Special functions and orthogonal polynomials

Workshop A6 at FoCM'11

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Organizers:
Peter A. Clarkson (University of Kent, United Kingdom)
Andrei Martínez-Finkelshtein (University of Almera, Spain)
Kerstin Jordaan (University of Pretoria, South Africa)

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   Bernhard Beckermann (University of Lille, France)

2. *Random matrix model with external source and a constraint equilibrium problem*
   Pavel M. Bleher (Indiana University-Purdue University, USA)

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16. *Orthogonal polynomials and quantum walks*
    Luis Velazquez (Universidad de Zaragoza, Spain)

Superlinear convergence of the rational Arnoldi method for matrix functions

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Abstract

An important problem arising in science and engineering is the computation of matrix functions $f(A)b$, where A is a large Hermitian matrix, b a vector of unit length, and f is a sufficiently smooth function, e.g., $A^{-1/2}b$ with a Markov function f. Here a popular method consists in projecting to so-called rational Krylov spaces, which mathematically is equivalent to interpolate f by some rational functions with fixed poles, and interpolation points given by so-called rational Ritz values. The asymptotic distribution of such rational Ritz values can be described via some constrained equilibrium problem with a particular external field [2]. By generalizing [3], we present a new Buyarov–Rakhmanov type formula for the rate of convergence containing the one given in [1], and allowing to explain superlinear convergence.

Joint work with Stefan Güttel.

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Random matrix model with external source and a constraint equilibrium problem

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Abstract

We study the Brézin-Hikami random matrix model with external source, in the case when the potential $V(x)$ is an even polynomial and the external source has two eigenvalues $\pm a$ of equal multiplicity. We find the limiting distribution of eigenvalues in terms of a constraint vector equilibrium problem, and we prove the universality of the limiting local eigenvalue correlations.

This is a joint project with Arno Kuijlaars and Steven Delvaux.

Don't be afraid of the 1001st (numerical) derivative

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Abstract

The accurate and stable numerical calculation of higher-order derivatives of holomorphic functions (as needed, e.g. in RMT, to extract higher-order gap probabilities from a generating function) turns out to be a surprisingly rich topic: there are connections to asymptotic analysis, the theory of entire functions, and to a problem in algorithmic graph theory.

Nonintersecting paths with a staircase initial condition

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Abstract

The talk will describe a model of discrete nonintesecting paths starting at equidistant points and ending at consecutive integers. The model is equivalent to a tiling model and as such can be viewed as one of placing boxes on a staircase. The process at the local scale, close to the starting points, does not fall in the universality class of the sine kernel. Instead, as the number of paths tends to infinity we obtain a new family of kernels describing the local correlations, one of whose limits is the extended sine kernel. One interesting feature of the model is number variance saturation.

This is joint work with Maurice Duits.

Partition function and free energy in the cubic random matrix model

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Abstract

We consider the partition function and the free energy of a unitary random matrix model with weight function $e^{-NV(z)}$, where $V(z) = z^2/2 - uz^3$ and $u > 0$ is a real parameter. For $0 \leq u < u_c$, where u_c is an explicit known value, the free energy admits an asymptotic expansion in powers of N^{-2}. The first two terms of this topological expansion are known from [2], and can be written in terms of hypergeometric functions. Near the critical value $u = u_c$, a double scaling limit leads to an asymptotic approximation in terms of a particular solution of Painlevé I, similarly to the quartic model analyzed in [3].

Joint work with Pavel Bleher

References

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Random graphs and Chebyshev polynomials

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Abstract

Random matrix theory abounds in results involving Chebyshev polynomials. A classical example is that, for most types of β-ensembles, global fluctuations for linear functionals are given by Gaussian processes whose covariance structure is given by appropriate Chebyshev polynomials.

In random graph theory, a similar phenomenon is being observed; this time, though, the connections to Chebyshev polynomials are more basic–in fact, they have deterministic roots.

I will survey some of these connections in the case of regular graphs, show how they extend to the case of bipartite biregular ones, mention the applications to random graphs, and enumerate a few open problems.

Continuous and discrete special functions from the self-dual Yang-Mills equations

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Abstract

Many integrable differential equations are known to arise as reductions of the self-dual Yang-Mills equations. In this talk, various discrete integrable equations and discrete special functions will be derived from the Bäcklund transformations of the self-dual Yang-Mills equations.

Universality in Hamiltonian PDEs

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Abstract

We consider Hamiltonian PDEs and study the behaviour of solutions near critical points. Such behaviour is locally independent on the initial data and it is described by a special solution of a Painlevé equation.

Quadratic decomposition of orthogonal polynomials, Stieltjes functions and Laguerre-Hahn linear functionals

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Abstract

A sequence of monic polynomials $(R_n)_{n\geq 0}$ (SMOP, in short) orthogonal with respect to a linear functional v is said to be quasi-symmetric if it satisfes the three term recurrence relation $R_{n+2}(x) = (x - (-1)^{n+1}\beta_0)R_{n+1}(x) - \gamma_{n+1}R_n(x), n \geq 0$, with initial conditions $R_0(x) = 1,\ R_1(x) = x - \beta_0$. Here, $\beta_0 \in \mathbf{C}$, and $\gamma_{n+1} \in \mathbf{C}^*, n \geq 0$.

Such a SMOP is characterized by the following quadratic decomposition $R_{2n}(x) = P_n(x^2)$, $R_{2n+1}(x) = (x - \beta_0)P_n^*(x^2), n \geq 0$, where $(P_n)_{n\geq 0}$ is a SMOP with respect to a linear functional u and $(P_n^*)_{n\geq 0}$ is the sequence of monic kernel polynomials of K-parameter β_0^2 associated with u.

We prove that v is a Laguerre-Hahn functional if and only if u is a Laguerre-Hahn sequence. The polynomial coefficients of the Riccati equation satisfied by the Stieltjes function corresponding to v are given in terms of those of u. In such a way, we can deduce the class of v in terms of the class of u.

As an application, we determine all quasi-symmetric Laguerre-Hahn functionals of class one.

This is a joint work with Belgacem Bouras. It has been supported by Ministerio de Ciencia e Innovación of Spain, under grant MTM2009-12740-C03-01.

Equilibrium problems for vector potentials with semidefinite interaction matrices and constrained masses

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Abstract

In this talk we prove existence and uniqueness of a solution to the problem of minimizing the logarithmic energy of vector potentials associated to a d-tuple of positive measures supported on closed subsets of the complex plane. The assumptions we make on the interaction matrix are weaker than the usual ones and we also let the masses of the measures vary in a compact subset of $\mathbb{R}_+^d$. We characterize the solution in terms of variational equations. Finally, we review a few examples taken from the recent literature that are related to our results.

Joint work with Bernhard Beckermann, Valery Kalyagin and Franck Wielonsky.

References

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Computing Painlevé II in the complex plane

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Abstract

Painlevé transcendents can be rewritten as Riemann–Hilbert problems, from which their asymptotics can be determined. But Riemann–Hilbert problems can also be solved numerically, using a method we have recently constructed. We apply this approach to Painlevé II, to obtain a numerical method which is fast and reliable, and accurate in the complex plane. By deforming the Riemann–Hilbert problem along the path of steepest descent, the approach remains accurate in the asymptotic regime as well.

The Gamma function from a complex perspective

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Abstract

In this talk we highlight some recent results for functions related to Euler's gamma function:

- The median $m(x)$ in the Gamma distribution is investigated as a function of the shape parameter x. The median is defined as
$$\int_0^{m(x)} e^{-t} t^{x-1}\, dt = \frac{1}{2}\Gamma(x).$$
It is obtained that m is convex and this proves a conjecture of Chen-Rubin in a continuous setting.
- The volume of the unit ball in n dimensional Euclidean space. Denoting this volume by Ω_n, it is obtained that
$$\Omega_{n+2}^{1/(n+2)\log(n+2)}$$
is a Hausdorff moment sequence. This is based on properties of the one parameter family
$$F_a(x) = \frac{\log\Gamma(x+1)}{x\log(ax)}, \quad a \geq 1.$$
These functions are extended to the complex plane cut along the negative real axis.

Complex methods play an important role in the proofs. If time permits remainders in asymptotic expansions of the double and triple gamma functions introduced by Barnes will be investigated.

The talk is based on joint work with Christian Berg and with Stamatis Koumandos.

Two-variable Wilson polynomials and the generic superintegrable system on the 3-sphere

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Abstract

In this talk, I will discuss representations of the symmetry algebra for the quantum superintegrable system on the 3-sphere with generic 4-parameter potential. Physically relevant irreducible representations of the quadratic algebra are represented via divided difference operators in two variables. We determined several ON bases for this model including spherical and cylindrical bases. These bases are expressed in terms of two variable Wilson and Racah polynomials with arbitrary parameters, as defined by Tratnik. The generators for the quadratic algebra are expressed in terms of recurrence operators for the one-variable Wilson polynomials. The quadratic algebra structure breaks the degeneracy of the space of these polynomials. I will also discuss how the analogous 2D system is embedded in the 3D system and how these models indicate a general relationship between 2nd order superintegrable systems and discrete orthogonal polynomials.

This work is joint with co-authors W. Miller Jr. and E.J. Kalnins and can be found on the arXiv:1010.3032.

Doubling measures and zeros of orthogonal polynomials

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Abstract

A measure μ on a compact subset E of the real line is called doubling if $\mu(2I) \leq L\mu(I)$ for all intervals $I \subset E$. We review some recent results (partially joint work with G. Mastroianni and T. Varga) on the behavior of the Christoffel functions as well as rough zero spacing of orthogonal polynomials with respect to doubling or locally doubling measures. The doubling property of μ seems to be the most general condition (substantiated by a converse result) under which zero spacing of classical orthogonal polynomials remains valid for the OP's associated with μ.

Orthogonal polynomials on a bi-lattice

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Abstract

We investigate generalizations of the Charlier and the Meixner polynomials on the lattice $\mathbb{N}$ and on the shifted lattice $\mathbb{N}+1-\beta$. We combine both lattices to obtain the bi-lattice $\mathbb{N}\cup(\mathbb{N}+1-\beta)$ and show that the orthogonal polynomials on this bi-lattice have recurrence coefficients which satisfy a non-linear system of recurrence equations, which we can identify as a limiting case of an (asymmetric) discrete Painlevé equation. We make some observations about the asymptotic behavior of the recurrence coefficients.

Joint work with Christophe Smet

Orthogonal polynomials and quantum walks

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Abstract

Quantum walks have become a candent research line in the last years due to their relevance in the understanding of quantum phenomena, and their important role in quantum computation and quantum information. Their classical counterpart, i.e., the random walks driven by stochastic matrices, have a close relation with the orthogonal polynomials on the real line which is known from the work of Karlin and McGregor in 1959. The cornerstone of this connection is the Jacobi matrix related to the recurrence of orthogonal polynomials on the real line. Such a relation has shown to be very fruitful, allowing for the use of spectral methods based on orthogonal polynomial techniques in the study of random walks.

The purpose of this talk is to show that a similar approach is possible for quantum walks too. We will see that quantum walks are connected to the orthogonal polynomials on the unit circle instead of the real line. The recently discovered canonical matrix representations of the unitary operators, known as CMV matrices, play an essential role in this connection. We will also show the advantages of the orthogonal polynomial representation of a quantum walk, especially regarding asymptotic properties of the quantum walk.

This is a joint work with F.A. Grünbaum (University of California, Berkeley), M.J. Cantero and L. Moral (Universidad de Zaragoza).