected with Padé and Hermite-Padé approximation. The rather complete understanding of the analytic theory of Padé approximants (due to pioneering work of H. Stahl) contrasts with the very few general results in **Hermite-Padé approximation**. **Sobolev OP** will continue playing important role in the research plans of this team, and we aim at several open problems, such as Mehler-Heine type asymptotics, varying Sobolev orthogonality, zero distribution and others. **Wronskians** constitute an important object of study, and we hope to contribute to their analytic theory.

Orthogonal polynomials on the unit circle (OPUC) play a unique role in the theory of OP. CMV matrices and Khrushchev theory have changed the way one looks at OPUC theory because of their impact in the theory itself, but also due to the multiple connections they have opened to other areas. Our main objective in this context is a further development of **CMV** and **Khrushchev approaches** to OPUC, based on the intertwining between them and also with their applications to quantum random walks (QRW). Other research lines in OPUC follow from the translation of techniques and results already fruitful for OP on the real line (OPRL), such as **generating functions** methods or Heine integral formula for Hankel

determinants. These topics open stimulating connections with functional equations, approximation theory, random matrix models, integrable systems and quantum field theory.
The experience and leading role of this research team in the development of OP theory explain the relatively high weight of this Activity in the whole working plan.

This Activity is divided into 9 tasks (initials of coordinating PI within brackets):
Task 1.1. Further development of the nonlinear steepest descent method, in particular, for higher-order matrices, required in the analysis of multiple orthogonality (Task 1.4) and random matrix models (Task 3.1), especially in connection with critical phenomena (phase transitions and Painlevé trascendents) [AMF].
Task 1.2. Asymptotics of OP with respect to logarithmic singularities and application to the study of the electronic structure of graphene [AMF].
Task 1.3. Algebraic and asymptotic properties of Sobolev OP (Mehler-Heine formulas, asymptotics of varying Sobolev OP), w.r.t. continuous, discrete and plane measures [JJMB].
Task 1.4. Wronskians of families of polynomials: zero localization and asymptotics [AMF].
Task 1.5. Asymptotic analysis of Padé, Hermite-Padé and general rational approximation to analytic functions, in particular, those with algebraic singularities [AMF].
Task 1.6. Khrushchev theory for matrix OPUC, as well as for OPRL [AMF].
Task 1.7. Further development and uses of diagrammatic techniques for OP [JJMB].
Task 1.8. Further development of generating function methods for OPUC and non-standard OP, and their connections with functional equations [JJMB].
Task 1.9. Development of Heine type integral representation for OPUC and applications to OPUC theory, highly oscillatory quadrature and random matrix models [AMF].

Researchers involved: being this activity crucial, it will require the efforts of all members of the team, in one sense or another. Namely:

- A. Martínez Finkelshtein should play a leading role in Tasks 1.1, 1.2, 1.4, 1.5, the latter in collaboration with P. Martínez González and A. Zarzo, who will take part also in Task 1.4 together with some researchers from other groups: A. Durán (U. Sevilla), E. Rakhmanov (U. South Florida), A. Aptekarev (Keldysh Institute of Applied Mathematics, Russia), S. Suetin (Steklov Mathematical Institute, Russia), M. Tiaglov (Shanghai Jiao Tong University), and D. Dimitrov, S. Ranga (UNESP).
- Sobolev OP (Task 1.3) are intensively studied by J.J. Moreno Balcázar and P. Martínez González.
- F.A. Grünbaum and L. Velázquez, which have already developed the QRW approach to Khrushchev formulas for matrix OPUC, are in charge of Tasks 1.6, 1.7, in collaboration with L. Moral and E. Huertas (U. Coimbra, Portugal).
- M.J. Cantero, in collaboration with A. Iserles (U. Cambridge, UK), is responsible of Tasks 1.8, 1.9.

The working plan includes several scientific exchanges: at least one stay of 15 days each year at KU Leuven (Belgium), Cambridge (UK), Coimbra (Portugal) and Berkeley (USA), two visits of F.A. Grünbaum to Spain, exchange visits of researchers from Almería and Zaragoza. There are also plans for an exchange with active groups from the US and Europe mentioned in the item “Related research teams”. Taking into account the intensity of the development of this topic and the international presence of this team, it is fundamental to keep the current high level of international collaboration.

Activity 2: electrostatics, equilibrium, and quadratic differentials
The tight connection of the logarithmic equilibrium and asymptotics, along with the recently introduced concept of **critical measures**, starting a whole program of extension of this notion to other fields, such as a well known class of ODE (equations of Heun’s type) or the so-called closed (or Striebel) **quadratic differentials**. This correspondence is not one-to-one, and describing the quadratic differentials associated precisely with positive critical measures is still open in the general case, something we plan to address. This question can be reduced also to problems about moduli of families of curves on Riemann surfaces.
Extending the **electrostatic interpretation** of the zeros of classical OP (for instance, to MOP) is currently a virtually open problem, in which members of this project have made important steps. It would allow, among other things, to find higher order asymptotics, as well as to deepen into the (not well understood) relation between OP and ODE.

A key feature of the support of a critical measure is its **S-property**, well understood in the scalar case, but in its infancy for the vector equilibria. We also plan to address this problem, which requires lifting the moduli problem to Riemann surfaces of higher genuses.

The study of the **support** of the equilibrium measures for one of these extremal problems has important applications in random matrix theory (phase transitions). We have started this study for the real case, but extending it to the complex plane would allow to reveal links with the Hele-Shaw or **laplacian growth**, and other problems.

Any advance in these problems will have direct impact on the goals outlined in Activity 1.

This Activity is divided into 3 tasks (initials of coordinating PI within brackets):

Task 2.1. Study of the critical measures on the plane and their connection with geometric function theory (quadratic differentials on a Riemann surface and problem of moduli). Extension of the notion of the S-property to vector equilibria [AMF].

Task 2.2. Electrostatic models for zeros of Heine-Stieltjes and MOP. Further asymptotics of polynomial solutions of ODE of Heun type, and relation with the monodromy problem [JJMB].

Task 2.3. Analysis of the dependence of the support of the equilibrium and critical measures from the intrinsic parameters of the system, description of the mechanisms of phase transitions, and connections with differenty notions of balayage [AMF].

Researchers involved: A. Martínez Finkelshtein should play a leading role in all goals of this Activity. Tasks 2.1, 2.3 are addressed in collaboration with E. Rakhmanov (U. South Florida, USA) and G. Silva (KU Leuven, Belgium). Task 2.2 is in collaboration with P. Martínez González and the research groups of D. Dimitrov and S. Ranga (UNESP, Brasil).

The working plan includes several scientific exchanges: at least one stay of 15 days each year at KU Leuven (Belgium), and a longer research stay at UNESP, two visits of F.A. Grünbaum to Spain, exchange visits of researchers from Almería and Zaragoza. There are also plans for an exchange with active groups from the US and Europe mentioned in the item “Related research teams”. Again, taking into account the intensity of the development of this topic and the international presence of this team, it is fundamental to keep the current high level of international collaboration.

Activity 3: Stochastic processes and Random Matrix models

This team is especially interested in a class of continuous stochastic processes (diffusion processes), given by **biorthogonal ensembles** and non-colliding point processes. These include different types of random matrices, as well as diffusion processes of many particles, an efficient modeling tool in physics and technology. We plan to continue our study of the interacting random matrix models and random matrix models with external source. Further achievements depend heavily on the progress of the matrix RH approach and the theory of vector equilibrium measures on the complex plane (Activities 1 and 2). We plan to extend our analysis also to normal, or non-Hermitian, matrix models.

The intertwining of ideas an achievements from Activities 1 and 4 allows us to expand our interest to other lines of research. For instance, Khrushchev formulas for OPUC can be generalized to give splitting rules for Schur functions related to abstract operators. This suggests the existence of splitting rules for Markov processes which we intend to explore.

This Activity is subdivided into 3 tasks (coordinating PI: AMF):

Task 3.1. Study of stochastic models (random matrices, non-colliding diffusion and other determinantal processes) associated to MOP, their asymptotics and universality.

Task 3.2. Asymptotics of normal matrix models, relation to the laplacian growth and other diffusion processes on the plane.

Task 3.3. Development of splitting techniques for Markov processes, based on Khrushchev type formulas for generating functions of first time return probabilities.

Researchers involved: A. Martínez Finkelshtein will play a leading role in Tasks 3.1, 3.2. Task 3.3 is basically responsibility of the same team involved in the closely related Tasks 1.7, 1.8: F. A. Grünbaum, L. Velázquez, in collaboration with E. Huertas (U. Coimbra).

The working plan includes scientific exchanges similar to those mentioned in Activities 1 and 2. Especially relevant are short visits to Coimbra, KU Leuven, Berkeley and IUPUI.

Activity 4: OP approach to quantum random walks (QRW)

The CGMV method, based on the use of OPUC, CMV matrices and Schur functions, provides efficient tools for analytical and qualitative analysis of QRW, as it follows from recent works of the group. Direct relations between spectral measures and limit laws would be of maximum interest. On the other hand, Schur functions have proved to be an invaluable tool for the study of return properties of QRW. Their analysis yields a number of deep and surprising consequences even in absence of interaction. More interesting settings should include external interactions or self-interactions, also in the context of QRW in general graphs. The general treatment of return properties requires the use of matrix Schur functions and matrix Khrushchev formulas codifying very useful splitting rules for QRW. Matrix Schur functions and matrix Khrushchev formulas are also of interest for the study of topological phases in QRW. Current analysis of this problem assume translation invariance, a requirement which prevents from a rigorous proof of the bulk-edge correspondence observed in **topological insulators and superconductors**: when crossing over from one phase to another, a bound state appears localized in the region connecting both phases. The importance of this result lies on the possible use of such bound states as information storage units in a future **quantum computer**. Our new approach avoids translation invariance, opening the possibility of a really general and rigorous study of **topological phases**.

This Activity is subdivided into 4 tasks (coordinating PI: JJMB):

Task 4.1. General relations between limit laws for QRW and spectral measures. In particular, analysis of singular continuous measures and measures giving strongly ballistic diffusion.
Task 4.2. Return properties of interacting QRW. Situations of experimental interest.
Task 4.3. CGMV approach to QRW on graphs.
Task 4.4. Use of Schur functions techniques and Khrushchev formulas for the classification of topological phases in QRW. Rigorous proof of bulk-edge correspondence.

Researchers involved: the main weight of this activity lies upon L. Velázquez and F.A. Grünbaum, with participation of A. Martínez Finkelshtein, and in collaboration with R.F. Werner and his group on Quantum Information (Leibniz U. Hannover) and A.H. Werner (Freie U. Berlin). There are also partial collaborations with J. Bourgain (IAS, Princeton), J. Wilkening (UC Berkeley), N. Konno (Yokohama National U.), E. Segawa (Tohoku U.).
The working plan includes also scientific exchanges among UC Berkeley, U. Zaragoza, Leibniz U. Hannover, Freie U. Berlin, involving F.A. Grünbaum and L. Velázquez, and collaborations with U. Almería and Yokohama National University.

Activity 5: Darboux, integrable systems, bispectral problems and signal theory

The idea is to study the possible extension of the strong connections among these topics, already known in the context of Jacobi matrices and OPRL, to **block-Jacobi matrices** and **matrix OPRL**, as well as to CMV matrices and OPUC. Apart from the interest that these results have by themselves, the hope is that widening the scope of Darboux transformations could help to shed light on the origin of these miraculous interconnections.

This Activity is subdivided into two tasks (coordinating PI: JJMB):

Task 5.1. Darboux transformations for block-Jacobi matrices, integrable systems, bispectral situations and the time-band-limiting problem in signal processing.
Task 5.2. Darboux for CMV, integrable systems, bispectral problems and signal theory.

Researchers involved: Task 5.1 is carried out basically by F.A. Grünbaum (UC Berkeley), who has already made the first steps in this direction. This researcher is also involved in Task 5.2, together with the team form U. Zaragoza (M.J. Cantero, L. Moral, L. Velázquez), which in collaboration with F. Marcellán (U. Carlos III, Madrid) is currently developing the Darboux transformations for CMV matrices.
The working plan includes also scientific exchanges among institutions of members of the team (UC Berkeley, U. Zaragoza) and the U. Carlos III de Madrid.

Activity 6: Image processing and clinical applications

The development of new diagnostic techniques (aberrometers, corneal topographers, OCT, etc.) has put forward the need of analysis of fine features of anatomic structures of the human eye from experimentally obtained data. This team has accumulated experience in the use of mathematical tools (especially, from approximation theory and scientific computation) in improving the diagnostic technology used in biomechanical engineering and ophthalmology (several papers and a patent, partially in collaboration with practitioners from the VISSUM Ophthalmological Corporation). We will continue exploring more efficient algorithms for objective measurements of the eye characteristics and diagnostics. A recently created Instituto de Investigaciones Neurobiológicas Neurobia has expressed interest in results of this research, putting their diagnostic equipment at this team's disposal. Some of the tasks related below might lead to patents and knowledge transfer to industry.

This Activity is subdivided into 4 tasks (coordinating PI: AMF):

Task 6.1. Further study and improvement of the reconstruction techniques of wavefronts and corneal topography, alternative or complementary to Zernike polynomials fitting.

Task 6.2. Study of applicability of methods of compressed sensing and sparse signal reconstruction in the setting of corneal topography.

Task 6.3. Simplification of ray-tracing procedures for a real-time modeling of the internal structure of an in-vivo eyes.

Task 6.4. Study of applicability of phase-recovery techniques in the reconstruction of wavefront aberrations, as a possible alternative to the Hartman-Shack aberrometry.

Researchers involved: this Activity is basically a responsibility of the reserchers from U. Almería (A. Martínez Finkelshtein, P. Martínez González, D. Ramos López) and UPM (A. Zarzo). We plan to incorporate in this Activity the graduate student Ana Belén Castaño, whose research will be centered at these problems. A crucial ingredient is our collaboration with the team of R. Iskander, from the Institute of Biomedical Engineering and Instrumentation, Wroclaw University of Technology, Poland, as well as with researchers from Queensland University of Technology, Australia (M.J. Collins, D. Alonso-Caneiro). The whole activity will be carried out in collaboration with Instituto de Investigaciones Neurobiológicas Neurobia ("EPO", "Promoting and Observing Entity", for our project).

The working plan includes further contacts with groups in optometry and ophthalmology, such as those led by R. Montés-Micó (U. Valencia) or N. López-Gil (U. Murcia).

C.1.5. Infrastructure

Human resources:

This team is comprised of 10 members, 9 of which are Ph. D., and 1 graduate student (with Master in Mathematics and B. Sc. in Computer Science); 5 are from the University of Almería (1 "Catedrático de Universidad" – Full Professor, 2 "Profesor Titular de Universidad" - Associate Professors, 1 post-graduate and 1 graduate student), 3 from the University of Zaragoza (two "Profesores Titulares de Universidad" and a "Profesor Contratado Doctor"), 1 from the Polytechnic University of Madrid ("Profesor Titular de Universidad "), and 1 from University of California, Berkeley, U.S.A. (Full Professor). Essentially, this is the same team of the projects MTM2005-08648-C02, MTM2008-06689-C02 and MTM2011-28952-C02, mentioned above. The CVs show the productivity and the high level of the team in all fields related to our research project, with publications in the top ranked international journals, according to the impact indices. Furthermore, the researchers of this team take part or lead several projects or research groups of local or regional character. Those team members not belonging to the coordination institution (U. Almería) exhibit a long collaboration record with the team, partially endorsed by the previous joint projects and publications. The only foreign member (Dr. Grünbaum) has being crucial in the development of one of the central lines of research. His membership in the project is motivated by the stable ongoing collaboration (materialized in several joint publications), and the complementary character of his research. All the members of the team are active researchers, with important international presence.

Two members are part of the editorial boards or of the advisory panel of first class specialized international journals: A. Martínez Finkelshtein is editor of the *Journal of Approximation Theory*, of *Computational Methods and Function Theory*, and of the *Journal of Spectral Theory* and *Random Matrix: Theory and Applications,* while A. Grünbaum has been editor (and currently is member of the advisory panel) of *Inverse Problems*. The relationship with outstanding international researchers, specified in the item "Related research teams", is intense and free flowing, expressed both in joint publications and mobility of researchers.

Among organizational aspects we may stress the participation of the team members in organization of several international scientific events in the last 3 years, such as:

- 12 International Conference "Computational and Mathematical Methods in Science and Engineering" (CMMSE). Minisymposium "Recent Trends on Orthogonal Polynomials and Special Functions", Murcia, July 2-5, 2012 (Organizing Committee).
- 12 International Symposium on Orthogonal Polynomials, Special Functions and Applications (OPSFA'12), Tunissia, March 25-29, 2013 (Scientific Committee).
- 13 International Symposium on Orthogonal Polynomials, Special Functions and Applications (OPSFA'13), USA, June 1-5, 2015 (in preparation), (Scientific Committee).

Material resources:
The main computational needs of this team (desktop computers, laptops, tablets, and printers) are partially covered by the existing inventory; the necessary and schedulled renovation of this equipment is included in this application.
A crucial ingredient, at least for Activity 6, is the access to opthalmological diagnostic equipment, partially provided by Instituto de Investigaciones Neurobiológicas Neurobia.

C.1.6. Timetable

The features of mathematical research prevent from elaborating detailed time distribution of activities and/or tasks beyond simple speculations. As it is well known, in Mathematics problems comprising our goals can be tackled sequentially or simultaneously, without possibility of even a short term forecast of which is the best option, or even to make realistic predictions of time necessary to achieve them. At any rate, the only possible time distribution has been made in several sections of this proposal.
This chronogram indicates the researchers involved in the project. The details of the tasks are given in item C.1.4.
Acronyms: AMF: Andrei Martínez Finkelshtein, **JJMB**: Juan José Moreno Balcázar, **LML**: Leandro Moral Ledesma, **MJCM**: María José Cantero Medina, **LVC**: Luis F. Velázquez Campoy, **AZA**: Alejandro Zarzo Altarejos, **PMG**: Pedro Martínez González **DRL**: Darío Ramos López, **ABCF**: Ana Belén Castaño Fernández, **AG**: F. Alberto Grünbaum.

Tasks	Center	Researchers	Period
1.1, 1.2, 1.3, 1.4, 1.5	U. Almería, UC Berkeley U. Pol. Madrid	AMF, JJMB, AG, AZA, PMG.	Spans the three years of the project (here and below)
1.6, 1.7, 1.8, 1,9	U. Zaragoza UC Berkeley	LVC, AG, MJCM, LML.	
2.1, 2.2, 2.3	U. Almería	AMF, JJMB, PMG	
3.1, 3.2, 3.3	U. Almería, U. Zaragoza, UC Berkeley	AMF, LVC, AG	
4.1, 4.2, 4.3, 4.4	U. Zaragoza UC Berkeley U. Almería	LVC, AG, AMF	
5.1, 5.2	UC Berkeley U. Zaragoza	AG, LVC, MJCM, LML	
6.1, 6.2, 6.3, 6.4	U. Almería, U. Pol. Madrid	AMF, DRL, AZA, PMG, ABCF	

C.1.7. Justification of inclusion of one graduate student.

The volume of the work is overwhelmingly large, so in order to maintain the competitive advantage of this team at least one graduate student would be crucial. The possible specializations could be the fastest advancing topics of the Riemann-Hilbert analysis or of the quantum random walks (or both).

It should be observed that a relatively low number of dissertations under the supervision of the members of this team (see Section C.3) is explained uniquely by the lack of financial support for graduate students: national members of this team belong to universities in the periphery, with very scarse proper resources for scholarships. Thus, the group has to rely completely on external sources of financial support, being able to obtain it so far only from Junta de Andalucía. Despite all our efforts, no scholarship from any national project (FPI) has been granted along these years (although, ironically, we did receive observations from the referees of the previous project saying that we should increase our "commitment towards training of young researchers").

C.2. IMPACTO ESPERADO DE LOS RESULTADOS

As already pointed out in section C.1.1, the research group of this project has intense and stable connections within Spain and abroad, being some of its members worldwide experts in some of the subjects, including developers or even creators of highly topical tools in the area. The development of these connections with the clear possibility of increasing them and the maintenance and improvement of the international leadership are two of the primary consequences expected form the project, whose results will be spread in several directions:

- From the **editorial perspective**, by means of their publication in highly ranked specialized journals with high impact and visibility, cited in recognized databases such as Thomson-Reuters (formerly ISI) Web of Science and Journal Citation Reports (see the "*Selected publications of the team related with the project*" in section C.1.1 for some relevant examples of such journals).
- In **Internet**, by means of the web site of this team hosted by the server of the University of Almería (www.ual.es), of the Institute Carlos I of Theoretical and Computational Physics (http://www.ugr.es/~carlos1), and of the Research Group "Approximation Theory and Orthogonal Polynomials" of Junta de Andalucía (http://www.ual.es/GruposInv/Tapo/), as well as in several international electronic preprint archives (e.g. arxiv.org, math.nist.gov/opsf, and others). Moreover, members of the team have their own web sites, with a detailed account of research activities and results (see e.g. http://www.ual.es/~andrei/).
- **Personally**, at conferences and meetings, both national and international, of scientific areas related to the project, that usually appear published in corresponding proceedings, as well as in national and foreign research institutions by means of talks and colloquia delivered by the team members.

On the other hand, as it is also described in section C.1.1, this project focuses both on basic and fundamental research in the theory of orthogonal polynomials and special functions, and on their applications in science and technology. So, the results may generate transfer of knowledge mainly from the outcome of Activity 6. This expectation is based both on the previous experience of the team (with already one pattent) and on the ongoing collaboration with the "Instituto de Investigaciones Neurobiológicas Neurobia", which is an E.P.O. ("Promoting and Observing Entity") for the project.

C.3. CAPACIDAD FORMATIVA DEL EQUIPO SOLICITANTE

Nine of ten members of the team are Ph. D., publishing regularly in the most prestigious journals included in the Science Citation Index; this activity has come to fruition in more than 200 publications (appeared or in process) in top journals of the field during the years spanned by the previous projects mentioned in previous sections; as well as in participation in 15 research projects (where we count only those where at least one non-foreign member has taken part): regional (5), national (7) and international (3). On the other hand, several members of the team combine their intense scientific production with scientific administration or collaboration within editorial boards of scientific journals. Two members teach within Spanish graduate programs: Postgrado Oficial de Matemáticas (AMF and JJMB, http://www.ugr.es/~doctomat/), Master en Física y Matemáticas (AMF, http://www.ugr.es/~fisymat/), and Programa de Doctorado Interuniversitario "Las aguas subterraneas y el medio ambiente" (AMF); the last two exhibit the Certificate of Quality by the Spanish Ministry of Education.

Regarding the Ph. D. dissertations advised by the team members, 4 have been defended along last 10 years, and 2 are in preparation (see our remarks in Section C.1.7 explaining this relatively low number of dissertations).

It should be stressed also that the team has a wide experience in coordination of the joint research; besides, geographic localization of the team members at 3 European universities and 1 in the USA gives clear advantages for training of young researchers. A wide network of connections with international researchers of acknowledged prestige, consolidated by this team along years allows graduate students to improve their level at foreign institutions of high quality, with special experience in topics related to or complementary to the research carried out for the dissertation.

C.4. IMPLICACIONES ÉTICAS Y/O DE BIOSEGURIDAD

None.